Zusammenfassung Topology I

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Contents

0 *Humble beginnings*

- simply connected = no "handle-shaped" holes
- contractible = no holes at all

Lemma 0.1 (Splitting lemma)**.** Given SES

$$0 \to X \xrightarrow{r} Y \xrightarrow{q} Z \to 0$$

the following are equivalent:

- SES splits on the left
- SES splits on the right
- $Y \cong X \bigoplus Z$

- let Y be a Hausdorff space, and let X be a compact space, and $\sim$ equivalence relation on X. Let $f : X \to Y$ be a surjective continuous mapping that is constant on the equivalence classes.
  - then f induces a surjective map $\overline{f} : X/ \sim \to Y$
  - if $\overline{f}$ is bijective, then it is a homeomorphism
- classification of 2-surfaces

	with boundary	w/o boundary
orientable	sphere, torus, etc	
non-orientable	Moebius strip	Klein bottle, real projective space

Notes from "Homology theory, lecture 10, Panov T. E."

- tensor product $\bigotimes$ – quotient of a free abelian group
- set of homomorphisms $\hom(G, H)$
  - H abelian $\iff$ $\hom(G, H)$ has a natural AbGrp structure
- covariant functor $\otimes G : H \mapsto H \otimes G, (H_1 \to H_2) \mapsto (H_1 \otimes G \to H_2 \otimes G)$
  - $H_n(X) \otimes \mathbb{Z} = H_n(X)$
- contravariant functor $\hom(-, G) : H \mapsto \hom(H, G), (H_1 \to H_2) \mapsto (\hom(H_2, G) \to \hom(H_1, G))$
- topological space X, singular n-chain complex $C_n^{\text{sing}}(X) = \mathbb{Z}\langle \sigma : \Delta^n \to X \rangle$

- $C_n^{\text{sing}}(X;G) = C_n^{\text{sing}}(X) \otimes G$
  - this is functor from TOP to Grp
  - singular n-chains with coefficients in G
- $C^n_{\text{sing}}(X) = \hom(C_n^{\text{sing}}(X), G)$
  - singular n-cochains with coefficients in G
  - cochain = function on singular n-chains with values in G
  - 0-dim cochain is a function on 0-dim chains in X with values in G (function)
- boundary homomorphism $\partial : C_n(X) \to C_{n-1}(X)$
- boundary homomorphism $\partial : C_n(X;G) \to C_{n-1}(X;G)$
  - same formulas
- coboundary homomorphism $\delta : C^{n-1}(X;G) \to C^n(X;G)$
  - $\langle \delta_{n-1}c, \sigma \rangle = \langle c, \partial_n \sigma \rangle$
  - δ is dual to ∂; the value of $\delta_{n-1}c$ on n-simplex σ is determined by the value of c on $n-1$ simplex $\partial\sigma$.
  - $(\delta_{n-1}c)(\sigma) = \sum_i (-1)^i c(\sigma \mid_{[v_0 \dots \hat{v}_i \dots v_n]})$
- {chain complex, homology} $\xrightarrow{\hom(-,G)}$ {cochain complex, cohomology}
- Rechenregeln
  - $H_n(\bigvee_i X_i) = \bigoplus_i H_n(X_i)$
  - $H^n(\bigvee_i X_i) = \prod_i H^n(X_i)$
- homological algebra
  - given $0 \to F \to G \to H \to 0$ apply $C_n(X) \bigotimes -$ and consider $0 \to C_n(X,F) \to C_n(X,G) \to C_n(X,H) \to 0$

Notes from "Homology theory, lecture 12, Panov T. E."

- cup product $\smile: H^p(X;R) \times H^q(X;R) \to H^{p+q}(X;R)$
- associative graded-commutative ring with 1 denoted by $H^*(X;R) = \bigoplus_{p \geq 0} H^p(X;R)$
- graded-commutative $\iff ab = (-1)^{ij} ba$
- $a \in C^p(X;R), b \in C^q(X;R)$, define $a \smile b \in C^{p+q}(X;R)$
  - given $p+q$-simplex $\sigma : \Delta^{p+q} = [v_0 \dots v_{p+q+1}] \to X$

- define $(a \smile b)(\sigma) = a(\sigma\ |_{0,p})b(\sigma\ |_{p,p+q})$ where a and b share one vertex p

- Rechenregeln:
  - identity element is $1 \in H^0(X;R)$, cochain with values 1 on each point of X
  - $\delta(a \smile b) = \delta(a \smile b) + (-1)^p(a \smile \delta b)$ (Leibniz property)
    * ausklammern + p transpositions to change δ with a
    * $\sigma : \Delta^{p+q+1} \to X$
      · $(\delta a \smile b)(\sigma) = \sum_i (-1)^i a(\sigma\ |_{0,p+1})b(\sigma\ |_{p+1,p+q+1})$
      · $(-1)^p(a \smile \delta b)(\sigma) = \sum_i (-1)^i a(\sigma\ |_{0,p})b(\sigma\ |_{p,p+q+1})$

- cohomology ring $H^*(X;R) = \bigoplus_{p\geq 0} H^p(X;R)$ is a graded-commutative ring with 1

1 *Axiomatic homology*

1.1 *Eilenberg-Steenrod axioms*

Vorlesung 1, 09.10.23

- homology theory $(\mathcal{H}_*, \partial *)$
  - functor $\mathcal{H}_* : \text{TOP}^2 \longrightarrow \mathbb{Z}$-graded R-modules
  - boundary operator $\partial_* : \mathcal{H}_* \to \mathcal{H}_{*-1} \circ I$
    * $I : (X, A) \to (A, \emptyset)$

- five axioms
  - homotopy invariance
    * homotopic maps induce the same map in homology
    * caution! the converse doesn't hold
  - long exact sequence of pairs
    $\ldots \to \mathcal{H}_n(A) \to \mathcal{H}_n(X) \to \mathcal{H}_n(X, A) \to \ldots$
  - excision
    * if $\overline{A} \subset B^\circ$ then $\mathcal{H}_n(X - A, B - A) \cong \mathcal{H}_n(X, B)$
  - dimension
    * $\mathcal{H}_0(\bullet) = R$
    * $\mathcal{H}_{\neq 0}(\bullet) = 0$
  - additivity
    * $X = \coprod_\alpha X_\alpha \implies H_n(X) = \bigoplus H_n(X_\alpha)$
    * caution! this implies that each X_α is open in X, otherwise you can get pathological examples (taking disjoint union of all points in the space implying homology of any space equal to homology of a point)

Natürlich fragt man sich – what the fuck? Jedem topologischen Raum X wird eine unendliche Kette an R-Moduln zugeordnet. Jede stetige Abbildung zwischen X, Y induziert ein Homomorphismus zwischen Homologien von solchen Ketten. Aus Exaktheit versuchen wir, neue Informationen über die Moduln in der Kette abzuleiten. Das ist die Grundidee.

1.2 *Mayer-Vietoris sequence*

- Mayer-Vietoris sequence
  - if $X = A^\circ \cup B^\circ$, then exists the following Mayer-Vietoris sequence $\mathcal{H}_n(A \cap B) \to \mathcal{H}_n(A) \oplus \mathcal{H}_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0$.
- inclusion classification for pairs (X, A)
  - given $r : X \to A$ and $\iota : A \hookrightarrow X$
  - r retraction $\iff r \circ \iota = \mathrm{id}_A$
  - r deformation retraction $\iff r \circ \iota \simeq \mathrm{id}_{\mathrm{rel}\ A}$
    * in other words, a homotopy between a retraction and the identity map on X.
    * in other words, A and X are homotopy equivalent
  - r neighborhood deformation retraction $\iff$ exists $U^\circ \supseteq A$ such that A is a deformation retract of U
    * in other words, $A \subseteq U^\circ \subseteq X$ and exists $r' : U^\circ \to A$ with $r' \circ i = \mathrm{id}_A$
    * tutor mentioned that HEP is equivalent to cofibration and NDR is similar to that too.
  - cofibration

Vorlesung 2, 11.10.23

- Mayer-Vietoris sequence for pushouts
  - $$\begin{array}{ccc} X_0 & \xrightarrow{i_2} & X_2 \\ \big\downarrow{\scriptstyle i_1} & & \big\downarrow{\scriptstyle j_2} \\ X_1 & \xrightarrow{j_1} & X \end{array}$$
  - if i_1 closed inclusion and (X_1, X_0) NDR
  - then j_2 is a closed inclusion and (X, X_2) NDR
  - $\mathcal{H}_n(X_1, X_0) \cong \mathcal{H}_n(X, X_2)$
  - and the usual Mayer-Vietoris sequence from above works too

1.3 *Category-theoretical constructions*

Vorlesung 3, 16.10.23

- five lemma
- cone
- suspension
- suspension isomorphism
- mapping cone
- pushout
- wedge $X \vee Y = {}^{X \coprod Y}\!/\!_{(x_0 \sim y_0)}$

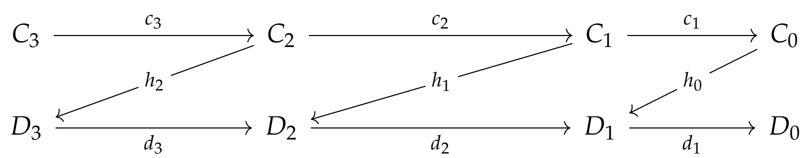
1.4 *Chain complexes, homology, chain homotopy*

Vorlesung 4, 18.10.23

- chain complex $C_* = (C_*, c_*)$
  - family of $\mathbb{Z}$-graded R-modules C_n
  - n-th differentials $c_n : C_n \to C_{n-1}$
    * $c_n \circ c_{n+1} = 0 \implies \operatorname{im} c_{n+1} \subseteq \ker c_n$
    * caution! not the other way around
  - finitely generated, projective, free, finitely generated projective, finitely generated free, ...
  - $\dim C_* = d \iff C_n = 0$ für $n > d$
  - finite = finitely generated and finite dimension
  - positiv $\iff C_n = 0$ für $n < 0$
  - $\cdots \longrightarrow C_{n+1} \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots$
- homology $H_n(C_*)$
  - $H_n(C_*) = \ker c_n \,/\, \operatorname{im} c_{n+1}$
    * $\operatorname{im} c_{n+1} \subset \ker c_n$ because $c_n \circ c_{n+1} = 0$
    * measures how far away from being exact C_* is
  - cycle $u \in C_n \iff c_n(u) = 0 \iff u \in \ker c_n$
    * cycles a, b homologous if $a - b$ boundary
    * abelianization of loops
      · cycles are loops without basepoint, since changing basepoint cyclically permutes its letters (Hatcher, p. 99)
  - boundary $u \in C_n \iff u \in \operatorname{im} c_{n+1}$
- chain map $f_* : C_* \to D_*$
  - family of maps $f_n : C_n \to D_n$ such that everything commutes

  - induces a homomorphism on homology $H(f_*) : H_(C_*) \to H_(D_*)$

- chain homotopy h_* from f_* to g_*
  - given chain maps $f_*, g_* : C_* \to D_*$
  - h_* is family of homomorphisms $h_n : C_n \to D_{n+1}$
  - $d_{n+1} \circ h_n + h_{n-1} \circ c_n = f_n - g_n$
  - if $f_* \simeq g_*$ then they induce the same homomorphisms on homology $H_n(f_*) = H_n(g_*)$



$$d_3 \circ h_2 + h_1 \circ c_2 = f_2 - g_2$$

Vorlesung 5, 23.10.23

- long homology sequence
  - SES of chain complexes C_*, D_*, E_* induces LES on their homologies in each dimension + induces a natural boundary operator
  - zig-zag lemma

- universal property of direct sum
  - direct sum of chain complexes induces direct sum on their homologies

2 *Simplicial complexes and simplicial homology*

- abstract simplicial complex $\Sigma = (V, \Sigma)$
  - V vertices, Σ simplices
    * **the vertex condition**: $v \in V$ are always contained in Σ
    * **the subset condition**: subsets $S \subseteq \Sigma$ are elements of Σ

0-simplex is a point
1-simplex is a line
2-simplex is a triangle
3-simplex is a solid tetrahedron

- ordering of Σ
  - let Σ_p be the set of p-simplices (= consist of $p + 1$ elements)
  - for each simplex $S \in \Sigma$, choose bijection $u(S) : [0, 1, \dots, p + 1] \to S$
  - switching two indices in the ordering introduces a minus sign, $[1, 2] = -[2, 1]$

We now define a covariant functor from the category of simplicial complexes and simplicial maps to the category of topological spaces and continuous maps.

- geometric simplicial complex $|\Sigma|$

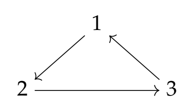
$|\Sigma|$ is a topological space.

  - the underlying set of $|\Sigma|$ is the set of all functions $\alpha : V \to [0,1]$ such that $\alpha(v)$ is positive for all v and the sum $\sum_{v \in V} \alpha(v)$ is always equal to 1
    - basically all possible "weights" in barycentric coordinates
  - for subset $S \in \Sigma$, the underlying set is the subset of Σ given by $|S| = \{\alpha \mid s \notin S \implies \alpha(s) = 0\}$
    - basically only uses the weights from $S \implies$ points lay in the convex hull
- incidence number $\text{inz}^p_{s,t}$
  - let s be a p-simplex and t a $(p-1)$-simplex
    - if $t \not\subseteq s$ then $\text{inz}^p_{s,t} = 0$
    - else $\text{inz}^p_{s,t}$ is sgn of the permutation of the ordering
- simplicial chain complex $C^{\text{simp}}_*(\Sigma; R)$
  - $C^{\text{simp}}_n(\Sigma)$ = free abelian on the set of n-simplices

6. Vorlesung, 25.10.23

- simplicial boundary operator c_n
  - $c_n : C^{\text{simp}}_n(\Sigma) \to C^{\text{simp}}_{n-1}(\Sigma)$

Example: $c_2 : (1,2,3) \mapsto (2,3) - (1,3) + (1,2)$



  - takes simplex to alternating sum of its boundary $(1, \ldots, n)$ to $\sum_{1 \le i \le n} (-1)^i (1, \ldots, \hat{i}, \ldots, n)$
- simplicial homology $H^{\text{simp}}_*(\Sigma; R)$
  - measures n-dimensional holes in X

7. Vorlesung, 30.10.23

  - abstract homology on $C^{\text{simp}}_*(\Sigma; R)$
  - verifying Eilenberg-Steenrod axioms

Remark: simplicial complices are very combinatorial in nature. Not very good for explicit computations by hand but good for computers.

3 *Singular chain complexes and singular homology*

Sometimes a triangulation might not exist, so we could do the next best thing available – consider maps from Δ_n to X instead.

- standard n-simplex $\Delta_n \subseteq R^{n+1}$
  - closed convex hull of $\{e_0, \ldots, e_n\} \subseteq R^{n+1}$
- k-th face is the image of i^n_k
  - i^n_k maps everything except the k-th element to Δ_n

- singular n-simplex $\sigma : \Delta_n \to X$
  - simplices are allowed to be "singular" e. g. constant map
- singular chain complex $C_*^{\text{sing}}(X) = (C_*^{\text{sing}}(X), \partial)$

$$\cdots \longrightarrow S_2(X) \longrightarrow S_1(X) \longrightarrow S_0(X) \longrightarrow 0$$

  - each $S_n(X)$ is a free R-module generated by the set of all singular n-simplexes in X
    - elements of $S_n(X)$ are called singular n-chains
    - finite linear combinations of $\sigma \in S_n(x)$ with coefficients from R
  - singular boundary operator $\partial : S_n(X) \to S_{n-1}(X)$
    - sends $\sigma : \Delta_n \to X$ to $\sum_i (-1)^i (\sigma \circ i_i^n)$
- induced chain map $C_*^{\text{sing}}(f) : C_*^{\text{sing}}(X) \to C_*^{\text{sing}}(Y)$
- singular homology $H_*^{\text{sing}}(X, R)$
  - free abelian of uncountable rank, unless X is a finite collection of points
- $H_1^{\text{sing}}(X) = \pi_1(X, x_0) / [\pi_1(X, x_0), \pi_1(X, x_0)]$

4 *CW-complexes and cellular homology*

8. Vorlesung, 06.11.23

4.1 *CW-complexes*

- **relative CW-complex** (X, A)
  - **topological pair** (X, A)
  - **relative CW-structure** on (X, A) is a filtering (ascending chain)

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

    - such that for each $n \geq 0$ exists *a* pushout that glues n-cells together

$$\begin{array}{ccc} \coprod_{i \in I_n} S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ {\scriptstyle \coprod_{i \in I_n} j_i} \downarrow & & \downarrow {\scriptstyle k_n} \\ \coprod_{i \in I_n} D^n & \xrightarrow[\coprod_{i \in I_n} Q_i^n]{} & X_n \end{array}$$

    - $X = \cup_{n \geq 0} X_n$ and has direct limit topology
      - $A \subset X$ closed $\iff$ $A \cap X_n$ closed in X_n for all n

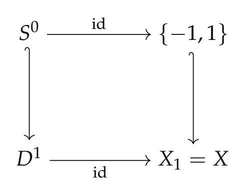
The pushouts in the definition of Wolfgang are not unique, e. g. there are many different pushouts for the same filtering giving the same spaces. Could be done differently, but the category behaves much better if you only request the existence of pushouts.

Example for n=0: X_1 is a discrete set of points (perhaps uncountably many points)

$$\begin{array}{ccc} \coprod S^{-1} = \emptyset & \longrightarrow & \emptyset = X_{-1} \\ \downarrow & & \downarrow \\ \coprod D^0 = \{\text{pt}\} & \longrightarrow & X_1 \end{array}$$

- Remarks:
  - $\coprod_{i \in I_n} D^n \setminus \coprod_{i \in I_n} S^{n-1}$ is homeomorphic to $X_n \setminus X_{n-1}$
    * each of I_n describes one path component of $X_n \setminus X_{n-1}$
    * $|I_n|$ = number of path components in $X_n \setminus X_{n-1}$
  - main ingredients:
    * n-skeleton X_n
    * open n-cell $e_i^n := Q_i^n(D^n \setminus S^{n-1}) \subset X_n$
    * closed n-cell $\overline{e_i^n}$
    * boundary $\overline{e_i^n} \setminus e_i^n$ or ∂e_i^n
    * characteristic map Q_i^n
    * gluing map q_i^n
- cellular map $f : (X, A) \to (Y, B)$
  - $f(X_n) \subseteq Y_n$ for all $n \geq -1$
- isomorphism of CW complexes
  - cellular maps $f : X \to Y$ and $g : Y \to X$
- **CW pair** (X, A)
  - if $e \cap A \neq \emptyset$ then $\bar{e} \subseteq A$
  - closed subspace $A \subset X$ + union of cells
  - A is a **CW subcomplex** of X with CW structure given by $A_n = X_n \cap A$
  - any CW pair is a relative CW complex but not vice versa

Example: $X = [-1, 1]$ has the following CW-structure $X_{-1} = \emptyset, X_0 = \{-1, 1\}, X_i = [-1, 1]$ for $i \geq 1$. I could also use $-\mathrm{id}$ as the gluing map.



- compactness lemmas about CW complexes
  - subset $C \subseteq X$ is closed $\iff$ for all $e : C \cap \bar{e}$ compact
  - subset $C \subseteq X$ is compact $\iff$ C closed and meets finitely many cells
  - subset $C \subseteq X$ is compact $\iff$ C closed and contained in a finite CW subcomplex
  - CW complex X is compact $\iff$ CW complex X is finite
- relative CW complex $(X, A) \implies (X, A)$ is a NDR
- cellular pushout

9. Vorlesung, 08.11.23

- Mayer-Vietoris for CW-complexes
- covering space has CW-structure $\iff$ base space has CW-structure

- X, Y CW-complexes and X or Y locally compact $\implies X \times Y$ inherits CW-structure
- examples of CW-complexes
  - S^n
  - $\mathbb{RP}^n$
  - $\mathbb{CP}^n$
  - T^2
- cellular approximation theorem
  - there is no difference between cellular maps and just normal maps between CW complexes

4.2 *Maps between spheres*

What do we know about spheres?

- built inductively
  - $v_n : \Sigma S^{n-1} \to S^n$ is a homeomorphism
- exists suspension isomorphism
  - $\sigma_n(X) : \mathcal{H}_{n-1}(S^{n-1}) \to \mathcal{H}_n(\Sigma S^{n-1}) = \mathcal{H}_n(S^n)$
- if $[S^d, S^d]$ set of homotopy classes of selfmaps in S^d then suspension induces isomorphism between $[S^d, S^d]$ and $[S^{d+1}, S^{d+1}]$ (Freudenthal)

4.3 *Cellular chain complex associated with (any) homology*

10. Vorlesung, 13.11.23

Given CW-complex (X, A) and a homology theory (any) $\mathcal{H}_*$ define

- cellular chain complex $C_*(X, A) = (\cup_{n=0}^{\infty} \mathcal{H}_n(X_n, X_{n-1}), \partial)$

$$\cdots \longrightarrow H_3(X_3, X_2) \longrightarrow H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0, A)$$

  - each $H_n(X_n, X_{n-1})$ is R-module of pairs in theory $\mathcal{H}_*$
  - cellular boundary operator $\partial : \mathcal{H}_n(X_n, X_{n-1}) \to \mathcal{H}_{n-1}(X_{n-1}, X_{n-2})$ from the triple (X_n, X_{n-1}, X_{n-2})

Yes, cellular chain complex really is defined in terms of homology groups...

- cellular homology $H^{\text{cell}}(X, A) = \mathbb{Z}[\{\text{number of } n\text{-cells}\}]$
- Hauptsatz:
  - if (X, A) a finite CW-complex or $\mathcal{H}_*$ satisfies disjoint union + dimension axiom then $H_n^{\mathcal{H}_*}(X, A) \cong \mathcal{H}_n(X, A)$

4.4 Computing cellular chain complexes

11. Vorlesung, 15.11.23

- choose pushouts for CW complex
- $H_k(X_n, X_{n-1}) \xleftarrow{\sim} H_k(\coprod_{I_n} D_n, \coprod_{I_n} S_{n-1}) \xleftarrow{\sim} \bigoplus_{I_n} H_k(D_n, S_{n-1}) \xrightarrow{\sim} \bigoplus_{I_n} H_k(S_n, \bullet) \xrightarrow{\sim} \bigoplus_{I_n} H_{k-n}(S_0, \bullet) \xrightarrow{\sim} \bigoplus_{I_n} H_{k-n}(\bullet) = \bigoplus_{I_n} H_0(\bullet)$ if $k = n$ and 0 otherwise.
  - I should probably elaborate on this some time later...
- **cellular boundary formula**

4.5 Uniqueness of homology theory for CW complexes

12. Vorlesung, 20.11.23

- if X is a finite CW complex then there exist only one unique homology theory satisfying the dimension axiom
- if X is an infinite CW complex then there exists only one unique homology theory satisfying the dimension axiom and the disjoint union axiom
- cell orientation

5 Euler characteristic

13. Vorlesung, 22.11.23

Let R be any PID, e. g. $\mathbb{Z}, \mathbb{Q}, \mathbb{F}_p$.

- preliminaries:
  - structure theorem for finitely generated modules over PID
    * $M \cong R^r \oplus R/p_1^{\alpha_1}R \oplus \ldots \oplus R/p_n^{\alpha_n}R$
    * $tors(M) = R/p_1^{\alpha_1}R \oplus \ldots \oplus R/p_n^{\alpha_n}R$
    * define rank of M as $rk_R(M) := r$
  - short exact sequence of finitely generated modules,
    * $0 \to M_0 \to M_1 \to M_2 \to 0$
    * $rk(M_1) = rk(M_0) + rk(M_2)$
    * if either M_1 or M_0, M_2 are finitely generated, then all of them are finitely generated
- Euler characteristic $\chi(C_\bullet)$ for finite chain complexes
  - finite chain complex $C_\bullet$ = finite dimension d + C_n's are finitely generated
  - $\chi(C_\bullet) = \sum_{n=0}^{\dim C_\bullet} (-1)^n \cdot rk(C_n) = rk(C_0) - rk(C_1) + rk(C_2) + \ldots$
  - equivalent with homology groups, $\chi(C_\bullet) = \sum (-1)^n \cdot rk(H_n(C_\bullet))$
- Euler characteristic $\chi(X)$ for finite CW complexes

- $\chi(X) = \sum(-1)^n |I_n|$
- compatible with cellular pushouts and direct products; "additive" and "multiplicative".

- universal additive invariant for finite CW complexes
- Euler characteristic for standard n-simplices *(Exercise)*
  - $\chi(\Delta_n) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}$

14. Vorlesung, 27.11.23

- classification of platonic solids (there are only five of them)

6 *Lefschetz numbers*

- Lefschetz numbers for chain complexes $\Lambda(f_*)$
  - chain endomorphism $f : C_* \to C_*$ induces a map f_* on $H_*(C_*)$
  - this is a map from R-modules, so you can calculate $\operatorname{tr}(A_n)$ in each dimension n
  - $\Lambda(f_*) = \sum_k (-1)^k \operatorname{tr}(A_k)$

15. Vorlesung, 29.11.23

- Lefschetz numbers for CW complexes $\Lambda(f_*)$
  - $\Lambda(f_*) = \sum_k (-1)^k \operatorname{tr}(f_k)$ where f_k is a cellular map $C_k(X) \to C_k(X)$
- both can be calculated directly from f or from induced map on homology f_*
- Lefschetz fixed-point theorem
  - if f endomorphism of a finite CW complex, then $\Lambda(f) = 0 \iff f$ has no fixed-point (cell mapped to itself)

7 *Cohomology*

15. Vorlesung, 04.12.23
$\mathcal{H}_*$ homology, $\mathcal{H}^*$ cohomology

- cohomology theory
  - functor $\mathcal{H}^* : (\mathrm{TOP}^2)^{op} \longrightarrow \mathbb{Z}$-graded R-modules
  - boundary operator $\delta^* : \mathcal{H}^* \circ I \to \mathcal{H}^{*+1}$
    - $I : (X, A) \to (A, \emptyset)$
  - and five axioms
- cochain complex $C^* = (C^*, \delta^*)$
  - n-th differentials $\delta^n : C^n \to C^{n+1}$
- cohomology of a cochain complex $H^n(C^*) = \ker \delta^n / \operatorname{im} \delta^{n-1}$
- dual cochain complex C_*

C^* cochain complex, C_* dual cochain complex

  - $C^n = \hom(C_n, R)$

7.1 Singular cohomology

- singular cochain complex $C^*_{\text{sing}} = \hom(C^{\text{sing}}_*, R)$
- singular cohomology $H^*_{\text{sing}} = H^n(C^*_{\text{sing}})$

7.2 Cellular cohomology

16. Vorlesung, 04.12.23

- cellular cochain complex $C^*_{\text{cell}} = \hom(C^{\text{cell}}, R)$
- cellular cohomology $H^*_{\text{cell}} = H^n(C^*_{\text{cell}})$
- sadly LES in homology does not induce LES in cohomology. sometimes it does, sometimes it doesn't $\implies$ universal coefficient theorem

7.3 Multiplicative structure

- Eilenberg MacLane space $K(A, n)$
  - CW complex with $\pi_n(X) = A$ and 0 otherwise
- n-th homotopy group $\pi_n(X) = [(S^n, *), (X, x)]$
- **multiplicative structure** (cup product)
  - assigns to X with $A, B \subseteq X$ family of bilinear maps

$$\smile : H^p(X, A) \times H^q(X, B) \to H^{p+q}(X, A \cup B)$$

- cross product

17. Vorlesung, 06.12.23

7.4 Cohomology ring of projective spaces

7.5 Cup product for CW complexes

8 Homological algebra

8.1 Tor *and* Ext *functor*

20.Vorlesung, 08.01.2023

- fundamental theorem of homological algebra = lifting R-module homomorphisms to R chain maps
- $[P_*, Q_*] \to \hom_R(M, N)$
- given $M_1 \to M_2, N_1 \to N_2$ have four functors
  - $M_1 \otimes_R N \to M_2 \otimes_R N$,
  - $N \otimes_R M_1 \to N \otimes M_2$,

- $\hom_R(M_1, N) \leftarrow \hom_R(M_2, N)$,
- $\hom_R(N, M_1) \to \hom_R(N, M_2)$

- free resolution of R-module M = exact sequence $\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$ with free F_i
  - because of fundamental theorem of finitely generated abelian groups any fin. gen. abelian M has free resolution $0 \to F_1 \to F_0 \to M$ where F_0 are generators and F_1 relations
  - has same homologies with trivial resolution $0 \to M \to 0$
- apply $\otimes_R N$ to free resolution and omit $M \otimes_R N$
  - extends category of modules to category of chain complexes
  - n-th homology group = $\mathrm{Tor}_n^R(M, N) = \ker(F_n \otimes N \to F_{n-1} \otimes N) / \operatorname{im}(F_{n+1} \otimes N \to F_n \otimes N\}$

$$\to F_2 \otimes N \to F_1 \otimes N \to F_0 \otimes N \to 0$$

- apply $\hom_r(-, N)$ and omit $\hom_R(M, N)$
  - n-th cohomology group = $\mathrm{Ext}_R^n(M, N)$ = $\ker(\hom(F_n, N) \to \hom(F_{n+1}, N)) / \operatorname{im}(\hom_R(F_{n-1}, N) \to \hom_R(F_n, N))$

$$0 \to \hom(F_0, N) \to \hom(F_1, N) \to \hom(F_2, N) \to$$

- properties
  - $\mathrm{Tor}, \mathrm{Ext}$ independent of resolution
  - $\mathrm{Tor}_0^R(M, N)) = M \otimes_R N$, $\mathrm{Ext}_R^0(M, N) = \hom_R(M, N)$
  - $\mathrm{Tor}_n^R(M, N) \cong \mathrm{Tor}_n^R(N, M)$ for commutative R
  - Tor commutes with $\oplus$
- universal coefficient theorem
  - $0 \to H_n(X) \otimes G \to H_n(X, G) \to \mathrm{Tor}(H_{n-1}(X), G) \to 0$
  - $0 \to H^n(X) \otimes G \to H^n(X, G) \to \mathrm{Tor}(H^{n+1}(X), G) \to 0$
  - $0 \to \mathrm{Ext}(H_{n-1}, G) \to H^n(X, G) \to \hom(H_n(X), G) \to 0$
- computing tips for $\mathrm{Tor}_i^R(M, N)$
  - find projective resolution $\ldots P_2 \to P_1 \to P_0 \to M \to 0$ of M and apply functor $\otimes_R N$ to it omitting $\to M \otimes_R N$, calculate hojmology of this chain comploex
  - find SES of R-modules $0 \to K \to M \to I \to 0$ and apply functor $\otimes_R N$, obtain a LES $\to \mathrm{Tor}_1^R(K, N) \to \mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^R(I, N) \to K \otimes_R N \to M \otimes_R N \to I \otimes_R N \to 0$
  - $\mathrm{Tor}_1^R(R/I, R/J) \cong (I \cap J)/IJ$.

8.2 *Universal coefficient theorem*

1. True or false?

- The homology groups of a free chain complex are free
- A bounded chain complex has only finitely many non-trivial homology groups
- The degree of a homeomorphism $f : S^n \to S^n$ is always $+1$
- Homology groups $H_n(\mathbb{RP}^n; \mathbb{Z}) \cong \mathbb{Z}/2$ for all $n > 0$
- For path-connected X trivial first fundamental group implies trivial first homology group
- Homology of $X \times Y$ is equal to $H_n(X) \otimes H_n(Y)$
- If $X = X_1 \cup X_2$ and X_1, X_2 and $X_1 \cap X_2$ acyclic, then X is acyclic
- If $\iota : X \to Y$ is an embedding, then $\iota_* : H_n(X) \to H_n(Y)$ is a monomorphism
- By Borsuk-Ulam, each map $f : S^n \to \mathbb{R}^n$ has a point $X \in S^n$: $f(-X) = -f(X)$
- There are no vector fields on S^{2n} without zeroes, $n > 0$

2. Chain complexes

- What is a chain complex over a commutative ring R?
- Give two non-trivial examples of chain complexes
- How are the homology groups $H_n(C_*)$ over a chain complex C_* defined?
- Compute the homology groups over $\mathbb{Z}$:

$$0 \to C_2 \to C_1 \to C_0 \to 0$$

where $C_0 \cong \mathbb{Z}$ generated by a, $C_1 \cong \mathbb{Z} \otimes \mathbb{Z}/2$ generated by b of infinite order and by c of order 2, $C_2 \cong \mathbb{Z}/4$ generated by d. Differentials are given by $\partial(b) = 3a, \partial(c) = 0, \partial(d) = c$

5. Assume $A_\bullet$, $B_\bullet$ are chain complexes over $\mathbb{Z}$, $A_i = 0$ for $i > N$, $A_N \cong \mathbb{Z} \cong B_N$, $B_i = 0$ for $i < N$. Under which conditions is

$$\cdots \leftarrow A_{N-2} \overset{\partial^A}{\leftarrow} A_{N-1} \overset{\partial^A}{\leftarrow} \mathbb{Z} \overset{\partial^B}{\leftarrow} B_{N+1} \overset{\partial^B}{\leftarrow} B_{N+2} \leftarrow \cdots$$

a chain complex $C_\bullet$? What is $H_N(C_\bullet)$ in this case?

3. Connecting homomorphisms

Let $\epsilon : 0 \to A_* \to B_* \to C_* \to 0$ a SES of chain complexes over some commutative ring R

- Define the connecting homomorphisms $\partial : H_n(C_*) \to H_{n-1}(A_*)$
- Point out two choices on which the definition depends
- Write down the LES of homology groups induced by ϵ
- How is the LES of a pair (X, X_0) for X_0 a subspace of X defined?

4. Eilenberg-Steenrod axioms

- Name all axioms for a homology theory $h_*(X, A)$
- Why does singular homology theory satisfy the dimension theorem?
- If h^a_*, h^b_* are two homology theories, is $h_* = h^a_* \oplus h^b_*$ a homology theory?

5. Relative homology groups

- Definte the relative homology groups of a pair of spaces $(X; A)$
- Write down the LES of a pair (X, A) including the homomorphisms in this sequence
- Use this sequence to compute the homology of $X = F_2$, the surface of genus 2, where A is the right half of X

6. Hurewicz isomorphism

- Define the Hurewicz homomorpism in degree 1
- Verify that it's well-defined and state which choices your definition depends on
- Formulate the Hurewicz theorem in degree 1
- Prove: if map $f : (X, x) \to (Y, y)$ induces an epimorphism f_* between the fundamental groups π_1, then it also induces and epimorphism between the first homology groups.

7. Chain maps

- What is a chain map of degree k between two chain complexes
- Prove: A chain map of degree k induces homomorphism between homology groups
- What is a chain hojmotopy between two chain maps of degree k
- Prove: Chain homotopic chain maps induce the same homomorphism

8. Simplicial approximation theorem

- Formulate the simplicial approximation theorem
- Prove: if X is a finite polyhedron of dimension $m < n$ then any map $f : X \to S^n$ is null-homotopic

9. Jordan-Brouwer

- Formulate the Jordan-Brouwer theorem giving the homology of the complement of a k-sphere S embedded in $\mathbb{R}^n$
- Define the linking number $Link(S, T)$ of a p-sphere S and a q-sphere T disjointly embedded in $\mathbb{R}^n$ where $n - 1 = p + q$
- Prove: if T is isotoped inside the complement of S to T' then $Link(S, T) = Link(S, T')$

10. Transfer

- Define the transfer homomorphism for a 2-fold covering $\pi : X' \to X$ in modulo 2 homology $H_*(-; \mathbb{F}_2)$
- Investigate the LES of coverings shown in the drawing below. Calculate all relvant homology groups, calculate the connecting homomorphism for coefficients in $\mathbb{F}_2$

11. Bonus questions

- When did Poincare develop the concept of homology groups?
- When did Eilenber and Steenrod formulate their axioms?