Growth of Functions

	constant	*logarithmic*	*linear*	*N-log-N*	*quadratic*	*cubic*	*exponential*
n	**O(1)**	**O(log n)**	**O(n)**	**O(n log n)**	**O(n^2)**	**O(n^3)**	**O(2^n)**
1	1	1	1	1	1	1	2
2	1	1	2	2	4	8	4
4	1	2	4	8	16	64	16
8	1	3	8	24	64	512	256
16	1	4	16	64	256	4,096	65536
32	1	5	32	160	1,024	32,768	4,294,967,296
64	1	6	64	384	4,069	262,144	1.84×10^{19}

Asymptotic Notations

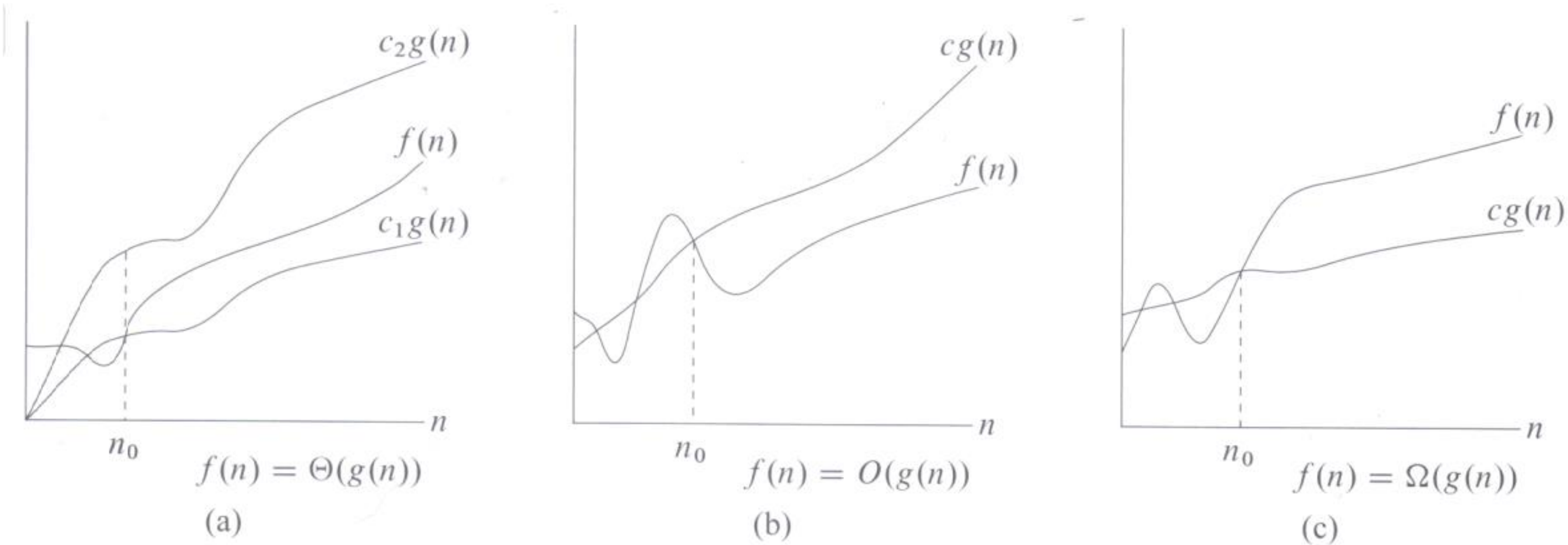
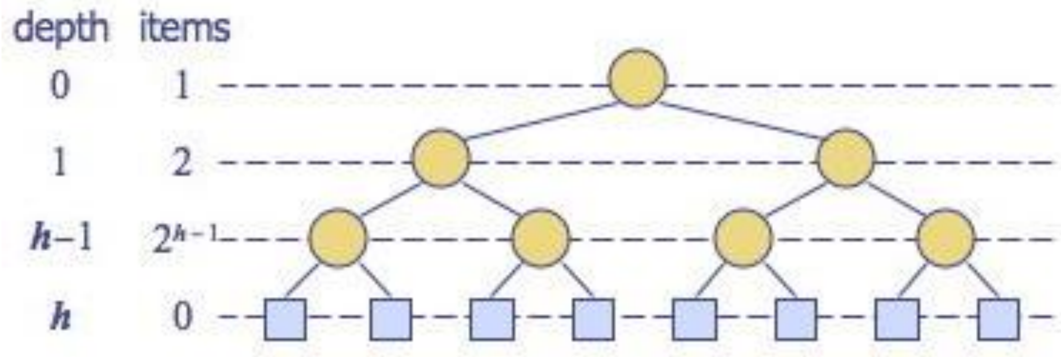


Figure 3.1 Graphic examples of the Θ, O, and Ω notations. In each part, the value of n_0 shown is the minimum possible value; any greater value would also work. **(a)** Θ-notation bounds a function to within constant factors. We write $f(n) = \Theta(g(n))$ if there exist positive constants n_0, c_1, and c_2 such that at and to the right of n_0, the value of $f(n)$ always lies between $c_1 g(n)$ and $c_2 g(n)$ inclusive. **(b)** O-notation gives an upper bound for a function to within a constant factor. We write $f(n) = O(g(n))$ if there are positive constants n_0 and c such that at and to the right of n_0, the value of $f(n)$ always lies on or below $cg(n)$. **(c)** Ω-notation gives a lower bound for a function to within a constant factor. We write $f(n) = \Omega(g(n))$ if there are positive constants n_0 and c such that at and to the right of n_0, the value of $f(n)$ always lies on or above $cg(n)$.

Binary Tree, Summations…

- **Binary Tree**



- **Summations**

Let $s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$.

Then $rs = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n$

Then $s - rs = a - ar^n$

Then $s(1-r) = a(1-r^n)$, so $s = a\dfrac{1-r^n}{1-r}$ (if $r \neq 1$).

Sums of Powers of Natural Numbers

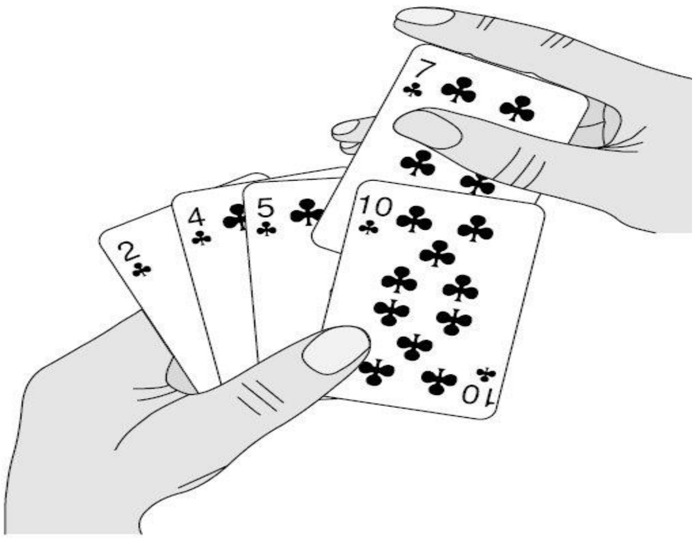
1. $1+2+3+\cdots+n = \sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

2. $1^2+2^2+3^2+\cdots+n^2 = \sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$

3. $1^3+2^3+3^3+\cdots+n^3 = \sum_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$

Sorting Algorithms

Insertion sort



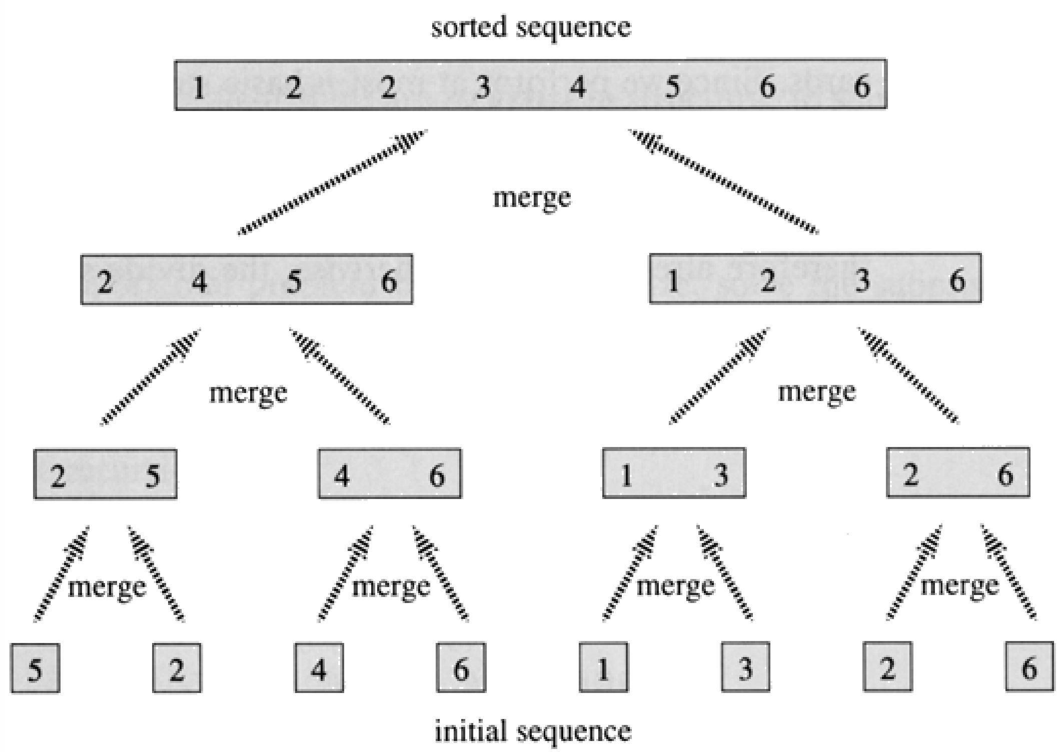
INSERTION-SORT(A)	*cost*	*times*
for $j = 2$ **to** $A.length$	c_1	n
$key = A[j]$	c_2	$n-1$
// Insert $A[j]$ into the sorted sequence $A[1\,..\,j-1]$.	0	$n-1$
$i = j-1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
$A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n}(t_j - 1)$
$i = i-1$	c_7	$\sum_{j=2}^{n}(t_j - 1)$
$A[i+1] = key$	c_8	$n-1$

Note: More Sorting algorithms (Bubble sort, selection sort) are covered in the class, take notes in class

Divide and Conquer: Merge Sort

```
MERGE(A, p, q, r)
n1 = q − p + 1
n2 = r − q
let L[1 .. n1 + 1] and R[1 .. n2 + 1] be new arrays
for i = 1 to n1
    L[i] = A[p + i − 1]
for j = 1 to n2
    R[j] = A[q + j]
L[n1 + 1] = ∞
R[n2 + 1] = ∞
i = 1
j = 1
for k = p to r
    if L[i] ≤ R[j]
        A[k] = L[i]
        i = i + 1
    else A[k] = R[j]
        j = j + 1
```

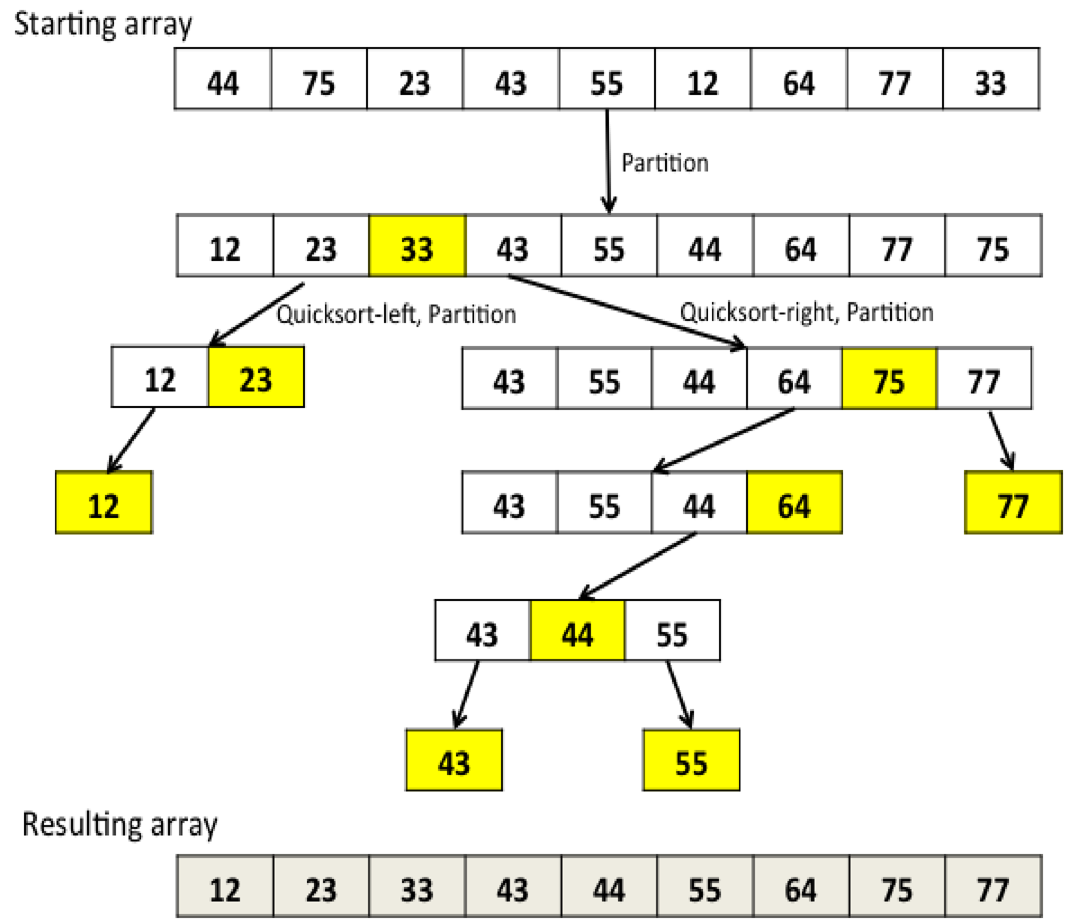
```
MERGE-SORT(A, p, r)
if p < r
    q = ⌊(p + r)/2⌋
    MERGE-SORT(A, p, q)
    MERGE-SORT(A, q + 1, r)
    MERGE(A, p, q, r)
```



Divide and Conquer: Quick Sort

```
QUICKSORT(A, p, r)
if p < r
    q = PARTITION(A, p, r)
    QUICKSORT(A, p, q − 1)
    QUICKSORT(A, q + 1, r)
```

```
PARTITION(A, p, r)
x = A[r]
i = p − 1
for j = p to r − 1
    if A[j] ≤ x
        i = i + 1
        exchange A[i] with A[j]
exchange A[i + 1] with A[r]
return i + 1
```



Masters Theorem

The Master Theorem

- if $T(n) = aT(n/b) + f(n)$ then

$$T(n) = \begin{cases} \Theta\left(n^{\log_b a}\right) & f(n) = O\left(n^{\log_b a - \varepsilon}\right) \\ \Theta\left(n^{\log_b a} \log n\right) & f(n) = \Theta\left(n^{\log_b a}\right) \\ \Theta\left(f(n)\right) & f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right) \text{ AND} \\ & af(n/b) < cf(n) \text{ for large } n \end{cases} \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\end{matrix}}\right\} \begin{matrix} \varepsilon > 0 \\ c < 1 \end{matrix}$$