Maximizing Growth Rates in Finite Conway's Game of Life

UNIVERSITÀ DI PAVIA

Ambrogio Maria Bernardelli

Department of Mathematics "F. Casorati", University of Pavia, Pavia 27100, Italy

CompOpt UNIPV

Motivation and Setting

Conway's Game of Life (GoL) is a deterministic cellular automaton traditionally studied on the infinite square grid. We consider **finite grids** with different boundary topologies, including rectangular grids, cylinders, and tori. We study the maximal growth rate of live cells over time, asking how fast the **population can increase** under the standard GoL rules when the space is finite. To the best of our knowledge, the problem of maximizing the growth rate in finite instances of GoL, and its formulation as a time-expanded Integer Linear Program (ILP), has not been systematically studied.

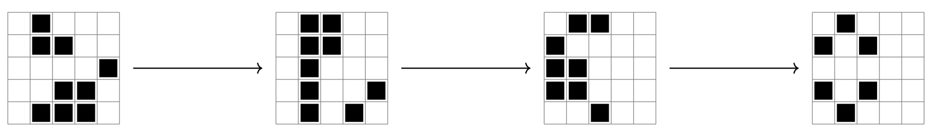


Figure 1. Example of a few iterations of GoL.

Axioms

- Any live cell with fewer than two live neighbours dies, as if by underpopulation.
- Any live cell with two or three live neighbours survives to the next generation.
- Any live cell with more than three live neighbours dies, as if by overpopulation.
- Any dead cell with exactly three live neighbours becomes a live cell, as if by reproduction; otherwise, it remains dead.

Growth Rate Maximization

Consider GoL on an $n \times n = n^2$ grid and let $X(T)$ denote the set of live cells at time T. Given an initial configuration $X(0)$, we define the **growth rate** after T steps as

$$g_T(X) = \frac{|X(T)| - |X(0)|}{|X(0)|}.$$

Fix an initial population size $m \in \{1, \dots, n^2\}$. We define the maximal growth rate

$$g_{T,m}(n) = \max\left\{ g_T(X) \mid X \in \mathrm{GoL}(n),\ |X(0)| = m \right\}.$$

Finally, we define the maximal growth rate on an n^2 grid as

$$\mathcal{G}_T(n) = \max_m g_{T,m}(n).$$

ILP Formulation

We formulate the problem as a **binary ILP** whose variables describe the state of each cell across time steps. Let $x_{i,j,t}$ be a binary variable indicating whether cell (i,j) is alive at time t. The GoL update rules induce nonlinear logical relations between consecutive time steps, which we linearize using auxiliary binary variables, obtaining the following model:

$$\begin{aligned}
\Gamma_{T,m}(n) = \max \quad & \sum_{(i,j)\in n^2} x_{i,j,T} \\
\text{s.t.} \quad & \sum_{(i,j)\in n^2} x_{i,j,0} = m, \\
& x_{i,j,t} + \sum_{(k,l)\in a(i,j)} x_{k,l,t} \geq 3x_{i,j,t+1} && \forall (i,j,t) \in I, \\
& \sum_{(k,l)\in a(i,j)} x_{k,l,t} \leq 3 + (M_{i,j} - 3)(1 - x_{i,j,t+1}) && \forall (i,j,t) \in I, \\
& \sum_{(k,l)\in a(i,j)} x_{k,l,t} \geq 4y_{i,j,t} && \forall (i,j,t) \in I, \\
& x_{i,j,t} + \sum_{(k,l)\in a(i,j)} x_{k,l,t} \leq 2 + (M_{i,j} - 1)(1 - z_{i,j,t}) && \forall (i,j,t) \in I, \\
& 1 - x_{i,j,t+1} \leq y_{i,j,t} + z_{i,j,t} && \forall (i,j,t) \in I, \\
& x_{i,j,t}, y_{i,j,t}, z_{i,j,t} \in \{0,1\} && \forall (i,j,t) \in I, \\
& x_{i,j,T} \in \{0,1\} && \forall (i,j) \in n^2.
\end{aligned}$$

Here $I = \{(i,j,t) \mid i,j = 1, \dots, n,\ t = 0, \dots, T-1\}$, $a(i,j)$ denotes the set of neighbours of cell (i,j), and $M_{i,j} = |a(i,j)|$. Note that

$$g_{T,m}(n) = -1 + \frac{1}{m}\Gamma_{T,m}(n).$$

The formulation contains $O(n^2T)$ binary variables and constraints and can be viewed as a **time-expanded** representation of the grid, where each layer corresponds to a time step. The strong logical coupling between neighbouring cells across time steps makes the resulting instances computationally challenging.

For fixed n and T, the model must be solved for several values of the initial population m, and some choices of m lead to significantly harder instances. Symmetries of the grid and the choice of boundary topology significantly influence the difficulty of the resulting instances.

Different Topologies

We study finite grids endowed with different boundary topologies. While the update rules remain unchanged, the neighbourhoods of boundary cells are modified according to the chosen topology. We consider: (s) the **square grid** with fixed boundaries, (c) the **cylindrical grid**, with horizontal periodicity and vertical fixed boundaries, (t) the **toroidal grid**, with both horizontal and vertical periodicity. Quantities referring to these cases are denoted by the superscripts s, c, and t, respectively.

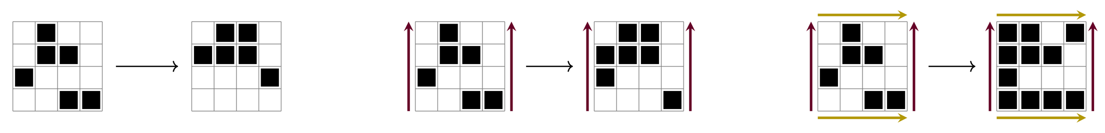


Figure 2. Different topologies depicted with the identification of the boundary.

Preliminary Computational Results

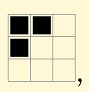
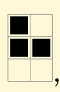
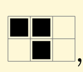
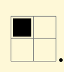
Using the ILP formulation, implemented with `Gurobi 13.0`, we compute optimal solutions for small values of n and T. For $T = 1$, periodic topologies (cylinder and torus) attain the value 2, while the square grid value is strictly smaller but increases with n. As n and T increase, the ILP quickly becomes harder to solve for some values of m, making larger instances **computationally challenging**. The table was obtained in a total running time of 9700 seconds, with the hardest instance (square, $n, T, M = 6, 3, 7$) requiring about 6266 seconds. Improving the computational performance of the ILP formulation, for instance through stronger formulations or **symmetry-breaking** techniques, is therefore of significant interest.

n	T	$\mathcal{G}^s_T(n)$	$\mathcal{G}^c_T(n)$	$\mathcal{G}^t_T(n)$
3	1	0.750	2.000	2.000
	2	0.750	2.000	0.000
	3	0.333	1.000	0.000
4	1	0.800	2.000	2.000
	2	0.750	2.000	1.500
	3	0.800	3.000	1.500
5	1	1.100	2.000	2.000
	2	0.857	2.000	2.000
	3	1.400	4.000	3.000
6	1	1.286	2.000	2.000
	2	1.286	2.000	2.000
	3	1.857	5.000	5.000

Observation 1

We exhibit a family of configurations $X^k \in n(k)^2$, with $n(k) = 3k + 2$, such that

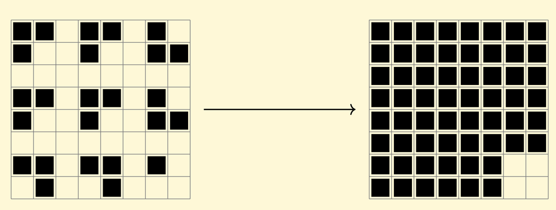
$$g^s_1(X^k) = \frac{6k^2 + 6k - 1}{3k^2 + 6k + 1} \longrightarrow 2 \qquad \text{as } k \to \infty.$$

The construction is based on four small building blocks, a = , b = , c = , d = . We assemble them into a $(3k+2)^2$ grid as follows:

$$X^k = \begin{bmatrix} a & \cdots & a & b \\ \vdots & & \vdots & \vdots \\ a & \cdots & a & b \\ c & \cdots & c & d \end{bmatrix}.$$

A direct count shows that $|X^k| = |a|k^2 + |b|k + |c|k + |d| = 3k^2 + 6k + 1$. After one iteration, all cells are alive except the four cells in the lower-right corner. More precisely, $Y^k_{n-1,n-1} = Y^k_{n-1,n} = Y^k_{n,n-1} = Y^k_{n,n} = 0$, and $Y^k_{i,j} = 1$ otherwise. Hence $|Y^k| = n^2 - 4 = (3k+2)^2 - 4$, and

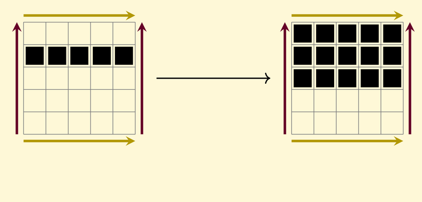
$$g^s_1(X^k) = \frac{|Y^k| - |X^k|}{|X^k|} = \frac{6k^2 + 6k - 1}{3k^2 + 6k + 1} = 2 - 3\,\frac{2k+1}{3k^2 + 6k + 1}$$



Observation 2

The following inequality holds:

$$\mathcal{G}^c_1(n) = \mathcal{G}^t_1(n) \geq 2 \quad \forall n \geq 3$$



Lemma

The following inequality holds

$$\mathcal{G}^\star_1(n) \leq 2 \quad \forall n \geq 3,\ \star \in \{s, c, t\}.$$

We start with the square topology. Let $X(0) = X$ and $X(1) = Y$. It is enough to prove that $|Y| \leq 3|X|$. Let $\tilde{X}$ be the embedding of X in a grid $\{0, \dots, n+1\}^2$. We have that

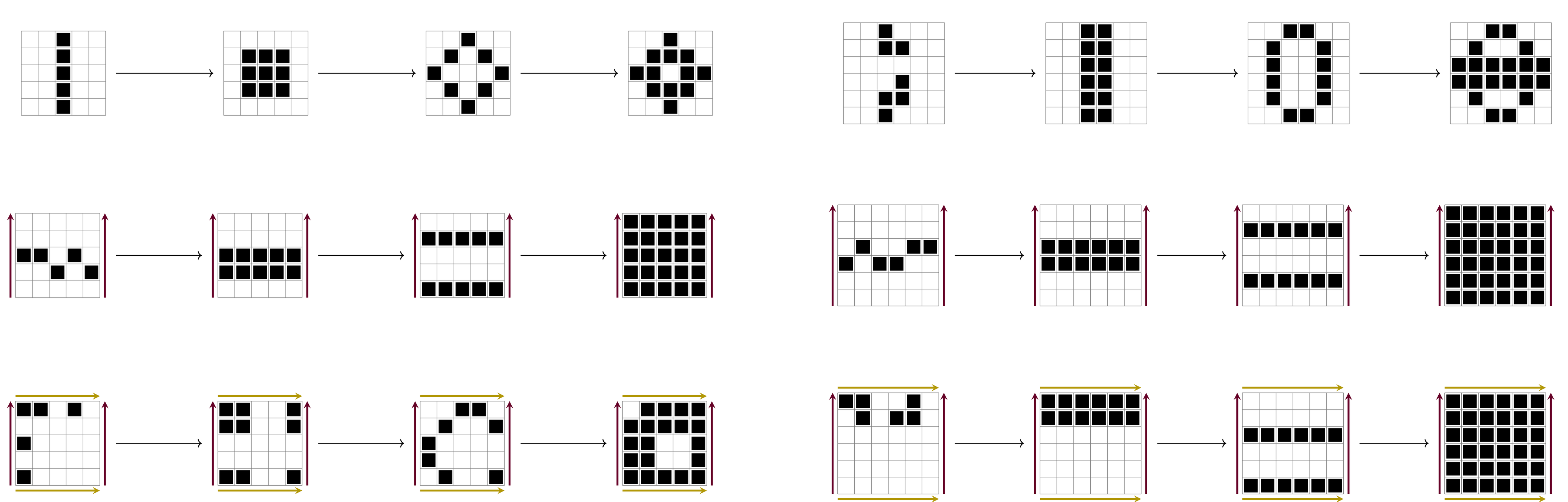
$$|X| = \sum_{i,j=1}^{n} X_{ij} = \frac{1}{8}\sum_{i,j=0}^{n+1} \sum_{(k,l)\in a(i,j)} \tilde{X}_{kl} \geq \frac{1}{8}\sum_{(i,j)\in A_0} \sum_{(k,l)\in a(i,j)} \tilde{X}_{kl} + \frac{1}{8}\sum_{(i,j)\in A_1} \sum_{(k,l)\in a(i,j)} \tilde{X}_{kl} \geq \frac{1}{8}\sum_{(i,j)\in A_0} 3 + \frac{1}{8}\sum_{(i,j)\in A_1} 2 =$$

$$\geq \frac{3}{8}|A_0| + \frac{2}{8}|A_1| = \frac{2}{8}(|A_0| + |A_1|) + \frac{1}{8}|A_0| \geq \frac{2}{8}(|A_0 \cup A_1|) + \frac{1}{8}(|A_0 \cup A_1| - |A_1|) \geq \frac{3}{8}|Y| - \frac{1}{8}|X|$$

from which the thesis follows. Note that $A_* := \{(i,j) \in n^2 \mid Y_{i,j} = 1, X_{i,j} = *\}$, $* = 0, 1$. For the cylindrical topology, it suffices to embed X in a rectangular grid, with one more row at the top and one more at the bottom, while for the toroidal topology no embedding is needed.

Optimal Solutions for T=3

Optimal configurations returned by the ILP for $n = 5$ and $n = 6$ (square, cylindrical, and toroidal topologies). These examples illustrate spatial structures achieving maximal growth.



Open Questions

The numerical evidence obtained from the ILP model suggests the following open problems.

- $\mathcal{G}^\star_T(n) \leq \mathcal{G}^\star_{T+1}(n) \quad \forall n \geq \bar{n},\ \star \in \{s, c, t\}$?
- $\exists u < 4 : \mathcal{G}^\star_2(n) \leq u \quad \forall n \geq 3,\ \star \in \{s, c, t\}$?

Takeaway. The maximal one-step growth rate appears to be **bounded by 2** and is achieved under cylindrical and toroidal topologies. Determining the behaviour for larger T and larger grids remains a challenging problem both combinatorially and computationally.

References

Gardner, M., The fantastic combinations of John Conway's new solitaire game 'life'. Mathematical Games. *Scientific American*, **223**(4), pp. 120–123, 1970.

Bosch, R. A., Integer programming and Conway's game of Life. *SIAM review*, **41**(3), pp. 594–604, 1999.