Ordinary Differential Equations

Autumn 2018

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http://www.jamescannon.net/teaching/ordinary-differential-equations

http://raw.githubusercontent.com/NanoScaleDesign/OrdinaryDifferentialEquations/master/ode.pdf

Contents

Chapter 0

Course information

0.1 This course

This is the Autumn 2018 Ordinary Differential Equations course studied by 2nd-year undegraduate international students at Kyushu University.

0.1.1 How this works

- In contrast to the traditional lecture-homework model, in this course the learning is self-directed and active via publicly-available resources.
- Learning is guided through solving a series of carefully-developed challenges contained in this book, coupled with suggested resources that can be used to solve the challenges with instant feedback about the correctness of your answer.
- There are no lectures. Instead, there is discussion time. Here, you are encouraged to discuss any issues with your peers, teacher and any teaching assistants. Furthermore, you are encouraged to help your peers who are having trouble understanding something that you have understood; by doing so you actually increase your own understanding too.
- Discussion-time is from 08:40 to 10:10 on Mondays at room Centre Zone 1409.
- Peer discussion is encouraged, however, if you have help to solve a challenge, always make sure you do understand the details yourself. You will need to be able to do this in an exam environment. If you need additional challenges to solidify your understanding, then ask the teacher. The questions on the exam will be similar in nature to the challenges. If you can do all of the challenges, you can get 100% on the exam.
- Every challenge in the book typically contains a **Challenge** with suggested **Resources** which you are recommended to utilise in order to solve the challenge. Occasionally the teacher will provide extra **Comments** to help guide your thinking. A **Solution** is also made available for you to check your answer. Sometimes this solution will be given in encrypted form. For more information about encryption, see section 0.3.
- For deep understanding, it is recommended to study the suggested resources beyond the minimum required to complete the challenge.
- The challenge document has many pages and is continuously being developed. Therefore it is advised to view the document on an electronic device rather than print it. The date on the front page denotes the version of the document. You will be notified by email when the document is updated. The content may differ from last-year's document.
- A target challenge will be set each week. This will set the pace of the course and defines the examinable material. It's ok if you can't quite reach the target challenge for a given week, but then you will be expected to make it up the next week.
- You may work ahead, even beyond the target challenge, if you so wish. This can build greater flexibility into your personal schedule, especially as you become busier towards the end of the semester.
- Your contributions to the course are strongly welcomed. If you come across resources that you found useful that were not listed by the teacher or points of friction that made solving a challenge difficult, please let the teacher know about it!

0.1.2 Assessment

In order to prove to outside parties that you have learned something from the course, we must perform summative assessments. This will be in the form of a mid-term exam (weighted 30%), coursework (weighted 15%), a satisfactory challenge-log (weighted 5%) and a final exam (weighted 50%).

Your final score is calculated as Max(final exam score, weighted score), however you must pass the final exam to pass the course.

0.1.3 What you need to do

- Prepare a challenge-log in the form of a workbook or folder where you can clearly write the calculations you perform to solve each challenge. This will be a log of your progress during the course and will be occasionally reviewed by the teacher.
- You need to submit a brief report at `https://goo.gl/forms/AqTAZ6D1exFbH1PW2` by 4am on the day of the class. Here you can let the teacher know about any difficulties you are having and if you would like to discuss anything in particular.
- Please bring a wifi-capable internet device to class, as well as headphones if you need to access online components of the course during class. If you let me know in advance, I can lend computers and provide power extension cables for those who require them (limited number).

0.2 Timetable

	Discussion	**Target**	**Note**
1	9 Oct	-	Tuesday class
2	15 Oct	3.2	
3	22 Oct	3.11	
4	29 Oct	4.8	
5	5 Nov	4.15	
6	12 Nov	4.18	
7	19 Nov		
8	3 Dec		
9	10 Dec	Midterm exam	
10	17 Dec		
11	7 Jan		
12	15 Jan		Coursework assignment
13	21 Jan		
14	28 Jan	Coursework	Coursework submission
15	4 Feb	Final exam	

Example: To keep pace with the course, you should aim to complete challenge 2 of chapter 3 by the 15th of October.

0.3 Hash-generation

Some solutions to challenges are encrypted using MD5 hashes. In order to check your solution, you need to generate its MD5 hash and compare it to that provided. MD5 hashes can be generated at the following sites:

- Wolfram alpha: (For example: md5 hash of "q1.00") `http://www.wolframalpha.com/input/?i=md5+hash+of+%22q1.00%22`
- `www.md5hashgenerator.com`

Since MD5 hashes are very sensitive to even single-digit variation, you must enter the solution *exactly.* This means maintaining a sufficient level of accuracy when developing your solution, and then entering the solution according to the format suggested by the question. Some special input methods:

Solution	**Input**
5×10^{-476}	5.00e-476
5.0009×10^{-476}	5.00e-476
$-\infty$	-infinity (never "infinite")
2π	6.28
i	im(1.00)
2i	im(2.00)
1 + 2i	re(1.00)im(2.00)
-0.0002548 i	im(-2.55e-4)
1/i = i/-1 = -i	im(-1.00)
$e^{i2\pi}$ $[= cos(2\pi) + isin(2\pi) = 1 + i0 = 1]$	1.00
$e^{i\pi/3}$ $[= cos(\pi/3) + isin(\pi/3) = 0.5 + i0.87]$	re(0.50)im(0.87)
Choices in order A, B, C, D	abcd

The first 6 digits of the MD5 sum should match the first 6 digits of the given solution.

0.4 Questions about the final exam

Will the final exam cover the entire course or only the course content between the mid-term exam and the end of the course?

The final exam will cover the entire course.

Do we need to memorise formulae like that for the Runge-Kutta method?

The aim is to test understanding rather than the ability to memorise formulae. So if you need to use the Runge-Kutta method I will supply the forumula for it. That said, you will need to remember basic methods that are fundamental to the basis of solving ODE's, such as the characteristic solutions to 2nd-order differential equations.

Will it be stated if we should use method X or method Y to solve an ODE?

If it doesn't specifically state how to solve a problem, then you're welcome to use whatever method you find easiest. If it states how you should solve a problem then you should use the method indicated.

Chapter 1

Hash practise

1.1 Hash practise: Integer

X = 46.3847
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

hash of aX = e77fac

1.2 Hash practise: Decimal

X = 49
Form: Two decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

hash of bX = 82c9e7

1.3 Hash practise: String

X = abcdef
Form: String.
Place the indicated letter in front of the number.
Example: aX where $X = abc$ is entered as aabc

hash of cX = 990ba0

1.4 Hash practise: Scientific form

X = 500,765.99
Form: Scientific notation with the mantissa in standard form to 2 decimal place and the exponent in integer form.
Place the indicated letter in front of the number.
Example: aX where $X = 4 \times 10^{-3}$ is entered as a4.00e-3

hash of dX = be8a0d

Chapter 2

Definitions

2.1 Order of a differential equation

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

What is the sum of the orders of the following equations?

$$\frac{dy}{dx}A = 5x^3 + 3 \tag{2.1}$$

$$cos(y)y'''(x) - y(x) = 25 \tag{2.2}$$

$$\frac{d}{dx}\frac{d^2y}{dx^2} = \frac{x^{-2}}{3} \tag{2.3}$$

Solution

X = Your solution
Form: Integer
Place the indicated letter in front of the number
Example: aX where $X = 46$ is entered as a46

hash of eX = 492585

2.2 Identifying linear and non-linear differential equations

Resources

- Video: `https://www.youtube.com/watch?v=bVy1F5T8QE4`

Comment

Being able to identify linear and non-linear ODE's will help you understand how to approach different problems.

Generally speaking, the differential equation is linear if the functions and orders of the differentials are linear. For example,

$$y'' - 4yx = lnx - y$$

can be shown to be linear. Rearranging to collect all the y-terms together:

$$y'' - 4yx + y = lnx$$

the dependent variable y and its derivatives are each of the first degree and depend only on a constant or the independent variable.

An example of a non-linear equation however would be

$$5 + yy' = x - y$$

or

$$yy' + y = x - 5$$

The fact that y' is multiplied by y results in a non-linear equation in y.

Challenge

Sum the points corresponding to the equations that are linear. Be sure to clearly justify your choices of linear or non-linear equations.

1 point: $\dfrac{dy}{dt} = 5t^3 + 3$.

2 points: $cos(y)y'''(t) - y(t) = 25$.

4 points: $\dfrac{d}{dt}\dfrac{d^2y}{dt^2} = \dfrac{t^{-2}}{3}$.

8 points: $y'(t) - sin(y(t)) = 0$.

16 points: $y'(t) - y(t) = 0$.

32 points: $ty'(t) - y(t) = 0$.

Solution

X = Your solution
Form: Integer
Place the indicated letter in front of the number
Example: aX where $X = 46$ is entered as a46

hash of rX = f5d2c0

2.3 Linear differential equations vs non-linear differential equations

Resources

- Wikipedia: `https://en.wikipedia.org/wiki/Nonlinear_system#Nonlinear_differential_equations`
- Wikipedia: `https://en.wikipedia.org/wiki/Linear_differential_equation`

Challenge

Write no-more than 1 short paragraph describing in qualitative terms the difference between a linear and non-linear differential equation.

Solution

Please compare with your partner in class and discuss with the teacher if you are unsure.

2.4 Valid solutions

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Use substitution to prove that

$$y = \frac{5}{5+x} \tag{2.4}$$

is a solution to the equation

$$xy' + y = y^2 \tag{2.5}$$

and state the value of x for which the solution is undefined.

Solution

Value of x for which solution is undefined:

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of tX = 829f33

2.5 Range of valid solutions

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Use substitution to prove that

$$y = -\sqrt{100 - x^2} \tag{2.6}$$

is a solution to the equation

$$x + yy' = 0 \tag{2.7}$$

and state the range of x for which the solution is valid. Enter the value of the lower range as the solution below.

Solution

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of yX = d96920

Chapter 3

1st-order differential equations

3.1 Determining a simple DE from a description

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`

Challenge

Newton's law of cooling states that the rate of cooling of an object is proportional to the temperature difference with the ambient surroundings. (a) Write a differential equation describing this situation. (b) Assuming a proportionality constant of 0.2 /hour, what is the rate of temperature change when the object is at 30 °C and the ambient temperature is 20 °C?

Solution

(units: $^{\circ}\mathrm{C\,h^{-1}}$)

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of qX = 4aca8d

3.2 Direction (Slope) fields

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/DirectionFields.aspx`
- Video 1: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/creating-a-slope-field`
- Video 2: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/differential-equations-intro/v/slope-field-to-visualize-solutions`

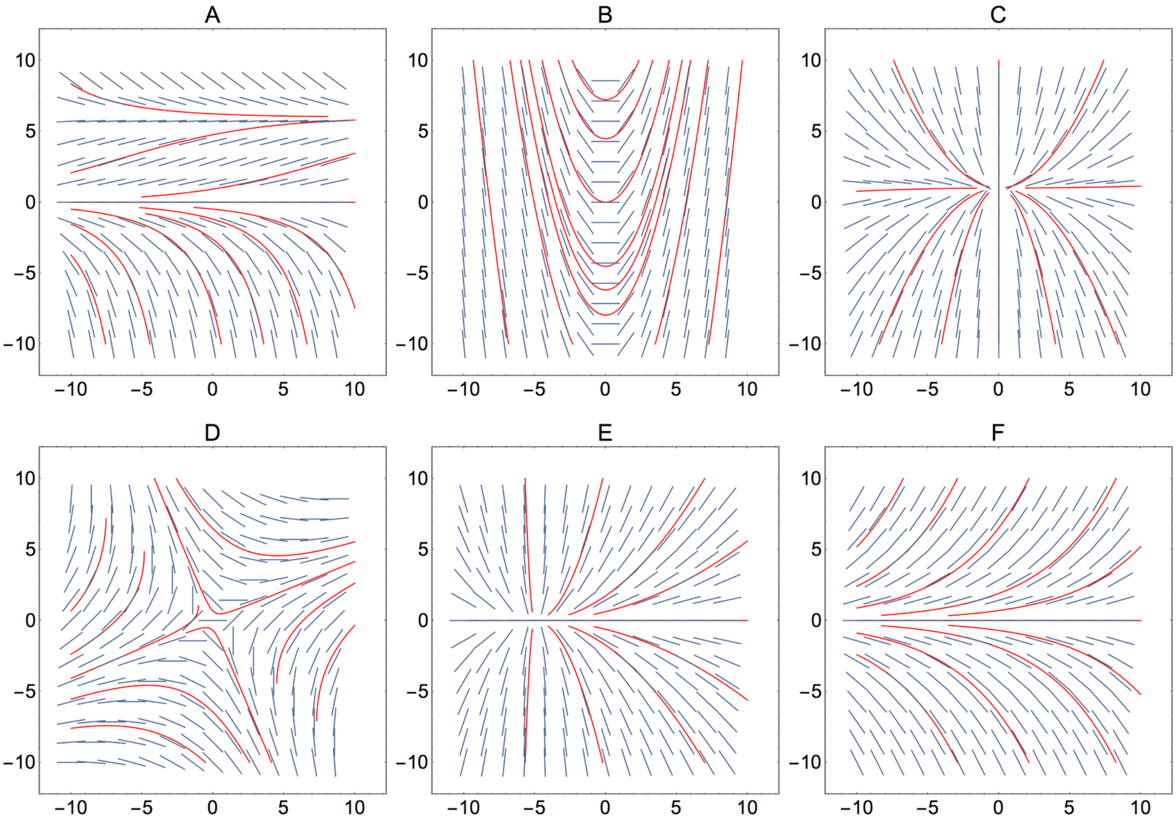
Comment

It is good practise to try drawing the below fields before looking at the next page. You need to be able to go in both directions (ie, drawing and recognising). You will not be given a glimps at the fields in the exam prior to being asked to draw them.

Question

Try drawing the slope field for at least 3 of the equations given below (your choice). Then, put the slope fields given on the next page in the same order as these equations.

1. $y' = x$
2. $y' = 0.2y$
3. $y' = 0.2y(1 - y/6)$
4. $y' = (x - y)/(x + y)$
5. $y' = 2(y - 1)/x$
6. $y' = 2y/(x + 5)$



Solution

X = Your solution
Form: String.
Place the indicated letter in front of the string.
Example: aX where X = abcdef is entered as aabcdef

Hash of qX = e93bfe

3.3 Separable equations I

Resources

- Video I: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction`
- Video II: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example`
- Text:`http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx`

Comment

Let's start with a fundamental equation:

$$\frac{dy}{dt} = y \tag{3.1}$$

This is saying that the slope (the rate of change of y) linearly depends on y. That is, that as the value of y increases, the slope also increases; a positive feedback loop. In fact, you get an exponentially-increasing function.

So one aim of this course is to be able to solve such equations mathematically. But I also want you to understand the "physical" meaning of the relation between y and its slope, and how this leads to such a fundamental function such as an exponential.

Challenge

Considering the equation

$$\frac{dy}{dt} = y \tag{3.2}$$

solve for y.

Solution

To check your answer, solve for $y(5)$ given the initial condition $y(0) = 1$.

$y(5) = 148.413$

3.4 Separable equations II

Resources

- Video I: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction`
- Video II: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example`
- Text:`http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx`

Challenge

a) Now consider what is meant, physically speaking, by the relation:

$$\frac{dy}{dt} = -y \tag{3.3}$$

Why does y tend to zero for increasing t?

b) Solve for y.

Solution

a) Please compare your solution with your partner or discuss with the teacher.

b) To check your answer, solve for $y(5)$ given the initial condition $y(0) = 1$.

$y(5) = 0.00674$

3.5 Separable equations III

Resources

- Video I: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction
- Video II: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example
- Text:http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx

Challenge

a) Now consider when the slope of y not only depends on y but also on t:

$$\frac{dy}{dt} = ty \tag{3.4}$$

b) or on a constant a:

$$\frac{dy}{dt} = ay \tag{3.5}$$

See how the feedback is greater or lesser, depending on the constant or variable placed in front of y?

Solution

a) Solve for $y(5)$ under the initial condition $y(0) = 1$

268,337

b) Solve for $y(5)$ under the initial condition $y(0) = 1$ and with $a = 2$

22,026.5

3.6 Separable equations IV

Resources

- Video I: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/separable-differential-equations-introduction
- Video II: https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/separable-equations/v/particular-solution-to-differential-equation-example
- Text:http://tutorial.math.lamar.edu/Classes/DE/Separable.aspx

Challenge

Determine $y(t)$ for

$$\frac{dy}{dt} = e^t \tag{3.6}$$

Again, think about what is happening here. Do you see the link with challenge 3.3? There we wrote in terms of y. Here we write in terms of e^t. Do you see they're the same thing?

Solution

To check your answer, solve for $y(3)$ given the initial condition $y(0) = 1$.

$y(3) = 20.09$

3.7 Rate of growth

Resources

- Video: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/logistic-differential-equation/v/modeling-population-with-differential-equations`

Comment

One interesting application of 1st-order differential equations is that of population growth.

Challenge

Assuming there is no-limit on growth, a given bacteria would be able to reproduce at such a rate that the amount of bacteria measured in mg increases by 20% every 25 hours. Derive an expression for the rate of growth.

Solution

To check your answer, calculate the rate of growth when there are 20 mg of bacteria. *To ensure accuracy, you will need to maintain a large degree of precision during your calculations.*

0.146 mg/hour

3.8 Logistic equation

Resources

- Videos: The 4 remaining logistic differential equation videos starting at: `https://www.khanacademy.org/math/differential-equations/first-order-differential-equations/logistic-differential-equation/v/logistic-differential-equation-intuition`

Comment

We considered exponential growth, but in real life there is often a limit to this. This is where the logistic equation is useful.

Challenge

Assuming there is no-limit on growth, a given bacteria would be able to reproduce at such a rate that the amount of bacteria measured in mg increases by 20% every 25 hours. However, due to environmental factors the limiting (maximum) amount of bacteria that can exist in the system at any one time is 400 mg. Assuming an initial amount of bacteria of 20 mg, how much time must one wait to reach 100 mg of bacteria?

Solution

253 hours

3.9 Autonomous differential equations

Resources

- Wikipedia: `https://en.wikipedia.org/wiki/Autonomous_system_(mathematics)`

Challenge

The logistic equation is an example of an autonomous differential equation. Add the points of the autonomous differential equations in the following list:

1 point: $y' = cos(y) - 5$

2 points: $y' = cos(y)/x - 5$

4 points: $y' = cos(y)/x - 5/x$

8 points: $y^2 = y'y + 5$

16 points: $xy' = 5y$

32 points: $y' = 1$

Solution

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of fX = 1227c7

3.10 The stability of solutions I

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx`
- Text: `http://www.math.psu.edu/tseng/class/Math251/Notes-1st%20order%20ODE%20pt2.pdf`

Challenge

Considering the logistic equation $N' = 0.2N(1 - N/6)$, make 3 separate lists containing any equilibrium, semi-stable and unstable y-values.

To check your answer, sum the value of each list. If there are no values in a list, enter -999 to check the result.

Solution

Stable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of gX = 4a4314

Semi-stable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of hX = 9df203

Unstable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of jX = 17cb7f

3.11 The stability of solutions II

Resources

- Text: `http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx`
- Text: `http://www.math.psu.edu/tseng/class/Math251/Notes-1st%20order%20ODE%20pt2.pdf`

Challenge

Considering the differential equation $y' = (y^2 - 16)(y + 3)^2$, make 3 separate lists containing any equilibrium, semi-stable and unstable y-values.

To check your answer, sum the value of each list. If there are no values in a list, simply enter "none" to check the result.

Solution

Stable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of kX = ffc446

Semi-stable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of zX = f76cc4

Unstable

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

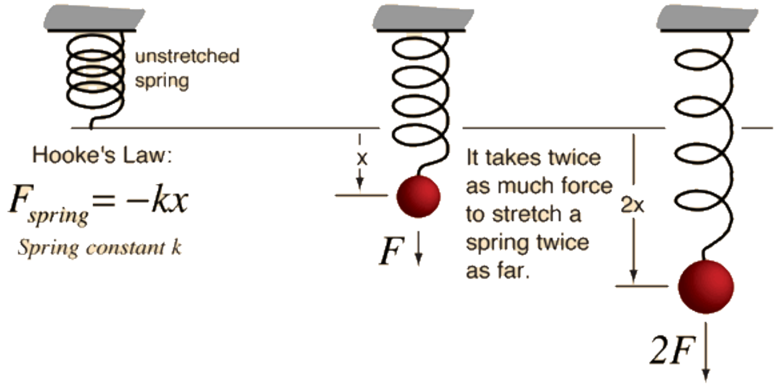
Hash of xX = bf947d

Chapter 4

2nd-order differential equations

4.1 (C1) Hooke's law

Comment



(Image from HyperPhysics by Rod Nave, Georgia State University)

Second-order differential equations deal with oscillations. Here we consider harmonic oscillation of a spring. The aim of this challenge is to give you the opportunity to think about how the terms of a 2nd-order ODE relate to force and stiffness in the context of a spring.

Equation 4.1 is a fundamental equation of mechanics describing oscillatory motion such as the spring here. Hooke's law states that the force leading to acceleration of the mass m is proportional to the stretching distance x. The proportionality constant is Hooke's constant, k.

$$mx'' + kx = 0 \tag{4.1}$$

or alternatively

$$mx'' = -kx \tag{4.2}$$

This leads to perfectly oscillating motion,

$$x(t) = \cos(\omega t) \tag{4.3}$$

which oscillates forever since there is no damping term.

Challenge

By considering the oscillatory motion (equation 4.3) as a solution of the 2nd-order differential equation given by Hooke's law (equations 4.1 and 4.2), determine the oscillation frequency ω in terms of the mass and spring constant.

Solution

To check your answer, calculate the oscillation frequency for a harmonic spring with a mass of 2 kg and spring-constant of 4 kg/s^2. Only enter numbers, without any units, in your answer.

X = Your solution
Form: Decimal to 2 decimal places.
Place the indicated letter in front of the number.
Example: aX where $X = 46.00$ is entered as a46.00

Hash of gX = 9553fe

4.2 (C2) Exponentials and trigonometry

Resources

- Text: https://www.phy.duke.edu/~rgb/Class/phy51/phy51/node15.html
- Text: http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf
- Video: https://www.youtube.com/watch?v=jhrJLmlwI98

Challenge

Write $\sin(x)$ and $\cos(x)$ in exponential form.

Solution

Please compare your solution with your partner or discuss with the teacher.

4.3 (C3) Characteristic equation: understanding

Resources

- Book (http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N) from page 113.

Comment

It is possible to add a damping term B to Hooke's law that is proportional to the velocity of the movement. You could imagine this as a friction term, with the force from friction becoming stronger as the velocity increases.

Challenge

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0 \tag{4.4}$$

Show that, assuming that all solutions to a 2nd-order differential equation of the form above will have solutions $y(t) = e^{rt}$, the value of r can in principle be determined by solving a quadratic equation of the form

$$Ar^2 + Br + C = 0 \tag{4.5}$$

Solution

If you are unsure of your derivation, please ask someone.

4.4 (C4,C5,C6) Characteristic equation: roots

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 106.
- Video (student suggestion): `https://www.youtube.com/watch?v=gdr4dSmzZ8Q`

Challenge

Sum the points of the differential equations that have characteristic equations with

- Real, distinct roots (C4)
- Complex roots (C5)
- Equal roots (C6)

1 point: $-3y'' - 5y' + 2y = 0$

2 points: $3y'' - 4y' + 3y = 0$

4 points: $3y'' - 6y' + 3y = 0$

8 points: $3y'' - 5y' + 2y = 0$

16 points: $3y'' - 5y' + 4y = 0$

32 points: $3y'' + 5y' + 2y = 0$

Solution

Real, distinct roots

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of iX = dc6ada

Complex roots

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of jX = 7c030b

Equal roots

X = Your solution
Form: Integer.
Place the indicated letter in front of the number.
Example: aX where $X = 46$ is entered as a46

Hash of kX = c90b44

4.5 (C7) Characteristic equation: real negative roots

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 118.
- Text: `http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf`

Challenge

1. Solve the following 2nd-order differential equation that has real roots:

$$y'' + 3y' + 2y = 0 \tag{4.6}$$

2. What is the effect of the damping term here?

Solution

1. $y(1) = 1.14$ given initial conditions $y(0) = 5$ and $y'(0) = -8$.
2. Please compare your answer with your partner or discuss with the teacher in class.

4.6 (C8) Characteristic equation: real positive roots

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 118.
- Text: `http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf`

Comment

Here we include a damping term again, but this time it is negative and this is enough to flip the sign of the roots.

Challenge

1. Solve the following 2nd-order differential equation that has real roots.

$$y'' - 3y' + 2y = 0 \tag{4.7}$$

2. What is the effect of the damping term here?

Solution

1. $y(1) = -47.13$ given initial conditions $y(0) = 5$ and $y'(0) = -8$.
2. Please compare your answer with your partner or discuss with the teacher in class.

4.7 (C9) Characteristic equation: real mixed roots

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 118.
- Text: `http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf`

Comment

Here we include the same negative damping term, but due to the coefficient of the other terms you obtain roots of mixed signs.

Challenge

Solve the following 2nd-order differential equation that has real roots.

$$-2y'' - 3y' + 2y = 0 \tag{4.8}$$

Solution

$y(1) = -5.38$ given initial conditions $y(0) = 5$ and $y'(0) = -20$

4.8 (C10) Characteristic equation: equal roots

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 127.
- Text: `http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf`

Comment

It is not necessary to follow the full derivation in the suggested resource.

Challenge

Solve the equation

$$y'' - 2y' + y = 0 \tag{4.9}$$

subject to the initial conditions $y(0) = 5$ and $y'(0) = 6$.

Solution

$y(1) = 16.310$

4.9 (C11) Characteristic equation: complex roots with B=0

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 122.
- Text: `http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf`

Challenge

1. Assuming there is no damping term (ie, $B = 0$) show that the roots for the differential equation

$$Ay'' + Cy = 0 \tag{4.10}$$

are $\pm i\sqrt{C/A}$.

2. Solve the following ODE:

$$y'' + 4\pi^2 y = 0 \tag{4.11}$$

subject to the initial conditions $y(0) = 4$ and $y'(0) = 10\pi$.

Solution

1. Please discuss in class if you are unsure of your solution.
2. Please check the solution on challenge-hub by solving for $y(0.4)$.

4.10 (C12) Characteristic equation: complex roots with non-zero B I

Resources

- Book (`http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N`) from page 122.
- Text: `http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf`

Challenge

Solve the following ODE:

$$y'' + y' + y = 0 \tag{4.12}$$

subject to initial conditions $y(0) = 8$ and $y'(0) = 2$. One of the integration constants is $4\sqrt{3}$. You will need to find the other one.

Solution

Please check the solution on challenge-hub by solving for $y(0.4)$.

4.11 (C13) Characteristic equation: complex roots with non-zero B II

Resources

- Book (http://tutorial.math.lamar.edu/getfile.aspx?file=B,1,N) from page 122.
- Text: http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf

Challenge

Solve the following ODE:

$$y'' - y' + y = 0 \tag{4.13}$$

subject to initial conditions $y(0) = 1$ and $y'(0) = 2$. One of the integration constants is $\sqrt{3}$. You will need to find the other one.

Solution

Please check the solution on challenge-hub by solving for $y(0.4)$.

4.12 (C14) Damping

Resources

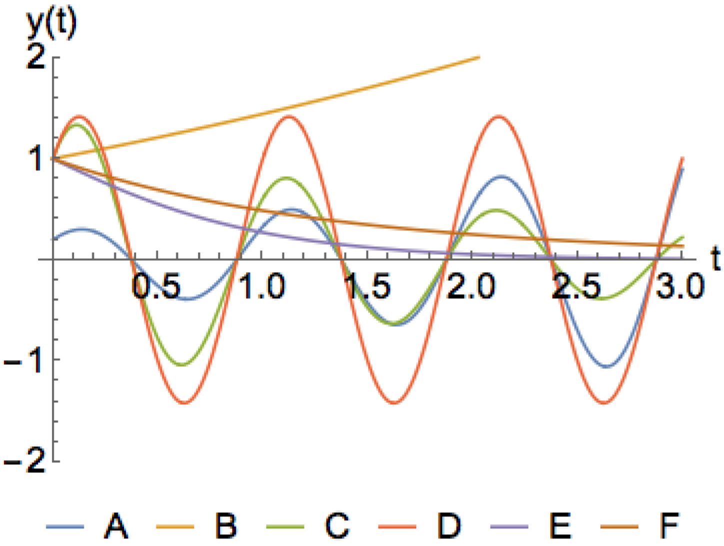
- Wikipedia: `https://en.wikipedia.org/wiki/Damping`
- Text: `http://www.its.caltech.edu/~roberto/FSRI/Lecture/fsri_math_2011_Aug_4.pdf`
- Video: `https://www.youtube.com/watch?v=99ZE2RGwqSM`

Comment

The video in the listed resources gives a particularly nice demonstration of the different forms of damping.

Challenge

Of the 6 functions shown in the graph, place the 3 that correspond to over-damped, critically damped and under-damped in the order mentioned in this sentence.



Solution

(eg, "abc")

Please check the solution on challenge-hub.

4.13 (C15) Damping and 2nd-order differential equations

Challenge

1. The 6 functions shown in the graph in challenge 4.12 may represent solutions of a 2nd-order differential equation $Ay'' + By' + Cy = 0$. Assuming $A > 0$ and $C > 0$, place the solutions A-F in the order shown below.

I. Solution of a 2nd-order differential equation with real negative roots.

II. Solution of a 2nd-order differential equation with real positive roots.

III. Solution of a 2nd-order differential equation with equal roots.

IV. Solution of a 2nd-order differential equation with complex roots and B=0.

V. Solution of a 2nd-order differential equation with complex roots and positive damping.

VI. Solution of a 2nd-order differential equation with complex roots and negative damping.

Solution

(eg, "abcdef")

X = Your solution
Form: String.
Place the indicated letter in front of the string.
Example: aX where X = abcdef is entered as aabcdef

Hash of uX = 33db25

4.14 (C16,C17,C18) Characteristic equation: exercises

(Note that if you encounter a square-root during your calculations such as $\sqrt{7}$, it is best to work with $\sqrt{7}$ rather than 2.65 in order to maintain accuracy until the final step where you need to evaluate it. If the equation becomes too messy (eg $e^{(\sqrt{7}-1)/\sqrt{3}}$) you can always substitute $m = (\sqrt{7}-1)/\sqrt{3}$, etc, to make things clearer.)

Challenge

1. Determine $y(1)$ for the equation

$$2y'' + 8y' + y = 0 \tag{4.14}$$

given the initial conditions $y(0) = 4$ and $y'(0) = 3$.

2. Determine $y(0.2)$ for the equation

$$2y'' + 4y' + 2y = 0 \tag{4.15}$$

given the initial conditions $y(0) = 4$ and $y'(0) = 2$.

3. Determine $y(0.1)$ for the equation

$$4y'' + 3y' + y = 0 \tag{4.16}$$

given the initial conditions $y(0) = 6$ and $y'(0) = 2$.

Solution

1. (C16)
2. (C17)
3. (C18)

4.15 (C19,C20,C21,C22) Non-homogeneous equations: Method of undetermined coefficients

Resources

- Video: All 4 Khan Academy videos starting at `https://www.khanacademy.org/math/differential-equations/second-order-differential-equations/undetermined-coefficients/v/undetermined-coefficients-1`

Comment

The 2nd-order equations we were considering until now were homogeneous equations (ie, the RHS was zero). We can now build upon this to expand our ability to solve non-homogeneous equations (ie, where the RHS of the equation is non-zero).

The Khan Academy videos give an excellent initial introduction to the subject, and so please do take the time to view and take notes about all four videos in the series.

In the 4th video Mr Kahn describes about how it is possible to add solutions if there are multiple terms on the right. This occasionally causes confusion. Consider for example:

$$y'' - 3y' - 4y = 2\sin x \tag{4.17}$$

This corresponds to the particular solution

$$y_p = A\sin x + B\cos x \tag{4.18}$$

A common point of confusion is about what to do in the case of something like

$$y'' - 3y' - 4y = 2\sin x + 2\cos x \tag{4.19}$$

Should you just write $y_p = (A\sin x + B\cos x) + (C\sin x + D\cos x)$? After all, you have two terms in equation 4.19 (ie, $2\sin x$ and $2\cos x$). You can note however that $A\sin x + C\sin x$ simplifies to $E\sin x$ where E is just another constant (in this case $A + B$) so in the end you will be left with $y_p = E\sin x + F\cos x$. So while it may be clearer to explicitly calculate coefficients for every term on the RHS, in many cases the terms will simplify.

Challenge

Find the general solution of the following non-homogeneous differential equations:

1. (C19) $y'' + 4y = 8$
2. (C20) $y'' + 4y = 8t^2 - 20t + 8$
3. (C21) $y'' + 4y = 5\sin 3t - 5\cos 3t$
4. (C22) $y'' + 4y = 24e^{-2t}$

(please just rate the challenges on challenge-hub after you have determined the answer for each one)

Solution

The solutions are contained in the list on the next page in no particular order. Your answers should match one of the solutions given. Please try to not look at the solutions before completing the questions, since this will facilitate deep understanding and reproduce a real-life/exam environment.

$y = C_1 \cos 2t + C_2 \sin 2t + 3e^{-2t}$
$y = C_1 \cos 2t + C_2 \sin 2t + 8e^{-2t}$
$y = C_1 \cos 2t + C_2 \sin 2t + 2t^2 - 5t + 1$
$y = C_1 \cos 2t + C_2 \sin 2t + 3t^2 + t + 3$
$y = C_1 \cos 2t + C_2 \sin 2t + \cos 3t - \sin 3t$
$y = C_1 \cos 2t + C_2 \sin 2t + 2$
$y = C_1 \cos 2t + C_2 \sin 2t + 5$

4.16 (C23-28) Method of undetermined coefficients II

Comment

The following pages go into more detail than the videos, considering a greater range of cases. You may note that here the particular solution is denoted by Y while Sal Khan denoted it as y_p in the videos.

The following notes were developed by Zachary S. Tseng at Pennsylvania State University, USA (http://www.math.psu.edu/tseng/). Included here with kind permission.

A (possible) glitch?

There is a complication that occurs under a certain circumstance…

Example: $$y'' - 2y' - 3y = 5e^{3t}$$

The old news is that $y_c = C_1 e^{-t} + C_2 e^{3t}$. Since $g(t) = 5e^{3t}$, we should be able to use the form $Y = A e^{3t}$, just like in the first example, right? But if we substitute Y, $Y' = 3A e^{3t}$, and $Y'' = 9A e^{3t}$ into the differential equation and simplify, we would get the equation

$$0 = 5e^{3t}.$$

That means there is no solution for A. Our method (that has worked well thus far) seems to have failed. The same outcome (an inability to find A) also happens when $g(t)$ is a multiple of e^{-t}. But, for any other exponent our choice of the form for Y works. What is so special about these two particular exponential functions, e^{3t} and e^{-t}, that causes our method to misfire? (Hint: What is the complementary solution of the nonhomgeneous equation?)

The answer is that those two functions are exactly the terms in y_c. Being a part of the complementary solution (the solution of the corresponding homogeneous equation) means that any constant multiple of either functions will ALWAYS results in zero on the right-hand side of the equation. Therefore, it is impossible to match the given $g(t)$.

The cure: The remedy is surprisingly simple: multiply our usual choice by t. In the above example, we should instead use the form $Y = A t e^{3t}$.

In general, whenever your initial choice of the form of Y has any term in common with the complementary solution, then you must alter it by multiplying your initial choice of Y by t, as many times as necessary but no more than necessary.

Example: $$y'' - 6y' + 9y = e^{3t}$$

The complementary solution is $y_c = C_1 e^{3t} + C_2 t e^{3t}$. $g(t) = e^{3t}$, therefore, the initial choice would be $Y = A e^{3t}$. But wait, that is the same as the first term of y_c, so multiply Y by t to get $Y = A t e^{3t}$. However, the new Y is now in common with the second term of y_c. Multiply it by t again to get $Y = A t^2 e^{3t}$. That is the final, correct choice of the general form of Y to use. (*Exercise*: Verify that neither $Y = A e^{3t}$, nor $Y = A t e^{3t}$ would yield an answer to this problem.)

Once we have established that $Y = A t^2 e^{3t}$, then $Y' = 2Ate^{3t} + 3At^2 e^{3t}$, and $Y'' = 2Ae^{3t} + 12Ate^{3t} + 9At^2 e^{3t}$. Substitute them back into the original equation:

$$(2Ae^{3t} + 12Ate^{3t} + 9At^2 e^{3t}) - 6(2Ate^{3t} + 3At^2 e^{3t}) + 9(At^2 e^{3t}) = e^{3t}$$

$$2Ae^{3t} + (12 - 12)Ate^{3t} + (9 - 18 + 9)At^2 e^{3t} = e^{3t}$$

$$2Ae^{3t} = e^{3t}$$

$$A = 1/2$$

Hence, $Y(t) = \frac{1}{2} t^2 e^{3t}$.

Therefore, $y = C_1 e^{3t} + C_2 te^{3t} + \frac{1}{2} t^2 e^{3t}$. Our “cure” has worked!

Since a second order linear equation's complementary solution only has two parts, there could be at most two shared terms with Y. Consequently we would only need to, at most, apply the cure twice (effectively multiplying by t^2) as the worst case scenario.

The lesson here is that you should always find the complementary solution first, since the correct choice of the form of Y depends on y_c. Therefore, you need to have y_c handy before you write down the form of Y. Before you finalize your choice, always compare it against y_c. And if there is anything those two have in common, multiplying your choice of form of Y by t. (However, you should do this ONLY when there actually exists something in common; you should never apply this cure unless you know for sure that a common term exists between Y and y_c, else you will not be able to find the correct answer!) Repeat until there is no shared term.

When $g(t)$ is a product of several functions

If $g(t)$ is a product of two or more simple functions, e.g. $g(t) = t^2 e^{5t}\cos(3t)$, then our basic choice (before multiplying by t, if necessary) should be a product consist of the corresponding choices of the individual components of $g(t)$. One thing to keep in mind: that there should be only as many undetermined coefficients in Y as there are distinct terms (after expanding the expression and simplifying algebraically).

Example: $$y'' - 2y' - 3y = t^3 e^{5t}\cos(3t)$$

We have $g(t) = t^3 e^{5t}\cos(3t)$. It is a product of a degree 3 polynomial†, an exponential function, and a cosine. Out choice of the form of Y therefore must be a product of their corresponding choices: a generic degree 3 polynomial, an exponential function, and both cosine and sine. Try

Correct form: $Y = (At^3 + Bt^2 + Ct + D)\,e^{5t}\cos(3t) +$
$(Et^3 + Ft^2 + Gt + H)\,e^{5t}\sin(3t)$

Wrong form: $Y = (At^3 + Bt^2 + Ct + D)\,E\,e^{5t}(F\cos(3t) + G\sin(3t))$

Note in the correct form above, each of the eight distinct terms has its own unique undetermined coefficient. Here is another thing to remember: that those coefficients should all be independent of each others, each uniquely associated with only one term.

In short, when $g(t)$ is a product of basic functions, $Y(t)$ is chosen based on:

i. $Y(t)$ is a product of the corresponding choices of all the parts of $g(t)$.
ii. There are as many coefficients as the number of distinct terms in $Y(t)$.
iii. Each distinct term must have its own coefficient, not shared with any other term.

† A power such as t^n is really just an n-th degree polynomial with only one (the n-th term's) nonzero coefficient.

Another way (longer, but less prone to mistakes) to come up with the correct form is to do the following.

Start with the basic forms of the corresponding functions that are to appear in the product, without assigning any coefficient. In the above example, they are $(t^3 + t^2 + t + 1)$, e^{5t}, and $\cos(3t) + \sin(3t)$.

Multiply them together to get all the distinct terms in the product:

$$(t^3 + t^2 + t + 1)\, e^{5t}(\cos(3t) + \sin(3t))$$
$$= t^3 e^{5t} \cos(3t) + t^2 e^{5t} \cos(3t) + t\, e^{5t} \cos(3t) + e^{5t} \cos(3t)$$
$$+ t^3 e^{5t} \sin(3t) + t^2 e^{5t} \sin(3t) + t\, e^{5t} \sin(3t) + e^{5t} \sin(3t)$$

Once we have expanded the product and identified the distinct terms in the product (8, in this example), then we insert the undetermined coefficients into the expression, one for each term:

$$Y = A\, t^3 e^{5t} \cos(3t) + B\, t^2 e^{5t} \cos(3t) + C\, t\, e^{5t} \cos(3t)$$
$$+ D\, e^{5t} \cos(3t) + E\, t^3 e^{5t} \sin(3t) + F\, t^2 e^{5t} \sin(3t) + G\, t\, e^{5t} \sin(3t)$$
$$+ H\, e^{5t} \sin(3t)$$

Which is the correct form of Y seen previously.

Therefore, whenever you have doubts as to what the correct form of Y for a product is, just first explicitly list all of terms you expect to see in the result. Then assign each term an undetermined coefficient.

Remember, however, the result obtained still needs to be compared against the complementary solution for shared term(s). If there is any term in common, then the entire complex of product that is the choice for Y must be multiplied by t. Repeat as necessary.

Example: $$y'' + 25y = 4t^3 \sin(5t) - 2e^{3t}\cos(5t)$$

The complementary solution is $y_c = C_1 \cos(5t) + C_2 \sin(5t)$. Let's break up $g(t)$ into 2 parts and work on them individually.

$g_1(t) = 4t^3 \sin(5t)$ is a product of a degree 3 polynomial and a sine function. Therefore, Y_1 should be a product of a generic degree 3 polynomial and both cosine and sine:

$$Y_1 = (At^3 + Bt^2 + Ct + D)\cos(5t) + (Et^3 + Ft^2 + Gt + H)\sin(5t)$$

The validity of the above choice of form can be verified by our second (longer) method. Note that the product of a degree 3 polynomial and both cosine and sine: $(t^3 + t^2 + t + 1)\times(\cos(5t) + \sin(5t))$ contains 8 distinct terms listed below.

$$\begin{array}{llll} t^3\cos(5t) & t^2\cos(5t) & t\cos(5t) & \cos(5t) \\ t^3\sin(5t) & t^2\sin(5t) & t\sin(5t) & \sin(5t) \end{array}$$

Now insert 8 independent undetermined coefficients, one for each:

$$\begin{aligned} Y_1 = {} & At^3\cos(5t) + Bt^2\cos(5t) + Ct\cos(5t) + D\cos(5t) + \\ & Et^3\sin(5t) + Ft^2\sin(5t) + Gt\sin(5t) + H\sin(5t) \end{aligned}$$

However, there is still one important detail to check before we could put the above expression down for Y_1. Is there anything in the expression that is shared with $y_c = C_1 \cos(5t) + C_2 \sin(5t)$? As we can see, there are – both the fourth and the eighth terms. Therefore, we need to multiply everything in this entire expression by t. Hence,

$$\begin{aligned} Y_1 &= t(At^3 + Bt^2 + Ct + D)\cos(5t) + \\ &\qquad t(Et^3 + Ft^2 + Gt + H)\sin(5t) \\ &= (At^4 + Bt^3 + Ct^2 + Dt)\cos(5t) + \\ &\qquad (Et^4 + Ft^3 + Gt^2 + Ht)\sin(5t). \end{aligned}$$

The second half of $g(t)$ is $g_2(t) = -2e^{3t}\cos(5t)$. It is a product of an exponential function and cosine. So our choice of form for Y_2 should be a product of an exponential function with both cosine and sine.

$$Y_2 = Ie^{3t}\cos(5t) + Je^{3t}\sin(5t).$$

There is no conflict with the complementary solution – even though both $\cos(5t)$ and $\sin(5t)$ are present within both y_c and Y_2, they appear alone in y_c, but in products with e^{3t} in Y_2, making them parts of completely different functions. Hence this is the correct choice.

Finally, the complete choice of Y is the sum of Y_1 and Y_2.

$$Y = Y_1 + Y_2 = (At^4 + Bt^3 + Ct^2 + Dt)\cos(5t) + (Et^4 + Ft^3 + Gt^2 + Ht)\sin(5t) + Ie^{3t}\cos(5t) + Je^{3t}\sin(5t).$$

Example: $$y'' - 8y' + 12y = t^2 e^{6t} - 7t\sin(2t) + 4$$

Complementary solution: $y_c = C_1 e^{2t} + C_2 e^{6t}$.

The form of particular solution is

$Y = (At^3 + Bt^2 + Ct)e^{6t} + (Dt + E)\cos(2t) + (Ft + G)\sin(2t) + H.$

Example: $$y'' + 10y' + 25y = t\,e^{-5t} - 7t^2 e^{2t}\cos(4t) + 3t^2 - 2$$

Complementary solution: $y_c = C_1 e^{-5t} + C_2\, t\, e^{-5t}$.

The form of particular solution is

$Y = (At^3 + Bt^2)e^{-5t} + (Ct^2 + Dt + E)\,e^{2t}\cos(4t) + (Ft^2 + Gt + H)\,e^{2t}\sin(4t)$
$+ I t^2 + J t + K.$

Example: Find a second order linear equation with constant coefficients whose general solution is

$$y = C_1 e^t + C_2 e^{-10t} + 4t^2.$$

The solution contains three parts, so it must come from a nonhomogeneous equation. The complementary part of the solution, $y_c = C_1 e^t + C_2 e^{-10t}$ suggests that $r = 1$ and $r = -10$ are the two roots of its characteristic equation. Hence, $r - 1$ and $r + 10$ are its two factors. Therefore, the characteristic equation is $(r - 1)(r + 10) = r^2 + 9r - 10$.

The corresponding homogeneous equation is, as a result,

$$y'' + 9y' - 10y = 0.$$

Hence, the nonhomogeneous equation is

$$y'' + 9y' - 10y = g(t).$$

The nonhomogeneous part $g(t)$ results in the particular solution $Y = 4t^2$. As well, $Y' = 8t$ and $Y'' = 8$. Therefore,

$$g(t) = Y'' + 9Y' - 10Y = 8 + 9(8t) - 10(4t^2) = 8 + 72\mathrm{t} - 40\, t^2.$$

The equation with the given general solution is, therefore,

$$y'' + 9y' - 10y = 8 + 72\mathrm{t} - 40\, t^2.$$

The 6 Rules-of-Thumb of the Method of Undetermined Coefficients

1. If an exponential function appears in $g(t)$, the starting choice for $Y(t)$ is an exponential function of the same exponent.

2. If a polynomial appears in $g(t)$, the starting choice for $Y(t)$ is a generic polynomial of the same degree.

3. If either cosine or sine appears in $g(t)$, the starting choice for $Y(t)$ needs to contain both cosine and sine of the same frequency.

4. If $g(t)$ is a sum of several functions, $g(t) = g_1(t) + g_2(t) + \ldots + g_n(t)$, separate it into n parts and solve them individually.

5. If $g(t)$ is a product of basic functions, the starting choice for $Y(t)$ is chosen based on:

   i. $Y(t)$ is a product of the corresponding choices of all the parts of $g(t)$.
   ii. There are as many coefficients as the number of distinct terms in $Y(t)$.
   iii. Each distinct term must have its own coefficient, not shared with any other term.

6. Before finalizing the choice of $Y(t)$, compare it against $y_c(t)$. If there is any shared term between the two, the present choice of $Y(t)$ needs to be multiplied by t. Repeat until there is no shared term.

Remember that, in order to use Rule 6 you always need to find the complementary solution first.

SUMMARY: Method of Undetermined Coefficients

Given $$a\,y'' + b\,y' + c\,y = g(t)$$

1. Find the complementary solution y_c.

2. Subdivide, if necessary, $g(t)$ into parts: $g(t) = g_1(t) + g_2(t) \ldots + g_k(t)$.

3. For each $g_i(t)$, choose the form of its corresponding particular solution $Y_i(t)$ according to:

$g_i(t)$	$Y_i(t)$
$P_n(t)$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0)$
$P_n(t)\,e^{at}$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0)\,e^{at}$
$P_n(t)\,e^{at}\cos\mu t$ and/or $P_n(t)\,e^{at}\sin\mu t$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_0)\,e^{at}\cos\mu t$ + $t^s(B_n t^n + B_{n-1} t^{n-1} + \ldots + B_0)\,e^{at}\sin\mu t$

Where $s = 0$, 1, or 2, is the **minimum** number of times the choice must be multiplied by t so that it shares no common terms with y_c.

$P_n(t)$ denotes a n-th degree polynomial. If there is no power of t present, then $n = 0$ and $P_0(t) = C_0$ is just the constant coefficient. If no exponential term is present, then set the exponent $a = 0$.

4. $Y = Y_1 + Y_2 + \ldots + Y_k$.

5. The general solution is $y = y_c + Y$.

6. Finally, apply any initial conditions to determine the as yet unknown coefficients C_1 and C_2 in y_c.

Challenge

The following challenges expand the range of problems to give you practise in a range of situations.

1. (C23) $y'' + 4y = 8\cos 2t$
2. (C24) $y'' + 2y' = 2te^{-t}$
3. (C25) $y'' + 2y' = 6e^{-2t}$
4. (C26) $y'' + 2y' = 12t^2$
5. (C27) $y'' - 6y' - 7y = 13\cos 2t + 34\sin 2t$
6. (C28) $y'' - 6y' - 7y = 8e^{-t} - 7t - 6$

Solution

Assuming that the constants you find in your solution are all equal to 1, check your answer by calculating $y(t = 0.4)$ in each case. To check your answer, please subsume all constants on any term into the constant that you set to 1. For example, instead of $y(t) = -2C_1e^{-t} + e^{-t}$, write $y(t) = C_1e^{-t}$ where the two e^{-t} terms have been combined and the -2 has been subsumed into the constant C_1, and then set $C_1 = 1$ to check the answer.

4.17 (C29) Method of undetermined coefficients: Determining the ODE I

Comment

This challenge gives you useful practise of going the other way; determining a differential equation that describes a given solution.

Challenge

Determine the 2nd-order linear differential equation which has the general solution

$$y = C_1 \cos 4t + C_2 \sin 4t - e^t \sin 2t \tag{4.20}$$

Solution

You will end up with differential terms on the left side and a function of t on the right side. Please compare your answer with your partner in class. To check your answer with challenge-hub, evaluate the right side of your equation by substituting the value $t = 0.4$.

4.18 (C30) Method of undetermined coefficients: Determining the ODE II

Comment

This challenge gives you useful practise of going the other way; determining a differential equation that describes a given solution.

Challenge

Determine the 2nd-order linear differential equation which has the general solution

$$y = C_1e^{-2t} + C_2te^{-2t} + t^3 - 3t \tag{4.21}$$

Solution

You will end up with differential terms on the left side and a function of t on the right side. Please compare your answer with your partner in class. To check your answer with challenge-hub, evaluate the right side of your equation by substituting the value $t = 0.4$.

Appendix A

Mid-term exam questions

A.1 2017

A.1.1 Questions

1

1. You are standing on a ferry travelling from Fukuoka to Busan when you drop a tennis ball into the sea. After the ball hits the water, it undergoes a deceleration until it reaches terminal velocity. Write a differential equation describing the acceleration of the ball under the water as a function of it's velocity at a given time t and the terminal velocity of the ball under the water v_T. You may assume:

- The ball falls vertically downwards.
- The trajectory of the ball is not disturbed by external factors such as waves or hungry fish.
- The density of the sea is constant so that the terminal velocity of the ball in the water is independent of depth.
- If you make any other assumptions, please state them clearly.

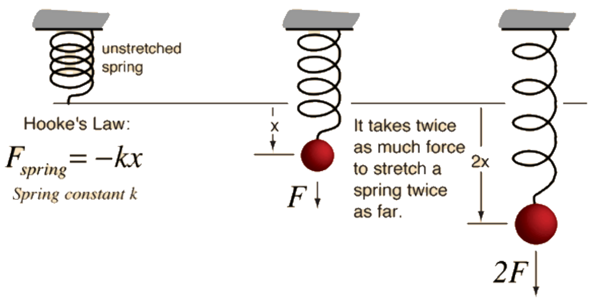
2. If the ball initially hits the water at a velocity of $2v_T$, write an expression for the velocity of the ball as a function of time.

2

From your seat on the ferry you notice a student trying to do their homework on the Laplace equation, but they are stuck. Luckily you brought your table of Laplace transforms with you (page 3). Help the student by determining $f(t)$ in the following functions:

1. $\mathcal{L}\{f(t)\} = \dfrac{10 - 10s}{s^2}$
2. $\mathcal{L}\{f(t)\} = \dfrac{e^{-3s}s}{s^2 + 16}$

3



In a play-area for children on the ferry there is a toy consisting of a mass attached to a spring hanging from the ceiling as shown in the figure above. Starting from the basic relation $F(x(t)) = -kx(t)$, derive a general expression for the velocity of the mass as a function of time.

4

As you approach the port in Busan the engine of the ferry switches to reverse. During the switching, the casing of the engine vibrates with the motion described by this 2nd-order differential equation:

$$3y'' - 4y' + y = te^t \tag{A.1}$$

The chief engineer wants to know the motion of the casing $y(t)$ as a function of time. Using the method of undetermined coefficients, find the general solution to this differential equation.

A.2 2016

A.2.1 Questions

1

Solve the following ODE for y given the condition $y(3) = 9e^9$.

$$\frac{x}{y}\frac{dy}{dx} - 1 = x^3 \tag{A.2}$$

2

The following equation is an autonamous equation:

$$y' = \frac{y^2}{5}(1 - \frac{y}{5}) \tag{A.3}$$

1. What key property does an autonamous equation have?
2. Determine the points of equilibrium and their stabilities.

3

Solve the following 2nd-order ODE's for y, and state what sort of damping they correspond to:

$$y'' + 5y' + 4y = 0 \tag{A.4}$$

$$y'' + 4y' + 4y = 0 \tag{A.5}$$

$$y'' + 3y' + 4y = 0 \tag{A.6}$$

4

Solve the following differential equation for y:

$$3x^2y + 2xy + y^3 + (x^2 + y^2)y' = 0 \tag{A.7}$$

Solutions can be found on the following page.

A.2.2 Solutions

1

$y = 3xe^{x^3/3}$

2

1. $y' = f(y)$
2. $y = 0$ (semi-stable), $y = 5$ (stable)

3

$y(t) = C_1 e^{-t} + C_2 e^{-4t}$, Overdamped

$y(t) = C_1 e^{-2t} + C_2 te^{-2t}$, Critically-damped

$y(t) = C_1 e^{-3t/2} Cos(\sqrt{7}t/2) + C_2 e^{-3t/2} Sin(\sqrt{7}t/2)$, Under-damped

4

$C = yx^2 e^{3x} + \frac{1}{3} y^3 e^{3x}$