Eigenvalues and Eigenvectors

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Basics

- Eigen (German)?
  - “proper”, “characteristic”

- Where is it used?
  - Find natural frequencies and mode shapes in dynamics systems
  - Solving differential equations
  - Reducing dimensions using principal component analysis
  - Obtaining principal stresses in mechanics
  - Google search engine algorithm →PageRank

Basics

- Matrix A applied to a column vector x → Ax which is a linear transformation of x.
- A special transform → $Ax = \lambda x$ where A is an n x n matrix, x is an n x 1 column vector ($x \neq 0$) and λ is a scalar.
- λ is an eigenvalue of the matrix A.
- The associated vector x is called an eigenvector corresponding to λ.

Basics

- Assume that we have Ax where A and x are two vectors. This means some kind of transformation of the vector x into another vector.

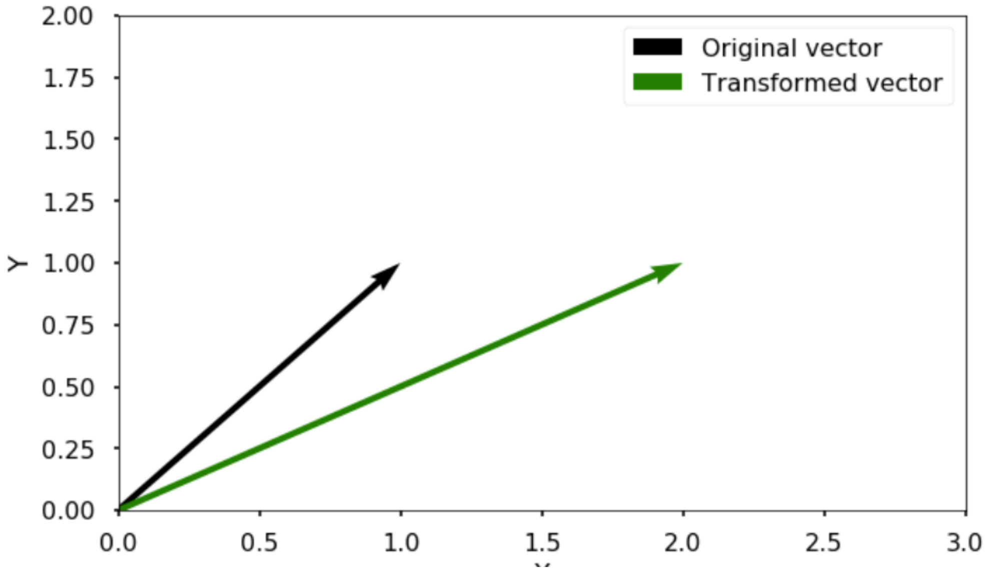
- For some vectors, the effect of the transformation, Ax, is only scaling (stretching, compressing and flipping).

- The eigenvectors are the vectors having this property and the eigenvalues λ are the scale factors.

Example

TRY IT Plot the vector x = [[1], [1]] and the vector $b = Ax$, where A = [[2, 0], [0, 1]]

The vector [[1], [1]] is transformed to
[[2], [1]]. Let's try to do the same exercise with a different vector [[1], [0]].

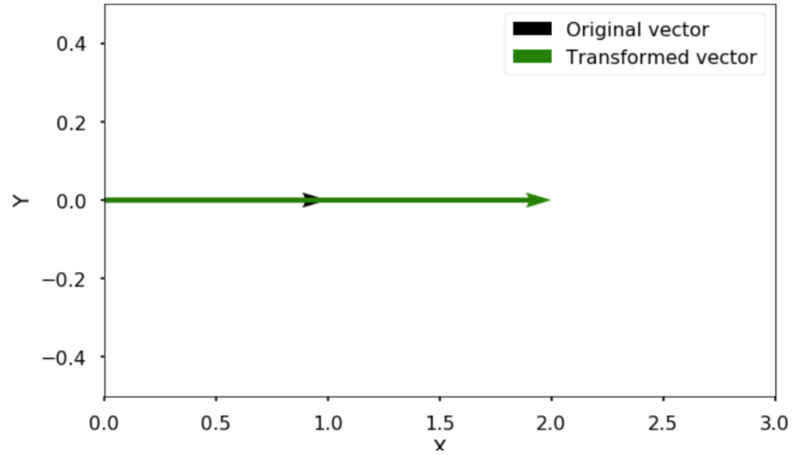


Example

TRY IT! Plot the vector x = [[1], [0]] and the vector $b = Ax$, where A = [[2, 0], [0, 1]]

Now we can see that with this new vector, the only thing changed after the transformation is the length of the vector, it is stretched. The new vector is [[2], [0]], therefore, the transform is

$$Ax = 2x$$



Example

TRY IT! Plot the vector x = [[1], [0]] and the vector $b = Ax$, where A = [[2, 0], [0, 1]]

Now we can see that with this new vector, the only thing changed after the transformation is the length of the vector, it is stretched. The new vector is [[2], [0]], therefore, the transform is

$$Ax = 2x$$

Therefore:
2 is the eigenvalue
X [[1],[0]] is the eigenvector

When x=[[0],[1]], it is also a eigenvector with 1 as the eigenvalue

Eigenvalue and Eigenvector

- So how do we find Eigenvalue?

- Characteristic Equations

- From $Ax = \lambda x$
  - Change the form to $(A - \lambda I)\, x = 0$
  - I is an identity matrix with the same dimension as A

Eigenvalue and Eigenvector

- $det(A - \lambda I) = 0$ → this equation is called a ***characteristic equation*** used for getting eigenvalues

In the case of a 2×2 matrix, the determinant is

$$\det|M| = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Eigenvalue and Eigenvector

- Obtain the eigenvalues for matrix [[0,2],[2,3]] → ($\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix}$)

- ***characteristic equation produces***

$$\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \begin{vmatrix} 0-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

Eigenvalue and Eigenvector

- ***characteristic equation produces :***

$$\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \begin{vmatrix} 0-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

- $ad - bc \rightarrow -\lambda(3-\lambda) - 2 \times 2 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$

- Two eigenvalues $\lambda = 4, \lambda = -1$

Eigenvalue and Eigenvector

- Eigenvectors for $\lambda = 4$

Let's get the first eigenvector when $\lambda_1 = 4$, we can simply insert it back to $A - \lambda I = 0$, where we have:

$$\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} - \begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix} = \begin{vmatrix} 0-4 & 2 \\ 2 & 3-4 \end{vmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvalue and Eigenvector

Therefore, we have two equations $-4x_1 + 2x_2 = 0$ and $2x_1 - x_2 = 0$, both of them indicate that $x_2 = 2x_1$. Therefore, we can have the first eigenvector as

$$x_1 = k_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

k_1 is a scalar vector ($k_1 \neq 0$), as long as we have the ratio between x_2 and x_1 as 2, it will be an eigenvector. We can verify the vector [[1], [2]] is an eigenvector by inserting it back:

$$\begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Class Assignment

- Now try $\lambda = -1$

- What is the eigenvector?

Python Code

Python

```
import numpy as np

# define matrix
A = np.array([
    [0, 2],
    [2, 3]
])

# eigenvalues and eigenvectors
eigenvalues, eigenvectors = np.linalg.eig(A)

print("Matrix A:")
print(A)

print("\nEigenvalues:")
print(eigenvalues)

print("\nEigenvectors:")
print(eigenvectors)
```

Python

```
Matrix A:
[[0 2]
 [2 3]]

Eigenvalues:
[-1.  4.]

Eigenvectors:
[[-0.89442719 -0.4472136 ]
 [ 0.4472136  -0.89442719]]
```

Notice that eigenvalues are unique;
Eigenvectors are not!