Probabalistic Well-Posedness of the Nonlinear Schrödinger Equation on $\mathbb{T}$

Abstract

In this paper, we will consider the nonlinear Schrödinger equation on the 1-dimensional torus $\mathbb{T}$ for $1 \leq p \leq 5$. By constructing a family of invariant measures, we will prove that almost every initial data leads to a globally well-posed solution.

Deterministic Preliminaries

Given a measurable initial data $u_0 : \mathbb{T} \to \mathbb{C}$, we investigate solutions u to the nonlinear Schrödinger equation (NLS)

$$\begin{cases} i\partial_t u + \Delta u &= u|u|^{p-1} \\ u(t=0) &= u_0 \end{cases}.$$

Of course, there is a question as to whether the equation locally or globally well-posed, and also the continuity of the data to solution map in a suitable choice of spaces. The canonical choice of spaces for the initial data are the Sobolev spaces H^s for $s \in \mathbb{R}$ with the norms

$$\|u\|_{H^s}^2 := \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} |\hat{u}(n)|^2, \quad \langle x\rangle^2 := 1 + |x|^2.$$

Foundational to the study of the NLS is the study of the ordinary Schödinger equation

$$\begin{cases} i\partial_t u + \Delta u &= 0 \\ u(t=0) &= u_0 \end{cases},$$

whose solution u is given by $u(t,x) = S(t)u_0(x)$, where $S(t)$ is the Schrödinger dispersion operator given by the Fourier multiplier $S(t)^\wedge(n) := e^{-in^2 t}$. Using this notation, it is possible to formulate the NLS as a fixed point of the operator

$$\Gamma u(t) := S(t)u_0 + \int_0^t S(t-s)u(s)|u(s)|^{p-1}\, ds.$$

However, it remains to be seen precisely the choice of s for which we have global and local well-posedness. There are two deterministic theorems relevant for this work, which can be found in Bourgain, 1993a and Bourgain, 1993b.

Theorem 1 (Cubic Global Well-posedness). *The NLS*

$$\begin{cases} iu_t + u_{xx} &= \pm u|u|^2 \\ u(t=0) &= u_0 \end{cases}$$

is globally well-posed for all initial data $u_0 \in H^s(\mathbb{T})$ when $s \geq 0$. Moreover, if u, v are two solutions to the NLS with initial data u_0 and v_0, respectively then

$$\|u(t) - v(t)\|_{H^s} \leq C^{|t|}\|u_0 - v_0\|_{H^s},$$

where $C = C(\|u_0\|_{L^2}, \|v_0\|_{L^2})$.

Theorem 2 (Local well-posedness for $3 < p \leq 5$). *The NLS*

$$\begin{cases} iu_t + u_{xx} &= \pm u|u|^{p-1} \\ u(t=0) &= u_0 \end{cases}$$

is locally well-posed on an interval $[0,\tau]$ when $3 < p \leq 5$ for initial data $u_0 \in H^s(\mathbb{T})$ for all $s > 0$. Moreover, $\tau \lesssim \|u_0\|_{H^s}^{-\theta}$ for some $\theta > 0$, and the mapping $u_0 \mapsto u(t)$ is Lipschitz in t.

The Gibbs Measure

Infinite Dimensional Gaussian Measures. In this section we are going to construct a measure μ on $H^1(\mathbb{T})$ which is invariant under the flow of the data-to-solution map of the NLS for select values of p. Before proceeding, we require some probabilistic lemmata. Throughout this section, we let $\mathcal{L}(B_1, B_2)$ be the set of bounded linear operators $B_1 \to B_2$, where B_1, B_2 are fixed Banach spaces.

Definition 3. A *normal* (or *Gaussian*) distribution on $\mathbb{R}^d$ is any Borel probability measure μ on $\mathbb{R}$ of the form

$$\frac{d\mu}{dx} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-(x-\mu)^\top \Sigma (x-\mu)\right)$$

with $\mu \in \mathbb{R}^d$ and symmetric PSD matrix Σ.

Definition 4. Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{B}_{\mathcal{H}}$ be the cylindrical σ-algebra. I.e., it is the σ-algebra generated by the algebra $\{T^{-1}A : T \in \mathcal{L}(\mathcal{H}, \mathbb{R}^n), A \subset \mathbb{R}^n \text{ is Borel}, n \geq 1\}$. A *Gaussian measure* γ is a probability measure defined on $\mathcal{B}_{\mathcal{H}}$ such that $\ell_\sharp \gamma$ is a normal distribution on $\mathbb{R}$ for each $\ell \in \mathcal{H}^*$.

Remark 5. Actually, it is possible to generalize any of the results we list here to the case when $\mathcal{H}$ is a separable Banach space. While this level of generality is useful in many contexts (e.g., it leads to a construction of Brownian motion), we will not need it, and refer the interested reader to Hairer, 2026

We now list a fundamental and nontrivial property of Gaussian measures known as *Fernique's theorem*, whose proof can be found in Hairer, 2026.

Theorem 6 (Fernique). *Let γ be a Gaussian measure on a separable Hilbert space $\mathcal{H}$. Then there exists a constant $\alpha > 0$ such that*

$$\int_{\mathcal{H}} \exp\left(\alpha \|x\|^2\right) \, d\gamma(x) < \infty.$$

Fernique's theorem implies several key estimates for Gaussian measures:

Theorem 7. *Let γ be a Gaussian measure on a separable Hilbert space $\mathcal{H}$.*

(1) For all integers $n \geq 1$, the n-th moment of γ exists, meaning

$$\int_{\mathcal{H}} \|x\|^n \, d\gamma(x) < \infty.$$

In fact, there is a constant $c > 0$ such that $\gamma(\{x : \|x\| > r\}) \leq 2e^{-cr^2}$ for all $r > 0$.

(2) For each $h \in \mathcal{H}$, there exists a unique $h^* \in \mathcal{H}$ such that

$$\int (h, x)(x, k) \, d\gamma(x) = (h^*, k)$$

for all $k \in \mathcal{H}$.

Using Fernique's theorem, we obtain the following important structure theorem for Gaussian measures. We include the proof because of a relevant construction.

Theorem 8 (Structure Theorem). *Given $h \in \mathcal{H}$, let $h^* \in \mathcal{H}$ be as in Theorem 7(2). Then the map $C_\gamma : \mathcal{H} \to \mathcal{H}$ defined by $h \mapsto h^*$ is symmetric, positive semi-definite, and trace class, meaning that if $\{e_n : n \geq 1\}$ is an orthonormal basis of $\mathcal{H}$, then*

$$\sum_{n \geq 1} |(C_\gamma e_n, e_n)| < \infty.$$

Conversely, if K is any other symmetric, positive semi-definite, and trace class operator on $\mathcal{H}$, then there is a Gaussian measure γ such that $C_\gamma = K$.

Proof. Since $\|C_\gamma h\|^2 \le \|h\|^2 \int \|x\|^2\, d\gamma(x)$, we see that C_γ is bounded, and clearly it is linear. Since

$$(h, C_\gamma k) = \overline{(C_\gamma k, h)} = \int \overline{(k,x)(x,h)}\, d\gamma(x) = \int (h,x)(x,k)\, d\gamma(x) = (C_\gamma h, k),$$

we obtain symmetry. Moreover, $(C_\gamma h, h) = \int |(h,x)|^2\, d\gamma(x) \ge 0$, which proves that C_γ is PSD. By monotone convergence and Parseval's theorem,

$$\sum_{n\ge 0} |(C_\gamma e_n, e_n)| = \int \sum_{n\ge 0} |(e_n, x)|^2\, d\gamma(x) = \int \|x\|^2\, d\gamma(x) < \infty.$$

For the converse, suppose that $K : \mathcal{H} \to \mathcal{H}$ is symmetric PSD and trace class. Let $g_1, g_2, \ldots$ be an IID sequence of standard Gaussian random variables defined on a probability space $(\Omega, \mathbb{P})$. Since K is trace class, it is compact, and so by the spectral theorem there is an orthonormal basis $\{e_n\}$ of eigenvectors with eigenvalues $\{\lambda_n\} \subset [0,\infty)$. Since K is trace class implies $\sum_k \lambda_k < \infty$, it holds $X := \sum_{n\ge 1} \sqrt{\lambda_n} g_n e_n$ exists as an element of $\mathcal{H}$ almost surely. Let

$$\gamma(A) = \mathbb{P}\left[X \in A\right].$$

Since for $a \in \mathcal{H}$ it holds $(a, X) = \sum_{n\ge 1} \sqrt{\lambda_n} g_n (a, e_n)$ is a Gaussian with mean 0 and variance $\sum_{n\ge 1} \lambda_n (a, e_n)^2 = (Ka, a)$, by Riesz representation, for any $T \in \mathcal{L}(\mathcal{H}, \mathbb{R}^n)$ it holds $T_\sharp \gamma$ is a multivariate Gaussian, so γ is a Gaussian measure. Finally, since the $\{g_n\}$ are independent,

$$\int (h,x)(x,g)\, d\gamma(x) = \mathbb{E}\left[(h,X)(X,g)\right] = \sum_{n=1}^{\infty} \lambda_n (h, e_n)(e_n, g) = (Kh, g),$$

proving $C_\gamma = K$. □

The structure theorem tells us that there is a one-to-one correspondence between symmetric, PSD, trace class operators on $\mathcal{H}$ and Gaussian measures on $\mathcal{H}$.

Example 9. Let $\mathcal{H} = H^s(\mathbb{T}^d)$ where $s \ge 0$, which has inner product

$$(f,g)_{H^s} := \sum_{k\in\mathbb{Z}^d} \langle k\rangle^{2s} \hat{f}(k)\overline{\hat{g}(k)}.$$

Let $K_\sigma = (1-\Delta)^{s-\sigma}$. Note that K_σ is diagonal with respect to the orthonormal basis $\phi_k(x) = \langle k\rangle^{-s} e^{ik\cdot x}$, with eigenvalue $\langle k\rangle^{2(s-\sigma)}$. The operator K_σ is trace class if and only if

$$\sum_{k\in\mathbb{Z}} \langle k\rangle^{2(s-\sigma)} < \infty,$$

which means $s < \sigma - d/2$. By Theorem 8, the measure γ with covariance K_σ is the push-forward of the random variable

$$\sum_{k\in\mathbb{Z}^d} \frac{g_k}{\langle k\rangle^{\sigma-s}} \phi_k = \sum_{k\in\mathbb{Z}^d} \frac{g_k}{\langle k\rangle^{\sigma}} e_k,$$

where $e_k(x) = e^{ik\cdot x}$ and $\{g_k\}_{k\in\mathbb{Z}^d}$ is an IID sequence of standard Gaussians.

For future reference, we codify Example 9 into the following theorem:

Theorem 10. *For each $\sigma < s - d/2$, there exists a Gaussian measure ρ_s on $H^\sigma(\mathbb{T}^d)$ whose covariance operator is $(1-\Delta)^{\sigma-s}$.*

While the utility of the measure in Theorem 10 will become apparent in the subsequence sections, we should still make a brief comment. Intuitively, it behaves like the Gaussian measure

$$\text{“}e^{-\|u\|_{H^s}^2}\, du\text{”}$$

at least when u is band limited. Of course, the concept of "du" is meaningless, because there is no infinite dimensional Lebesgue measure. But when a random sample u has only the frequencies $\hat{u}(0), \ldots, \hat{u}(k)$, then

$$\rho_s\left(\{u \in H^\sigma : \hat{u}(0) \in A_1, \ldots, \hat{u}(k) \in A_k\}\right) = c_k \int_{A_1} \cdots \int_{A_k} \exp\left(-\sum_{j=0}^{k} \langle j \rangle^{2s} |z_j|^2\right) dz_1 \cdots dz_k$$

where c_k is a scaling constant and dz denotes the Lebesgue measure on $\mathbb{C}$.

Invariant Gibbs Measures. Using the probabalistic machinery constructed in the preceding section, we now construct a Gibbs measure invariant under the action of the NLS. To introduce the context, in the the finite dimensional setting, let $H : \mathbb{R}^{2n} \to \mathbb{R}^n$ be a C^2 function (called the *Hamiltonian*). We write coordinates in $\mathbb{R}^{2n}$ as $(q_1, \ldots, q_n, p_1, \ldots, p_n) =: (q, p)$. The *Hamilton-Jacobi* equations associated to H are the system of ODE given by

$$\partial_t p_j = -\frac{\partial H}{\partial q_j}, \quad \partial_t q_j = \frac{\partial H}{\partial p_j}.$$

If one endows $\mathbb{R}^{2n}$ with the symplectic form $\omega((q,p),(q',p')) = \sum_{j=1}^n p_j' q_j - p_j q_j'$, then we can interpret the Hamilton-Jacobi equations as solutions to the gradient flow

$$\partial_t (q, p) = \nabla_\omega H(q, p)$$

where ∇_ω is the symplectic gradient. Crucially, if $S_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the map which sends an initial data $(q_0, p_0) \in \mathbb{R}^{2n}$ to the solution $(q(t), p(t))$ of the Hamilton-Jacobi equations with initial data (q_0, p_0) at time $t \geq 0$, then the *Gibbs measure*

$$d\mu := e^{-H(q,p)}\, dq\, dp$$

is invariant under S_t for all times t since $\partial_t H(q, p) = 0$. Of course, we need to assume that $e^{-H} \in L^1(\mathbb{R}^{2n})$ and that the solution map S_t exists when $t \neq 0$. Formally, this means that $\mu(S_t A) = \mu(A)$ for all $t \geq 0$ and measurable $A \subset \mathbb{R}^{2n}$.

The point is that we want to interpret the NLS as a gradient flow with respect to a suitable symplectic form, so that we can construct an invariant Gibbs measure. We follow the exposition of Tao, 2006. Suppose that $iu_t + u_{xx} = \lambda u |u|^{p-1}$ on $H^s(\mathbb{T})$ with $u(t = 0) = u_0$, where $\lambda = 1$ corresponds to the defocusing NLS and $\lambda = -1$ corresponds to the focusing NLS. Then, provided that u is suitably well-behaved, the function

$$H(u) = \frac{1}{2} \int |u_x|^2\, dx + \frac{\lambda}{p+1} \int |u|^{p+1}\, dx$$

is constant in time. This suggests H is our desired Hamiltonian. If we endow $H^s(\mathbb{T})$ with the symplectic form

$$\omega(f, g) = \operatorname{Im} \int \overline{f} g\, dx,$$

we can equivalently write the NLS as $\partial_t u = \nabla_\omega H(u)$, and therefore we endeavor to construct the Gibbs measure

$$\text{“}d\mu = e^{-H(u)}\, du.\text{”}$$

Such a measure obviously does not exist as stated, and this leads to the one the primary difficulties in the construction and applications of this technique.

Before proceeding, let us begin to address the foundational question: why do we care about invariant measures? The short answer is that they allow us to use the tools of ergodic theory and measure theory to make statistical claims about the behavior and existence of solutions. Consider the following example, which will serve as a template in later investigations (the uninterested reader can skip to Theorem 13).

Lemma 11. *Let H be a Hamiltonian and suppose that $e^{-H} \in L^1(\mathbb{R}^{2d})$, and suppose further that for each $R > 0$ and initial data $|(q_0, p_0)| \leq R$, a solution $(q(t), p(t))$ exists to the Hamilton-Jacobi ODE associated to H on the time interval $[-\delta, \delta]$, where $\delta \sim R^{-\theta}$ for some $\theta > 0$. Let μ be the Gibbs probability measure associated to H, and suppose that $R^\theta \mu(B_R^c) \to 0$ as $R \to \infty$, where B_R is the ball of radius R at the origin.*

Then for any $\epsilon > 0$ and $T > 0$ there exists a set $\Omega \subset \mathbb{R}^{2d}$ depending only on ϵ and T such that $\mu(\Omega^c) < \epsilon$, and for each $(q_0, p_0) \in \Omega$, there exists a solution to the Hamilton-Jacobi ODE on the time interval $[-T, T]$.

Proof. Let

$$\Omega_1 = B_R \cap S(\delta)(B_R) \cap S(-\delta)(B_R),$$

where S_t is the data-to-solution map on B_R for $|t| \leq \delta$. If $(q_0, p_0) \in \Omega$, then we claim that a solution to the Hamilton-Jacobi ODE exists on $[-T, T]$. Indeed, a solution $S(t)(q_0, p_0)$ exists for $|t| \leq \delta$. Since $(q_0, p_0) = (S(-\delta)\tilde{q}_0, S(-\delta)\tilde{p}_0)$ for some $(\tilde{q}_0, \tilde{p}_0) \in B_R$, we see that $(S(\delta)q_0, S(\delta)p_0) \in B_R$ and therefore a solution exists on the interval $[-\delta, \delta]$ to the Hamilton-Jacobi ODE with initial data $(S(\delta)q_0, S(\delta)p_0)$. By repeating this argument but replacing δ with $-\delta$, we see that a solution to the Hamilton-Jacobi ODE with initial data (q_0, p_0) on the interval $[-2\delta, 2\delta]$. Therefore on Ω_1 the data-to-solution map can be extended to times in the interval $[-2\delta, 2\delta]$. Therefore we construct a set Ω_2 by $\Omega_2 = \Omega_1 \cap S(-2\delta)(B_R) \cap S(2\delta)(B_R)$, and by repeating the same argument we see that on Ω_2 the solution-to-data map can be extended to times on the interval $[-3\delta, 3\delta]$. Repeating this argument a total of $\lfloor T/\delta \rfloor$ times, we define

$$\Omega := \bigcap_{|j| \leq \lfloor T/\delta \rfloor} S(j\delta)(B_R),$$

and notice that for initial data $(q_0, p_0) \in \Omega$ there is a solution to the Hamilton-Jacobi ODE on the interval $[-\delta(\lfloor T/\delta \rfloor + 1), \delta(\lfloor T/\delta \rfloor + 1)] \supset [-T, T]$. By the invariance of the Gibbs measure, we have

$$\mu(\Omega^c) \leq 2\frac{T}{\delta}\mu(B_R^c) \lesssim TR^\theta \mu(B_R^c).$$

As soon as R is large enough that $TR^\theta \mu(B_R^c) < \epsilon$, we see that Ω satisfies the requirements of the lemma. □

Corollary 12. *Suppose that H is a Hamiltonian whose Hamilton-Jacobi ODE and Gibbs probability measure satisfy the conditions of Lemma 11. Then if (Q, P) is sampled from μ, almost surely the the Hamilton-Jacobi ODE with initial data (Q, P) has a global-in-time solution.*[1]

Proof. Fix $\epsilon > 0$ and for each $j \geq 1$ use Lemma 11 to find a set Ω_j such that $\mu(\Omega_j^c) < \epsilon 2^{-j}$ and for initial $(q_0, p_0) \in \Omega_j$ the Hamilton-Jacobi ODE has a solution on the interval $[-2^j, 2^j]$. Putting $U_\epsilon = \bigcap_j \Omega_j$, we see that for initial $(q_0, p_0) \in \Omega_\epsilon$ there is a global-in-time solution to the Hamilton-Jacobi ODE and moreover

$$\mu(U_\epsilon^c) \leq \sum_j \mu(\Omega_j^c) \leq \epsilon.$$

Letting $U = \bigcup_j U_{2^{-j}}$, we see that the Hamilton-Jacobi ODE is globally well-posed on U and

$$\mu(U^c) \leq \mu(U_{2^{-j}}) \leq 2^{-j}$$

for all $j \geq 1$, hence $\mu(U^c) = 0$. □

[1] Ironically, in real world applications, one often tries numerically find solutions to the Hamilton-Jacobi ODE (or a stochastic variant) to obtain the sample (Q, P) from the Gibbs measure.

The point of Lemma 11 and Corollary 12 is that, if one can construct an invariant measure for the Hamilton-Jacobi flow locally-in-time, then provided this measure satisfies suitable decay conditions one can use this to claim that almost all initial data yield a global time solution to the Hamilton-Jacobi flow. While this is overkill for finite dimensional problems due to the well-established literature for the Hamilton-Jacobi ODE, the utility is more pronounced in the infinite dimensional setting we now discuss.

Henceforth, for sufficiently nice measurable functions u on the torus $\mathbb{T}$, we let

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} |u_x|^2 + \frac{\lambda}{p+1}\int_{\mathbb{T}} |u|^{p+1}$$

with $\lambda \in \{-1, 1\}$ corresponding to the focusing and defocusing NLS, respectively. Notice that by conservation of mass, we expect the formal measure

$$\exp\left(-\frac{1}{2}\|u\|_2 - H(u)\right)\, du$$

to also be invariant under the solution flow of the NLS. However, observe that $\frac{1}{2}\|u\|_2 + H(u) = \frac{1}{2}\|u\|^2_{H^1(\mathbb{T})} + \frac{1}{p+1}\int_{\mathbb{T}} |u|^{p+1}$. By Theorem 10 with $s = 1$, for $\sigma < 1/2$ we can find a Gaussian measure ρ_1 on $H^\sigma(\mathbb{T})$ whose covariance operator is $(1-\Delta)^{\sigma-1}$. We thus are motivated to define our Gibbs measure by

$$d\mu = \exp\left(\frac{\lambda}{p+1}\int |u|^{p+1}\right) d\rho_1(u).$$

This *nearly* works, but we still need to control $\frac{\lambda}{p+1}\int |u|^{p+1}$. By conservation of mass, we therefore consider the family of measures

$$d\mu_R := \mathbf{1}_{\{\|u\|_2 \leq R\}}(u) \exp\left(\frac{\lambda}{p+1}\int |u|^{p+1}\right) d\rho_1(u), \quad R > 0.$$

This mechanism allows control over $\frac{\lambda}{p+1}\int |u|^{p+1}$ for select values of p without ruining invariance properties.

Theorem 13 (Lebowitz et al., 1988, Bourgain, 1994). *For $R > 0$, let μ_R be the measure on $H^{\frac{1}{2}-}(\mathbb{T})$ defined by*

$$d\mu_R := \mathbf{1}_{\{\|u\|_2 \leq R\}}(u) \exp\left(\frac{1}{p+1}\int |u|^{p+1}\right) d\rho_1(u).$$

Then:

(1) For $1 \leq p < 5$, μ_R is a finite measure for all $R > 0$.
(2) If $p = 5$, μ_R is finite for all $0 < R \ll 1$.

The first two lemmas are standard; the nontrivial result is Lemma 16.

Lemma 14 (χ^2 Tail Bound). *Suppose that $X_1, \ldots, X_n$ are IID standard Gaussians in $\mathbb{C}$. Then there are constants $K, c > 0$ such that for all $\lambda > 0$ and $n \geq 1$, it holds*

$$\mathbb{P}\left[\sum_{j=1}^n |X_j|^2 > \lambda\right] \leq K e^{-c\frac{\lambda}{n}}.$$

Lemma 15. *If $c \geq 1 + \epsilon$ for some $\epsilon > 0$, $\alpha \geq 1$, and $k \geq 0$, then*

$$\sum_{j \geq k} e^{-\alpha c^j} \leq (\log(1+\epsilon))^{-1} e^{-c^k},$$

LEMMA 16. *Suppose that $c > 1/p$ for $2 < p < 6$. Then for all $M > 0$ and λ sufficiently large,*

$$\rho_1(\{u : \|u\|_p > \lambda,\ \|u\|_2 \le M\}) \lesssim e^{-c\lambda^p}$$

where the implied constant depends only on c and p. If $p = 6$, then the same conclusion holds but with the added restriction $0 < M \ll 1$.

PROOF. In Example 9, it was shown that ρ_1 is the push-forward of a probability measure $\mathbb{P}$ under the map

$$\omega \longmapsto u^\omega := \sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle} e_n,$$

where $e_n(x) = e^{inx}$ and $\{g_n\}$ is a sequence of IID standard Gaussians. Using the probabilistic convention of omitting the sample ω in notation, decompose the random function u into $u = u_{<k} + u_{\ge k}$, where

$$u_j = \sum_{2^j \le |n| < 2^{j+1}} \frac{g_n}{\langle n\rangle} e^{inx}, \quad u_{<k} = \sum_{j<k} u_j.$$

So,

$$\mathbb{P}\left[\|u\|_p > \lambda, \|u\|_2 \le M\right] \le \mathbb{P}\left[\|u_{<k}\|_p > \lambda/2, \|u\|_2 \le M\right] + \mathbb{P}\left[\|u_{\ge k}\|_p > \lambda/2, \|u\|_2 \le M\right].$$

So we should prove the desired estimate for both probabilities. For the low frequencies, by Bernstein's inequality

$$\|u_{<k}\|_p \le C 2^{k(2^{-1}-p^{-1})} \|u_{<k}\|_2 \le C 2^{k(2^{-1}-p^{-1})} M.$$

Choosing k such that $\|u_{<k}\|_p \le \lambda/2$, we see that

$$\mathbb{P}\left[\|u_{<k}\|_p > \lambda/2, \|u\|_2 \le M\right] = 0.$$

Note that this implies

$$k \sim \frac{2p}{p-2} \log_2\left(\frac{\lambda}{2CM}\right).$$

Next, we note that

$$\mathbb{P}\left[\|u_{\ge k}\|_p > \lambda/2, \|u\|_2 \le M\right] \le \mathbb{P}\left[\|u_{\ge k}\|_p > \lambda/2\right] \le \sum_{j\ge k} \mathbb{P}\left[\|u_j\|_p > \lambda_j\right]$$

where $\sum_j \lambda_j = \lambda/2$. By Bernstein again,

$$\|u_j\|_p \le C 2^{j(1/2-1/p)} \|u_j\|_2,$$

so that

$$\begin{aligned}
\mathbb{P}\left[\|u_j\|_p > \lambda_j\right] &\le \mathbb{P}\left[\|u_j\|_2 \ge C^{-1}\lambda_j 2^{j(p^{-1}-1/2)}\right] \\
&\le \mathbb{P}\left[\sum_{2^j \le |n| \le 2^{j+1}} \frac{|g_n|^2}{\langle n\rangle^2} \ge C^{-2}\lambda_j^2 2^{j(2p^{-1}-1)}\right] \\
&\le \mathbb{P}\left[\sum_{2^j \le |n| \le 2^{j+1}} |g_n|^2 \ge C^{-2}\lambda_j^2 2^{j(2p^{-1}+1)}\right] \\
&\lesssim \exp\left(-\kappa C^{-2} 2^{-j} \lambda_j^2 2^{(2p^{-1}+1)j}\right),
\end{aligned}$$

where $\kappa > 0$ is a universal constant according to Lemma 14. Choosing $\lambda_j = \frac{\lambda}{2}(1-2^{-r})2^{-r(j-k)}$ for some $r < p^{-1}$, condensing the constants into a new label α, Lemma 15 yields

$$\sum_{j\ge k} \mathbb{P}[\|u\|_p > \lambda_j] \le \sum_{j\ge k} \exp(-\alpha\lambda^2 2^{2(k-j)r} 2^{(2p^{-1}+1)j}) \lesssim \exp\left(-\alpha\lambda^2 2^{(2p^{-1}+1)k}\right)$$

as soon as λ is large enough that $\alpha\lambda^2 2^{2kr} \geq 2$, and the implied constant is independent of λ. Using the definition of k and condensing the exponents yields

$$\sum_{j\geq k} \mathbb{P}[\|u\|_p > \lambda_j] \lesssim \exp(-\alpha\lambda^{\frac{4p}{p-2}} M^{-2\frac{p+2}{p-2}}).$$

If $p < 6$, then $\frac{4p}{p-2} > p$ and so for λ sufficiently large we will have $\alpha\lambda^{\frac{4p}{p-2}} M^{-2\frac{p+2}{p-2}} \geq c\lambda^p$, thus yielding the claim of the lemma when $p < 6$. If $p = 6$, then $\frac{4p}{p-2} = 6$ and $\alpha\lambda^6 M^{-4} \geq c\lambda^6$ provided that M is sufficiently small, completing the proof. □

Proof of Theorem 13. We begin in the case $1 \leq p < 5$. By Lemma 16, there exists $M > 0$ such that $\lambda > M$ implies

$$\rho_1\left(\{\|u\|_p > \lambda,\ \|u\|_2 \leq R\}\right) \lesssim e^{-\frac{2\lambda^{p+1}}{p+1}}.$$

Then if $\frac{C^{p+1}}{p+1} = M$, it holds

$$\begin{aligned}
&\int_{H^\sigma(\mathbb{T})} \mathbf{1}_{\{\|u\|_2\leq R\}}(u)\exp\left(\frac{1}{p+1}\int |u|^{p+1}\right) d\rho_1(u)\\
&= \int_0^\infty \rho_1\left(\left\{u : \|u\|_2 \leq R,\ \exp\left(\frac{1}{p+1}\int |u|^{p+1}\right) > \lambda\right\}\right) d\lambda\\
&= \underbrace{\int_0^C \rho_1\left(\left\{u : \|u\|_2 \leq R,\ \exp\left(\frac{1}{p+1}\int |u|^{p+1}\right) > \lambda\right\}\right) d\lambda}_{=:I_1}\\
&+ \int_{\frac{C^{p+1}}{p+1}}^\infty \lambda^p e^{\frac{\lambda^{p+1}}{p+1}} \rho_1\left(\{\|u\|_p > \lambda,\ \|u\|_2 \leq R\}\right) d\lambda\\
&\lesssim I_1 + \int_M^\infty \lambda^p e^{-\frac{\lambda^{p+1}}{p+1}}\, d\lambda < \infty.
\end{aligned}$$

To deal with the case $p = 5$, just choose M small enough as in Lemma 16 and repeat the same argument as for $p < 5$. □

Global Well-posedness of NLS on $\mathbb{T}$

In this section, we will prove that the NLS is almost surely globally well-posed with respect to the measures μ_K described in Theorem 13, and as a consequence deduce their invariance under the NLS. To do this, we consider finite dimensional approximations. We restrict our analysis to just the focusing NLS, as the proofs carry out essentially the same in the defocusing case. We primarily follow the exposition of Oh, 2025, with some modification.

Definition 17 (FNLS). For $N \geq 1$, the finite nonlinear Schrödinger equation is the system of ODE defined by

$$\begin{cases} iu_t + u_{xx} &= P_{\leq N}(|P_{\leq N}u|^{p-1}P_{\leq N}u)\\ u(t=0) &= u_0\end{cases}$$

Here $P_{\leq N}u$ denotes the projection of u onto its first N frequencies.

We note that mass $\int |u|^2$ is conserved by the FNLS, and the Hamiltonian at level N is given by

$$H_N(u) = \frac{1}{2}\int |P_{\leq N}u_x|^2 - \frac{1}{p+1}\int |P_{\leq N}u|^{p+1}.$$

Henceforth, we shall fix $1 \leq p \leq 5$, and let

$$R_K(u) = \mathbf{1}_{\{\|u\|_2\leq K\}}(u)\exp\left(\frac{1}{p+1}\int |u|^{p+1}\right),$$

so that $d\mu_K = R_K\, d\rho_1$, with the understanding that $K \ll 1$ if $p = 5$. There is a Gibbs measure associated to the FNLS:

Definition 18 (Finite Dimensional Gibbs Measure). For each $N \geq 1$ and $K > 0$, and $0 \leq \sigma < 1/2$, let $R_{K,N}(u) = R_K(P_{\leq N} u)$ and $\mu_{K,N}$ be the measure on $H^\sigma(\mathbb{T})$ defined by

$$d\mu_{K,N} = R_{K,N}\, d\rho_1.$$

We note that any solution to the FNLS has $\hat{u}(t,k) = 0$ for $|k| > N$, since $P_{>N} u$ solves the ordinary Schrödinger equation with initial data 0.

Lemma 19. *The FNLS is locally well-posed for every $N \geq 1$ and initial data $u_0 \in H^{\frac{1}{2}-}(\mathbb{T})$. Moreover, the time interval of existence $[-\tau, \tau]$ satisfies $\tau \sim (1 + \|u_0\|_{H^\sigma})^{-\theta}$, and this bound is independent of N.*

Proof. Suppose that u is a solution. Then $P_{\leq N} u$ is a solution to a finite system of ODE, and $P_{>N} u$ solves the linear Schrödinger equation with initial data $P_{>N} u_0$, thus a local time solution exists by Picard's theorem. See Oh, 2025 for more details on the existence time bound. □

Theorem 20. *The measure $\mu_{K,N}$ is invariant under the dynamics of the FNLS at level N.*

Proof. Let $\Phi(t)$ be the Schrödinger dispersion operator at time t, and let $\Psi_N(t)$ be the data-to-solution map for the nonlinear ODE

$$\begin{cases} iu_t + u_{xx} &= P_{\leq N}(|P_{\leq N} u|^{p-1} P_{\leq N} u) \\ u(t=0) &= P_{\leq N} u_0 \end{cases}$$

Given initial data $u_0 \in H^\sigma(\mathbb{T})$, it holds that $\Psi_N(t) P_{\leq N} u_0 + \Phi(t) P_{>N} u_0$ solves the FNLS at level N. Let $\Phi_N(t) = \Psi_N(t) P_{\leq N} + \Phi(t) P_{>N}$. Write

$$\mu_{K,N}(A) = \mathbb{E}\left[\mathbf{1}_A(u)\, R_{K,N}(u)\right], \quad u = \sum_n \frac{g_n}{\langle n \rangle} e_n$$

where the $\{g_n\}$ are IID standard Gaussians on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Therefore, to check invariance of $\mu_{K,N}$ under $\Phi_N(t)$, it is enough to show that for all $t \in \mathbb{R}$ and Borel sets $A \subset H^\sigma(\mathbb{T})$,

$$\mathbb{E}\left[\mathbf{1}_A(\Phi_N(t) u)\, R_{K,N}(u)\right] = \mathbb{E}\left[\mathbf{1}_A(u)\, R_{K,N}(u)\right]$$

Let $\mathcal{F}_{>N}$ be the σ-algebra generated by the random variables $\{g_n : |n| > N\}$ and $\mathcal{F}_{\leq N}$ be the σ-algebra generated by $\{g_n : |n| \leq N\}$. Note that $\mathcal{F}_{>N}$ and $\mathcal{F}_{\leq N}$ are independent σ-algebras, hence $P_{\leq N} u$ and $P_{>N} u$ are independent random functions. Therefore,

$$\begin{aligned} \mathbb{E}[\mathbf{1}_A(u) R_{K,N}(u) \,|\, \mathcal{F}_{>N}] &= \mathbb{E}[\mathbf{1}_A(P_{\leq N} u + P_{>N} u) R_{K,N}(P_{\leq N} u) \,|\, \mathcal{F}_{>N}] \\ &= \mathbb{E}[\mathbf{1}_{P_{\leq N} A}(P_{\leq N} u) R_{K,N}(P_{\leq N} u)] \mathbf{1}_{P_{>N} A}(P_{>N} u) \end{aligned}$$

Next, since $R_{K,N}(\Phi_n(t) u) = R_{K,N}(\Psi_N(t) P_{\leq N} u)$, $\Psi_N(t) P_{\leq N} u$ is $\mathcal{F}_{\leq N}$ measurable, and $\Phi(t) P_{>N} u$ is $\mathcal{F}_{>N}$ measurable, it follows

$$\begin{aligned} \mathbb{E}\left[\mathbf{1}_A(\Phi_N(t) u)\, R_{K,N}(u) \mid \mathcal{F}_{>N}\right] &= \mathbb{E}\left[\mathbf{1}_{P_{\leq N} A}(\Psi_N(t) P_{\leq N} u)\, R_{K,N}(\Psi_N(t) P_{\leq N} u) \mid \mathcal{F}_{>N}\right] \mathbf{1}_{P_{>N} A}(\Phi(t) P_{>n} u) \\ &= \mathbb{E}\left[\mathbf{1}_{P_{\leq N} A}(\Psi_N(t) P_{\leq N} u)\, R_{K,N}(\Psi_N(t) P_{\leq N} u)\right] \mathbf{1}_{P_{>N} A}(\Phi(t) P_{>n} u) \\ &= \mathbb{E}\left[\mathbf{1}_{P_{\leq N} A}(P_{\leq N} u)\, R_{K,N}(P_{\leq N} u)\right] \mathbf{1}_{P_{>N} A}(\Phi(t) P_{>n} u). \end{aligned}$$

We used the facts that $R_{K,N}(u) = R_{K,N}(\Psi_N(t)P_{\leq N}u)$ and

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{P_{\leq N}A}\left(\Psi_N(t)P_{\leq N}u\right)R_{K,N}\left(\Psi_N(t)P_{\leq N}u\right)\right] &= Z_N^{-1}\int_{P_{\leq N}A}\mathbf{1}\{\|\Psi_N P_{\leq N}u\|_2 \leq K\}\\ &\quad \exp\left(-\|\Psi_N P_{\leq N}u\|_2^2 - H_N(\Psi_N P_{\leq N}u)\right)\prod_{|n|\leq N} d\hat{u}(n)\\ &= Z_N^{-1}\int_{P_{\leq N}A}\mathbf{1}\{\|P_{\leq N}u\|_2 \leq K\}\\ &\quad \exp\left(-\|P_{\leq N}u\|_2^2 - H_N(P_{\leq N}u)\right)\prod_{|n|\leq N} d\hat{u}(n)\\ &= \mathbb{E}\left[\mathbf{1}_{P_{\leq N}A}\left(P_{\leq N}u\right)R_{K,N}\left(P_{\leq N}u\right)\right]\end{aligned}$$

since the $2N+1$-dimensional mass and Hamiltonian are conserved under Ψ_N. Finally, since $e^{-in^2t}g_n =_d g_n$ (with $=_d$ denoting equivalence in distribution) and $\{e^{-in^2t}g_n : |n| > N\}$ are IID standard Gaussians,

$$\Phi(t)P_{>N}u = \sum_{|n|>N}\frac{e^{-in^2t}g_n}{\langle n\rangle}e_n =_d P_{>N}u,$$

and our prior calculations then show

$$\begin{aligned}&\mathbb{E}\left[\mathbf{1}_A\left(\Phi_N(t)u\right)R_{K,N}\left(u\right) \mid \mathcal{F}_{>N}\right]\\ &= \mathbb{E}\left[\mathbf{1}_{P_{\leq N}A}\left(P_{\leq N}u\right)R_{K,N}\left(P_{\leq N}u\right)\right]\mathbf{1}_{P_{>N}A}(\Phi(t)P_{>n}u)\\ &=_d \mathbb{E}\left[\mathbf{1}_{P_{\leq N}A}\left(P_{\leq N}u\right)R_{K,N}\left(P_{\leq N}u\right)\right]\mathbf{1}_{P_{>N}A}(P_{>n}u)\\ &= \mathbb{E}[\mathbf{1}_A(u)R_{K,N}(u) \mid \mathcal{F}_{>N}].\end{aligned}$$

Undoing the conditioning by taking expectations and using the equivalence in distribution, we get the desired conservation:

$$\mathbb{E}\left[\mathbf{1}_A\left(\Phi_N(t)u\right)R_{K,N}\left(u\right)\right] = \mathbb{E}\left[\mathbf{1}_A\left(u\right)R_{K,N}\left(u\right)\right], \quad t \in \mathbb{R}.$$

□

To establish invariance, we need to show that solutions u_N to the FNLS approximate solutions to the NLS (in a suitable sense), and also that the measures $\mu_{K,N}$ approximate μ_N (again in a suitable sense). This will require a few lemmas.

As in the proof of Theorem 20, we denote by Φ_N the data-to-solution map for the FNLS at frequency level N. The first lemma is a difficult technical lemma whose proof we omit:

Lemma 21. *Suppose that $u_0 \in H^\sigma(\mathbb{T})$ for some $0 < \sigma < 1/2$ and $\|u_0\|_{H^\sigma} \leq A$ for some $A > 1$. Suppose further that there exists a sequence $\{u_{0,N} : N \geq 1\}$ such that $u_{0,N} \to u_0$ in H^σ, $\|u_{0,N}\|_{H^\sigma} \leq A$ for all N, and $\|\Phi_N(t)u_{0,N}\|_{H^s} \leq A$ for all $|t| \leq T$. Then, if u solves the NLS with initial data u_0 on $[-T,T]$, for $\sigma_1 < \sigma$, it holds*

$$\|u - \Phi_N(t)u_{0,N}\|_{C_T H_x^{\sigma_1}} \lesssim e^{c_1 A^{c_2} T}\left(\|u_0 - u_{0,N}\|_{H^{\sigma_1}} + N^{\sigma_1 - \sigma}\right)$$

for all N sufficiently large and $|t| \leq T$.

Proof. See Oh, 2025. □

Lemma 22. *Fix $1 \leq p \leq 5$. If $p < 5$, then for $0 \leq s < 1/2$, it holds*

$$\lim_{N\to\infty} R_{K,N} = R_K \qquad \text{in } L^q\left(H^s(\mathbb{T}), \rho_1\right),\ q \geq 1.$$

In particular, taking $q = 1$, then $\mu_{K,N} \to \mu_K$ in the total variation sense as $N \to \infty$. If $p = 5$, then the same conclusion holds but with the added restriction $K \ll 1$.

PROOF. Let $\sigma = \frac{1}{2} - \frac{1}{p+1}$. First, we note that by Sobolev embedding,

$$\left| \int |P_{\leq N} u|^{p+1} - \int |u|^{p+1} \right| \lesssim \|u\|_{p+1}^{p} \|P_{>N} u\|_{H^\sigma(\mathbb{T})},$$

and since $0 \leq \sigma < 1/2$ it holds $\|P_{>N}u\|_{H^\sigma} \to 0$ a.s. It follows that $R_{K,N}(u) \to R_K(u)$ a.s. hence in probability. By a simple modification of the proof of Theorem 13, it holds $M := \sup_N \|R_{K,N}\|_{L^q(H^\sigma,\rho_1)} < \infty$. Fixing $\epsilon > 0$ and letting $A_{N,\epsilon} = \{u : |R_{K,N}(u) - R_K(u)| \leq \epsilon\}$ it holds

$$\|R_{K,N} - R_K\|_q \leq \epsilon \rho_1(A_{N,\epsilon}) + M\rho_1(A_{N,\epsilon}^c).$$

As $N \to \infty$, convergence in probability implies $\rho_1(A_{N,\epsilon}^c) \to 0$ and thus

$$\limsup_{N\to\infty} \|R_{K,N} - R_K\|_q \leq \epsilon.$$

But ϵ was arbitrary, and we are done. □

LEMMA 23 (FNLS DECAY BOUND). *Let $N \geq 1$, and $1 \leq p \leq 5$. If $0 \leq \sigma < 1/2$, for each $T > 0$ and $\epsilon > 0$, there is a set of initial data $\Omega \subset H^\sigma(\mathbb{T})$ depending only on T and ϵ such that:*

(1) $\mu_{N,K}(\Omega^c) < \epsilon$ (if $p = 5$ then $K \ll 1$)
(2) If $u_0 \in \Omega$, then

$$\|\Phi_N(t)u_0\|_{H^\sigma} \lesssim \left(\log \frac{T}{\epsilon}\right)^{1/2}, \quad |t| \leq T,$$

where the implied constant is independent of both t and N.

PROOF. Let B_R be the ball of radius $R > 0$ in H^σ. The FNLS is locally well-posed on an interval $[-\delta, \delta]$ with $\delta \lesssim \|u_0\|_{H^s}^{-\theta}$ for a constant θ independent of N, and by shrinking δ if necessary, we can assume that $\|\Phi_N(t)\| \leq 2R$ if $u_0 \in B_R$.

By proceeding as in Lemma 11 we can construct the set

$$\Omega = \bigcap_{|j| \leq \lfloor T/\delta \rfloor} \Phi_N(j\delta)(B_R).$$

It follows from the invariance of $\mu_{N,K}$ that

$$\mu_{N,K}(\Omega^c) \lesssim TR^\theta \mu_{N,K}(B_R^c).$$

Next, we observe that

$$\mu_{N,K}(B_R^c) \leq \|R_{K,N}\|_{L^2(\rho_1)} \rho_1(B_R^c)^{\frac{1}{2}}$$

and $\sup_N \|R_{K,N}\|_{L^2(\rho_1)} < \infty$ from Lemma 22. By Fernique's theorem (Theorem 7(1)), $\rho_1(B_R^c) \lesssim e^{-cR^2}$ for all R large enough, and so for $TR^\theta e^{-cR^2} \ll \epsilon$ (such R can be chosen independently of N from the bound on $\|R_{K,N}\|_{L^2(\rho_1)}$) it holds $\mu_{N,K}(\Omega^c) < \epsilon$.

To prove (2), the above calculation shows that by choosing $R^2 \lesssim \log \frac{T}{\epsilon}$, for $u_0 \in \Omega$ we have $\|\Phi_N(j\delta)u_0\|_{H^\sigma} \leq R$ for $|j| \leq \lfloor T/\delta \rfloor$ and thus $\|\Phi_N(t)u_0\|_{H^\sigma} \leq 2R$ for $|t| \leq T$. This yields (2). □

As in the proof of Corollary 12, we can deduce that the FNLS is $\mu_{N,K}$ a.e. globally well-posed. Moreover, we have the following:

COROLLARY 24. *For each $j \geq 1$ and $N \geq 1$, there exists a set of initial data $\Sigma_N^j \subset H^\sigma(\mathbb{T})$ such that*

(1) $\mu_{N,K}((\Sigma_N^j)^c) < 2^{-j}$
(2) If $u_0 \in \Sigma_N^j$, then

$$\|\Phi_N(t)u_0\|_{H^\sigma} \lesssim \sqrt{j + \log(1 + |t|)}.$$

PROOF. For each $k \geq 1$ construct a set Ω_k as in Lemma 23 satisfying $\mu_{N,K}(\Omega_k) < 2^{-(j+k)}$ and a solution to the FNLS exists for times $|t| \leq 2^k$ with decay bound described by the Lemma. Let $\Sigma_N^j = \bigcap_k \Omega_k$. Then $\mu((\Sigma_N^j)^c) < 2^{-j}$, and for $t > 0$, fixing $k \geq 1$ such that $2^{k-1} < |t| \leq 2^k$, it holds

$$\|\Phi_N(t)u_0\|_{H^\sigma} \lesssim \sqrt{\log(2^j 2^{2k})} \lesssim \sqrt{j + \log(1+|t|)},$$

and we are done. □

We are now ready for the first grand theorem of this section:

THEOREM 25. *The NLS is μ_K-a.s. globally well-posed on $H^\sigma(\mathbb{T})$ for any $0 \leq \sigma < 1/2$ and $1 \leq p \leq 5$.*

PROOF. Let Σ_N^j be the sets constructed as in Corollary 24 and let Σ^j be the set of $u \in H^\sigma$ such that there is a subsequence $\{N_k\}$ and $u_{N_k} \in \Sigma_{N_k}^j$ satisfying $u_{N_k} \to u$ in H^σ as $k \to \infty$. Since Lemma 22 implies $\mu_{N,K} \to \mu_K$ in the weak sense,

$$\mu_K(\Sigma^j) \geq \limsup_{N\to\infty} \mu_{N,K}(\Sigma_N^j) \geq 1 - 2^{-j}.$$

Therefore, if $\Sigma = \bigcup_j \Sigma_j$, it holds $\mu_K(\Sigma) = 1$. If $u_0 \in \Sigma$, then $u_0 \in \Sigma^j$ for some j, and so there is a sequence $\{u_{N_k}\}$ with $u_{N_k} \in \Sigma_{N_k}^j$ and $u_{N_k} \to u$ in H^σ. By Lemma 21, it holds

$$\lim_{k\to\infty} \Phi_{N_k}(t) u_{N_k}$$

converges to a solution to the NLS with initial data u when t is sufficiently small (see Theorems 1 and 2), and by a similar argument employed in Lemma 11 the solution map can be extended to all time t. □

COROLLARY 26. *The measures μ_K are invariant under the solution map $\Phi(t)$ of the NLS.*

PROOF. It is enough to show that for every $f \in C_b(H^\sigma)$, it holds

$$\int f(\Phi(t)u)\, d\mu_K(u) = \int f(u)\, d\mu_K(u).$$

To this end, by the invariance of the $\mu_{K,N}$,

$$\begin{aligned}\left|\int f(\Phi(t)u)\, d\mu_K(u) - \int f(u)\, d\mu_K(u)\right| &= \lim_{N\to\infty}\left|\int f(\Phi(t)u)\, d\mu_{K,N}(u) - \int f(\Phi_N(u))\, d\mu_{K,N}(u)\right| \\ &\leq \limsup_{N\to\infty} \|f\circ\Phi - f\circ\Phi_N\|_{L^2(\rho_1)} \|R_{K,N}\|_{L^2(\rho_1)}.\end{aligned}$$

Taking $A_{n,\epsilon} = \{u : |F(\Phi_N(u)) - F(\Phi(u))| \leq \epsilon\}$, and the fact $\|R_{K,N}\| \leq M$, we see that

$$\limsup_{N\to\infty} \|f\circ\Phi - f\circ\Phi_N\|_{L^2(\rho_1)} \|R_{K,N}\|_{L^2(\rho_1)} \leq M\epsilon + 2\|f\|_\infty \rho_1(A_{N,\epsilon}^c).$$

As $N \to \infty$, we obtain

$$\left|\int f(\Phi(t)u)\, d\mu_K(u) - \int f(u)\, d\mu_K(u)\right| \leq M\epsilon.$$

But $\epsilon > 0$ was arbitrary, and we are done. □

Further Directions

The primary purpose of this paper was to construct a "Gibbs" measure on $\mathbb{T}$ on which the NLS is globally well-posed for $3 \leq p < 5$. One wonders if this technique can be modified. Namely,

(1) Can we use this technique to prove a.s. GWP for the NLS on $\mathbb{T}^d$ for $d \geq 2$,
(2) can we replace the manifold $\mathbb{T}^d$ with an arbitrary compact manifold $M \subset \mathbb{R}^d$ (after perhaps imposing some additional boundary conditions),
(3) and can we study other nonlinear dispersive PDE with appropriately defined Gibbs measures?

The answer to all 3 of these questions is affirmative. In addressing problem (1), the primary difficulty in extending the theory is that the Gibbs measure requires Sobolev regularity $s = 1$. For $(1-\Delta)^{\sigma-1}$ to be trace class on $H^\sigma(\mathbb{T}^d)$, we require $0 \leq \sigma < 1 - d/2$, which forces $d = 1$. Unfortunately, swapping out the underlying space of initial data is not helpful, since the same conditions arise for instance when H^σ is replaced by the general Sobolev spaces $W^{\sigma,p}$ and any of the Besov spaces. Thus the technique employed here does not obviously generalize to $d \geq 2$. In the defocusing case, Bourgain, 1996 constructed the Φ_2^4 measure on $\mathbb{T}^2$ using Wick renormalization and this measure performs the same role as the Gibbs measure here.

In general, swapping out the manifold is a tricky subject. When the manifold is compact and suitable boundary conditions enforced, then it is possible to perform a similar argument employed here since the Laplace operator has a discrete spectrum and is thus trace class under suitable conditions. In fact, by Weyl's law, if M is any bounded domain in $\mathbb{R}^d$ then the operator $(1-\Delta)^{s-\sigma}$ is trace class in $H_0^s(M)$ if $s < \sigma - d/2$. It does not appear that there is a canonical choice of Gibbs measure on $\mathbb{R}^d$. When $d \geq 3$, one approach is to use "template" initial data, as illustrated for the cubic NLS in Bényi et al., 2015. However, it appears to be an open question the "correct" choice of invariant measure on $\mathbb{R}^d$.

While constructing the "correct" Gibbs measure is in general hard, luckily the technique generalizes fairly well to other PDE with Hamiltonian dynamics. Examples where success has been found include the nonlinear wave, KdV, and Benjamin-Ono equations. Interestingly, stationary measure type arguments have also been applied to study the Navier-Stokes equation.

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