Linear Programming: Formulations

Example 1:

A manufactures has two resources R_1 and R_2 And is making two products P_1 and P_2 each unit of P_1 requires 1 unit of R_1 and 4 units of R_2. each unit of P_2 requires 2 unit of R_1 and 1 units of R_2. Manufacturer has 8 units of R_1 and 14 units of R_2. The profit for P_1 is 4$ per unit and for P_2 it is 7$ per unit. Find how many units of both products should be produced to get the maximum profit.

Example 2:

A calculator company produces a scientific calculator and a graphing calculator. Long-term projections indicate an expected demand of at least 100 scientific and 80 graphing calculators each day. Because of limitations on production capacity, no more than 200 scientific and 170 graphing calculators can be made daily. To satisfy a shipping contract, a total of at least 200 calculators much be shipped each day.

If each scientific calculator sold results in a $2 loss, but each graphing calculator produces a $5 profit, how many of each type should be made daily to maximize net profits?

Linear Programming: Formulations (do it yourself)

1. A farmer has 10 acres to plant in wheat and rye. He has to plant at least 7 acres. However, he has only $1200 to spend and each acre of wheat costs $200 to plant and each acre of rye costs $100 to plant. Moreover, the farmer has to get the planting done in 12 hours and it takes an hour to plant an acre of wheat and 2 hours to plant an acre of rye. If the profit is $500 per acre of wheat and $300 per acre of rye how many acres of each should be planted to maximize profits?

2. A gold processor has two sources of gold ore, source A and source B. In order to kep his plant running, at least three tons of ore must be processed each day. Ore from source A costs $20 per ton to process, and ore from source B costs $10 per ton to process. Costs must be kept to less than $80 per day. Moreover, Federal Regulations require that the amount of ore from source B cannot exceed twice the amount of ore from source A. If ore from source A yields 2 oz. of gold per ton, and ore from source B yields 3 oz. of gold per ton, how many tons of ore from both sources must be processed each day to maximize the amount of gold extracted subject to the above constraints?

3. A publisher has orders for 600 copies of a certain text from San Francisco and 400 copies from Sacramento. The company has 700 copies in a warehouse in Novato and 800 copies in a warehouse in Lodi. It costs $5 to ship a text from Novato to San Francisco, but it costs $10 to ship it to Sacramento. It costs $15 to ship a text from Lodi to San Francisco, but it costs $4 to ship it from Lodi to Sacramento. How many copies should the company ship from each warehouse to San Francisco and Sacramento to fill the order at the least cost?

Linear Programming: Simplex Method

Maximize z: $6X_1 + 5X_2$

Such that,

$$X_1 + X_2 \leq 5$$
$$3X_1 + 2X_2 \leq 12$$

Add slack variables (s_1 and s_2) to convert inequality into equality,

$$X_1 + X_2 + s_1 = 5$$
$$3X_1 + 2X_2 + s_2 = 12$$

Simplex Method: Theoretical Basis

Iternation 1

First Basic feasible solution is obtained by getting the maximum feasible values for slack variables

$$S_1 = 5 - X_1 - X_2 \quad \text{------(1)}$$
$$S_2 = 12 - 3X_1 - 2X_2 \quad \text{-----(2)}$$

And $z = 6X_1 + 5X_2$

Here basic feasible solution is $z = 0$ for $X_1 = 0$, $X_2 = 0$, $S_1 = 5$, $S_2 = 12$

Iteration 2:

We observe the variable with highest coefficient in z, that variable is X_1 , we then try to increase z by increasing X_1, **the maximum feasible value of X_1** is 4 from eqn (2), increasing X_1 beyond 4 will make s_1 negative thus violating the constraint of all the variables being positive, therefore,

$X_1 = 4 - 2/3X_2 - 1/3s_2$ ----(3) // substituting for X_1 in (2)

and z

$S_1 = 5 - X_1 - X_2 = 1 - 1/3X_2 + 1/3s_2$ -----(4)

And $z = 24 + X_2 - 2s_2$

Simplex Method: Theoretical Basis

Iternation 3

We observe the variable with highest positive coefficient in z, that variable is X_2 , we then try to increase z by increasing this variable**, the maximum feasible value of X_2** is 3 from eqn (4), increasing X_1 beyond 3 will make s_2 negative thus violating the constraint of all the variables being positive, therefore,

X_2 **=** 3 **+** s_2-3s_1

X_1 = 2 + 2s_1 -s_2 // substituting in (3) and z

And z =27 - 3s_1- s_2

Notice that z can not be increased further because the coefficients of all the variables (s_1 and s_2) are negative and increasing them will decrease the value of z. Therefore, the final solution is z = 27 for X_2 **=** 3 **a**nd X_1 = 2 and value of slack variables are zero.

Simplex Method: Tabular Form

Problem 1: Maximize z: $6X_1+5x_2$

s.t.

$$x_1+x_2 \leq 5$$

$$3x_1+2x_2 \leq 12$$

$$x_1,x_2 \geq 0$$

Introduce slack variables to convert it into equality,

$$x_1+x_2 +s_1 = 5$$

$$3x_1+2x_2 +s_2 =12$$

$$x_1,x_2 ,s_1,s_2 \geq 0$$

In objective function, contribution of s_1 and s_2 are 0.

The first basic feasible solution is found by setting orignal variables (X_1 and X_2) to 0. We start with s_1 and s_2.

Simplex Method: Problem 1 contd.

Corresponding contribution of variables in Z, objective function

C_j			6	5	0	0		
			X_1	X_2	s_1	s_2	RHS	L (leaving)
0	s_1		1	1	1	0	5	
0	s_2		3	2	0	1	12	
		Z_j	0	0	0	0	0	
		$C_j - Z_j$	6	5	0	0	0	

Z_j values are computed by multiplying corresponding contribution of variables in objective function with corresponding coefficients of variables in the equation, such as, 0*1 + 0*3 = 0, and so on.

$C_j – Z_j$ is computed by subtracting Z_j from corresponding C_j values.

Simplex Method: Problem 1 contd.

Corresponding contribution of variables in Z, objective function

			C_j	6	5	0	0			
				X_1	X_2	s_1	s_2	RHS	L (leaving)	
0	s_1			**1**	**1**	**1**	**0**	**5**	**5/1 = 5**	R_1
0	s_2			**3**	**2**	**0**	**1**	**12**	**12/3 =4** →	R_2
s_1		Z_j		0	0	0	0	0		
		$C_j - Z_j$		6	5	0	0	0		
s_2										

Entering variable is the one with highest positive $C_j - Z_j$ values, in this step it is X_1. The leaving variable is the one with lowest corresponding (RHS/corresponding coefficient) of entering variable, in this case 1 and 3 are those coefficients. Thus S_2 leaves. Corresponding element, 3 (circled in the table) in this case is the pivot element and Row R2 is pivot Row.

Simplex Method: Problem 1 contd.

Corresponding contribution of variables in Z, objective function

C_j			6	5	0	0			
			X_1	X_2	s_1	s_2	RHS	L (leaving)	
0	s_1		**1**	**1**	**1**	**0**	**5**	**5/1 = 1**	R_1
0	s_2		**3**	**2**	**0**	**1**	**12**	**12/3 =4**	R_2
		Z_j	0	0	0	0	0		
		C_j - Z_j	6	5	0	0			
0	s_1		0	1/3	1	-1/3			R_3
6	X_1		1	2/3	0	1/3			R_4

X_1 enters the basis and s_2 leaves. We divide the R_2 by pivot element to get the value corresponding to X_1 1

Next, Now we have to get a 0 above the corresponding 1 in the last step, to achieve that, we subtract R_4 From R_1 to get R_3

Simplex Method: Problem 1 contd.

		C_j	6	5	0	0			
			X_1	X_2	s_1	s_2	RHS	L (leaving)	
0	s_1		**1**	**1**	**1**	**0**	**5**	**5/1 = 1**	R_1
0	s_2		**3**	**2**	**0**	**1**	**12**	**12/3 =4**	R_2
		Z_j	0	0	0	0	0		
		$C_j - Z_j$	6	5	0	0			
0	s_1		0	1/3	1	-1/3	1	1/1/3=3 →	R_3
6	X_1		1	2/3	0	1/3	4	4/2/3 = 6	R_4
		Z_j	6	4	0	2	24		
		C_j-Z_j	0	1	0	-2			

We computer corresponding Z_j and $C_j – Z_j$ values. Like in the last step we identify entering variable as X_2 since corresponding C_j-Z_j value is highest positive among all. To identify leaving variable we divide corresponding RHS value by corresponding column coefficient values of entering variable. So variable X_2 enters and variable S_1 leaves. Pivot element becomes 1/3 and pivot row is R_3

Simplex Method: Problem 1 contd.

		C_j	6	5	0	0			
			X_1	X_2	s_1	s_2	RHS	L (leaving)	
0	s_1		**1**	**1**	**1**	**0**	**5**	**5/1 = 1**	**R_1**
0	s_2		**3**	**2**	**0**	**1**	**12**	**12/3 =4**	**R_2**
		Z_j	0	0	0	0	0		
		$C_j – Z_j$	6	5	0	0			
0	s_1		0	1/3	1	-1/3	1	1/1/3=3	**R_3**
6	X_1		1	2/3	0	1/3	4	4/2/3 = 6	**R_4**
		Z_j	6	4	0	2	24		
		C_j-Z_j	0	1	0	-2			
5	X_2		0	1	3	-1	3		**R_5**
6	X_1		1	0	-2	1	2		**R_6**
		Z_j	6	5	3	1	27		
		C_j-Z_j	0	0	-3	-1			

We get value 1 corresponding to entering variable X_2 by dividng the pivot row by pivot element. Therefore, $R_5 = R_3/(1/3)$
Now we need 0 above the corresponding X_2 values, to get R_6, therefore, $R_6 = R_4 – (2/3)R_5$. Next we compute Z_j and C_j - Z_j

Simplex Method: Problem 1 contd.

		C_j	6	5	0	0			
			X_1	X_2	s_1	s_2	RHS	L (leaving)	
0	s_1		**1**	**1**	**1**	**0**	**5**	**5/1 = 1**	$\mathbf{R_1}$
0	s_2		**3**	**2**	**0**	**1**	**12**	**12/3 =4**	$\mathbf{R_2}$
		Z_j	0	0	0	0	0		
		$C_j – Z_j$	6	5	0	0			
0	s_1		0	1/3	1	-1/3	1	1/1/3=3	$\mathbf{R_3}$
6	X_1		1	2/3	0	1/3	4	4/2/3 = 6	$\mathbf{R_4}$
		Z_j	6	4	0	2	24		
		C_j-Z_j	0	1	0	-2			
5	X_2		0	1	3	-1	3		$\mathbf{R_5}$
6	X_1		1	0	-2	1	2		$\mathbf{R_6}$
		Z_j	6	5	3	1	27		
		C_j-Z_j	0	0	-3	-1			

All $C_j – Z_j$ values are either zero or negative, therefore, we found our solution. The maximum Z is the final Z_j value, in this case 27 and corresponding values for X_2 and X_1 are corresponding RHS values, in this case they are 2 and 3.

Simplex Method: Tabular Form (Problem 2)

Problem 1: Minimize z: $3X_1+4x_2$

s.t.

$$2x_1+x_2 \geq 5$$

$$x_1+3x_2 \geq 12$$

Introduce slack variables to convert it into equality,

$$x_1+x_2 - s_1 = 5$$

$$3x_1+2x_2 - s_2 =12$$

We need to convert this problem into maximization problem.

Recall that minimization of z is same as maximization of –z (negative z). Therefore, our problem becomes,

Maximize z: $-3X_1- 4x_2$

s.t.

$$2x_1+x_2 \geq 5$$

$$x_1+3x_2 \geq 12$$

Simplex Method: Problem 2 contd..

Now we introduce slack variables to convert inequality into equality.

$$2x_1+x_2 - s_1 = 5$$

$$x_1+3x_2 - s_2 = 12$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Notice the coefficient of slack variable is negative because of greater than inequality.

To get the first basic feasible solution, we set original variables to 0 but in this case the resulting values of slack variables s_1 = -5 and s_2= -12 are negative violating the last constraint. To avoid that we introduce artificial variables, a_1, a_2.

$$2x_1+x_2 - s_1 + a_1 = 5$$

$$x_1+3x_2 - s_2 + a_2 = 12$$

$$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$$

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Simplex Method: Problem 2 contd…

Now we introduce slack variables to convert inequality into equality.

$$2x_1+x_2 - s_1 = 5$$

$$x_1+3x_2 - s_2 = 12$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Notice the coefficient of slack variable is negative because of greater than inequality.

To get the first basic feasible solution, we set original variables to 0 but in this case the resulting values of slack variables s_1 = -5 and s_2= -12 are negative violating the last constraint. To avoid that we introduce artificial variables, a_1, a_2.

$$2x_1+x_2 - s_1 + a_1 = 5$$

$$x_1+3x_2 - s_2 + a_2 = 12$$

$$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$$

Notice in this case first basic feasible solution is possible by setting all non artificial variables to 0, we get positive feasible values of artificial variables.

Simplex Method: Problem 2 contd..

Notice the contribution of slack variables and artificial variables to objective function z is 0.

This problem will be solved in two phases, in the first phase we will get first basic feasible solution without artificial variable.

To do that, we wil maximize: $-a_1 - a_2$

s.t.

$$2x_1+x_2-s_1+a_1=5$$
$$x_1+3x_2-s_2+a_2=12$$
$$x_1,x_2,s_1,s_2,a_1,a_2 \geq 0$$

Simplex Method: Problem 2, phase 1

		C_j	0	0	0	0	-1	-1		
			X_1	X_2	s_1	s_2	a_1	a_2	RHS	L
-1	a_1		**2**	**1**	**-1**	**0**	**1**	**0**	**8**	**8**
-1	a_2		**1**	**3**	**0**	**-1**	**0**	**1**	**12**	**4** →
		Z_j	-3	-4	1	1	-1	-1	-20	
		$C_j - Z_j$	3	4	-1	-1	0	0		
-1	a_1		5/3	0	-1	1/3	1	-1/3	4	12/5 →
0	X_2		1/3	1	0	-1/3	0	1/3	4	12
		Z_j	-5/3	0	1	-1/3	-1	1/3		
		$C_j - Z_j$	5/3	0	-1	1/3	0	-4/3		
0	X_1		1	0	-3/5	1/5	3/5	-1/5	12/5	
0	X_2		0	1	1/5	-2/5	-1/5	-4/5	16/5	

Entering variables, leaving variables are shown with arrows and pivot elements are showing with circles. We performed the same steps as the last problem.
Notice now we have a basic solution without artificial variable. Therefore, we import this solution to the next phase.

Simplex Method: Problem 2, phase 2.

We will use this basic solution in the original objective function (the one with -z). We import the basic solution by dropping columns corresponding to artificial variables and continue

	C_j	-3	-4	0	0		
		X_1	X_2	s_1	s_2	RHS	L
-3 X_1		**1**	**0**	**-3/5**	**1/5**	**12/5**	
-4 X_2		**0**	**1**	**1/5**	**-2/5**	**16/5**	
	Z_j	-3	-4	1	1	-20	
	$C_j - Z_j$	0	0	-1	-1		

Notice all the C_j-Z_j values are negative or zero, therefore, we are done. -20 is the maximum value of –z therefore, the minimum value of z is 20 for X_1 = 12/5 and X_2 = 16/5.