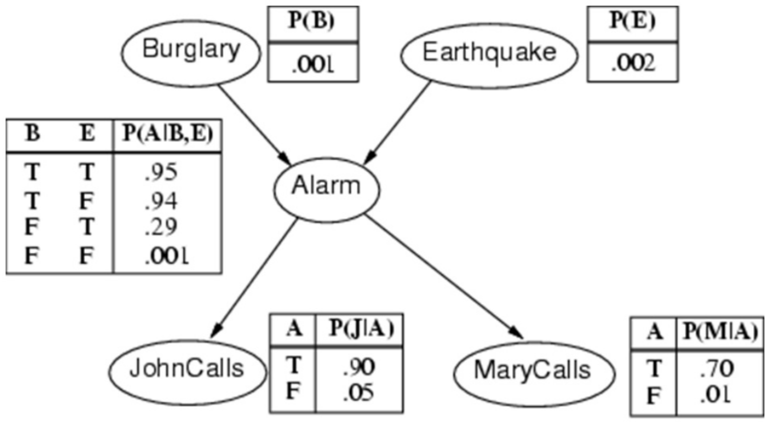
CS4410 - Review Final

Question 1. Show the perceptron that calculates the parity of its three inputs.

Question 2. Observe the Bayesian Network figure as below and compute:



a) The probability of an alarm.

b) Probability of an alarm given there was a burglary.

c) The probability of a burglary given there was an alarm and there was also an earthquake.

d) Knowing that John called, what is the probability of an alarm?

e) Knowing that Mary called, what is the probability of an alarm?

Solution. Before we solve the problem, we recall a very important theorem:

"Each node in the Bayesian Network is conditionally independent of its nondescendants, given its parents."

a) First, we have the following identity.

$$\begin{aligned}(B\cap E)\cup(\overline{B}\cap E)\cup(B\cap\overline{E})\cup(\overline{B}\cap\overline{E}) &= [(B\cup\overline{B})\cap E]\cup[(B\cup\overline{B})\cap\overline{E}]\\ &= E\cup\overline{E}\\ &= S\end{aligned}$$

In other words, our sample space can be divided into 4 separate components: $B\cap E, \overline{B}\cap E, B\cap\overline{E}, \overline{B}\cap\overline{E}$. Therefore, by applying the law of total probability, the probability of an alarm can be calculated using the following formula.

$$\begin{aligned}P(A) &= P(A,B,E)+P(A,\overline{B},E)+P(A,B,\overline{E})+P(A,\overline{B},\overline{E})\\ &= P(A|B,E)P(B,E)+P(A|\overline{B},E)P(\overline{B},E)+P(A|B,\overline{E})P(B,\overline{E})+P(A|\overline{B},\overline{E})P(\overline{B},\overline{E})\\ &= P(A|B,E)P(B)P(E)+P(A|\overline{B},E)P(\overline{B})P(E)+P(A|B,\overline{E})P(B)P(\overline{E})+P(A|\overline{B},\overline{E})P(\overline{B})P(\overline{E})\\ &\qquad\text{(since } B, E \text{ are independent)}\\ &= 0.95\times0.001\times0.002+0.29(1-0.001)0.002+0.94\times0.001(1-0.002)+0.001(1-0.001)(1-0.002)\\ &\approx 0.002516442\end{aligned}$$

b) The probability of an alarm given there was a burglary is

$$\begin{aligned}P(A|B) &= P(A,E|B)+P(A,\overline{E}|B) && \text{(law of total probability)}\\ &= P(A|E,B)P(E|B)+P(A|\overline{E},B)P(\overline{E}|B)\\ &= P(A|E,B)P(E)+P(A|\overline{E},B)P(\overline{E}) && \text{(since } E, B \text{ are independent)}\\ &= 0.95\times0.002+0.94\times(1-0.002)\\ &= 0.94002\end{aligned}$$

c) The probability of a burglary given there was an alarm and there was also an earthquake is

$$
\begin{aligned}
P(B|A,E) &= \frac{P(A,E|B)P(B)}{P(A,E)} && \text{(Bayes rule)} \\
&= \frac{P(A|E,B)P(E)P(B)}{P(A,E)} && \text{(from result of part b)} \\
&= \frac{P(A|E,B)P(E)P(B)}{P(A,E,B)+P(A,E,\overline{B})} && \text{(law of total probability)} \\
&= \frac{P(A|E,B)P(E)P(B)}{P(A|E,B)P(E,B)+P(A|E,\overline{B})P(E,\overline{B})} \\
&= \frac{P(A|E,B)P(E)P(B)}{P(A|E,B)P(E)P(B)+P(A|E,\overline{B})P(E)P(\overline{B})} && \text{(since } E,B \text{ are independent)} \\
&= \frac{0.95 \times 0.002 \times 0.001}{0.95 \times 0.002 \times 0.001 + 0.29 \times 0.002 \times (1-0.001)} \\
&\approx 0.00327
\end{aligned}
$$

d) Given John called, the probability of an alarm is

$$
\begin{aligned}
P(A|J) &= \frac{P(J|A)P(A)}{P(J)} && \text{(Bayes rule)} \\
&= \frac{P(J|A)P(A)}{P(J|A)P(A)+P(J|\overline{A})P(\overline{A})} && \text{(law of total probability)} \\
&= \frac{0.9 \times 0.002516442}{0.9 \times 0.002516442 + 0.05 \times (1-0.002516442)} \\
&\approx 0.0434
\end{aligned}
$$

e) Given Mary called, the probability of an alarm is

$$
\begin{aligned}
P(A|M) &= \frac{P(M|A)P(A)}{P(M)} && \text{(Bayes rule)} \\
&= \frac{P(M|A)P(A)}{P(M|A)P(A)+P(M|\overline{A})P(\overline{A})} && \text{(law of total probability)} \\
&= \frac{0.7 \times 0.002516442}{0.7 \times 0.002516442 + 0.01 \times (1-0.002516442)} \\
&\approx 0.1501
\end{aligned}
$$

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Question 3. Predict the rating for items 5, 6, and 7 by user 1. Show all your steps. User list and their ratings for various items is as below:

User	**Item1**	**Item2**	**Item3**	**Item4**	**Item5**	**Item6**	**Item7**
User1	5	3	4	1	?	?	?
User2	5	3	4	1	5	2	5
User3	5	?	4	1	5	3	?
User4	1	3	2	5	1	4	2
User5	4	?	4	4	4	?	4

Solution. Let us define

- r_{xi} be the rating score that user x rates item i.
- r_x be the vector of user x's ratings. In other words, r_x is the set of rating scores that the user x rates for all items.
- $\overline{r_x}$ be the average ratings of user x.
- $sim(x, y)$ be the similarity measure between user x and user y.
- s_{xy} be the set of items rated by **both** user x and user y.

First, we calculate the average rating for every user:

$$\begin{aligned}
\overline{r_{u_1}} &= \frac{5+3+4+1}{4} = \frac{13}{4} \\
\overline{r_{u_2}} &= \frac{5+3+4+1+5+2+5}{7} = \frac{25}{7} \\
\overline{r_{u_3}} &= \frac{5+4+1+5+3}{5} = \frac{18}{5} \\
\overline{r_{u_4}} &= \frac{1+3+2+5+1+4+2}{7} = \frac{18}{7} \\
\overline{r_{u_5}} &= \frac{4+4+4+4+4}{5} = 4
\end{aligned}$$

Because we want give a recommendation for *user1* , we must consider the similarities between *user1* and remaining users, as shown below.

$$\begin{aligned}
sim(u_1, u_2) &= \frac{\sum_{s \in s_{u_1 u_2}} (r_{u_1,s} - \overline{r_{u_1}})(r_{u_2,s} - \overline{r_{u_2}})}{\sqrt{\sum_{s \in s_{u_1 u_2}} (r_{u_1,s} - \overline{r_{u_1}})^2} \sqrt{\sum_{s \in s_{u_1 u_2}} (r_{u_2,s} - \overline{r_{u_2}})^2}} \\
&= \frac{\left(5 - \frac{13}{4}\right)\left(5 - \frac{25}{7}\right) + \left(3 - \frac{13}{4}\right)\left(3 - \frac{25}{7}\right) + \left(4 - \frac{13}{4}\right)\left(4 - \frac{25}{7}\right) + \left(1 - \frac{13}{4}\right)\left(1 - \frac{25}{7}\right)}{\sqrt{\left(5 - \frac{13}{4}\right)^2 + \left(3 - \frac{13}{4}\right)^2 + \left(4 - \frac{13}{4}\right)^2 + \left(1 - \frac{13}{4}\right)^2} \sqrt{\left(5 - \frac{25}{7}\right)^2 + \left(3 - \frac{25}{7}\right)^2 + \left(4 - \frac{25}{7}\right)^2 + \left(1 - \frac{25}{7}\right)^2}} \\
&\approx 0.977 \\
sim(u_1, u_3) &= \frac{\sum_{s \in s_{u_1 u_3}} (r_{u_1,s} - \overline{r_{u_1}})(r_{u_3,s} - \overline{r_{u_3}})}{\sqrt{\sum_{s \in s_{u_1 u_3}} (r_{u_1,s} - \overline{r_{u_1}})^2} \sqrt{\sum_{s \in s_{u_1 u_3}} (r_{u_3,s} - \overline{r_{u_3}})^2}} \\
&= \frac{\left(5 - \frac{13}{4}\right)\left(5 - \frac{18}{5}\right) + \left(4 - \frac{13}{4}\right)\left(4 - \frac{18}{5}\right) + \left(1 - \frac{13}{4}\right)\left(1 - \frac{18}{5}\right)}{\sqrt{\left(5 - \frac{13}{4}\right)^2 + \left(4 - \frac{13}{4}\right)^2 + \left(1 - \frac{13}{4}\right)^2} \sqrt{\left(5 - \frac{18}{5}\right)^2 + \left(4 - \frac{18}{5}\right)^2 + \left(1 - \frac{18}{5}\right)^2}} \\
&\approx 0.979 \\
sim(u_1, u_4) &= \frac{\sum_{s \in s_{u_1 u_4}} (r_{u_1,s} - \overline{r_{u_1}})(r_{u_4,s} - \overline{r_{u_4}})}{\sqrt{\sum_{s \in s_{u_1 u_4}} (r_{u_1,s} - \overline{r_{u_1}})^2} \sqrt{\sum_{s \in s_{u_1 u_4}} (r_{u_4,s} - \overline{r_{u_4}})^2}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(5-\frac{13}{4}\right)\left(1-\frac{18}{7}\right)+\left(3-\frac{13}{4}\right)\left(3-\frac{18}{7}\right)+\left(4-\frac{13}{4}\right)\left(2-\frac{18}{7}\right)+\left(1-\frac{13}{4}\right)\left(5-\frac{18}{7}\right)}{\sqrt{\left(5-\frac{13}{4}\right)^2+\left(3-\frac{13}{4}\right)^2+\left(4-\frac{13}{4}\right)^2+\left(1-\frac{13}{4}\right)^2}\sqrt{\left(1-\frac{18}{7}\right)^2+\left(3-\frac{18}{7}\right)^2+\left(2-\frac{18}{7}\right)^2+\left(5-\frac{18}{7}\right)^2}} \\
&\approx -0.993 \\
sim(u_1,u_5) &= \frac{\sum_{s\in s_{u_1u_5}}(r_{u_1,s}-\overline{r_{u_1}})(r_{u_5,s}-\overline{r_{u_5}})}{\sqrt{\sum_{s\in s_{u_1u_5}}(r_{u_1,s}-\overline{r_{u_1}})^2}\sqrt{\sum_{s\in s_{u_1u_5}}(r_{u_5,s}-\overline{r_{u_5}})^2}} \\
&= \frac{\left(5-\frac{13}{4}\right)(4-4)+\left(4-\frac{13}{4}\right)(4-4)+\left(1-\frac{13}{4}\right)(4-4)}{\sqrt{\left(5-\frac{13}{4}\right)^2+\left(4-\frac{13}{4}\right)^2+\left(1-\frac{13}{4}\right)^2}\sqrt{(4-4)^2+(4-4)^2+(4-4)^2}} \\
&= undefine
\end{aligned}
$$

Now, let's say that we would like to select 2 users most similar to user 1. Based on the previous similarity measures, those 2 users should be u_2, u_3. Now, we rely on u_2, u_3 to give a prediction for u_1:

$$
\begin{aligned}
r_{u_1,i_5} &= \frac{\sum_{y\in\{2,3\}} sim(u_1,u_y)\times r_{u_y i_5}}{\sum_{y\in\{2,3\}} sim(u_1,u_y)} \\
&= \frac{sim(u_1,u_2)r_{u_2,i_5}+sim(u_1,u_3)r_{u_3,i_5}}{sim(u_1,u_2)+sim(u_1,u_3)} \\
&= \frac{0.977\times 5+0.979\times 5}{0.977+0.979} \\
&= 5 \\
r_{u_1,i_6} &= \frac{\sum_{y\in\{2,3\}} sim(u_1,u_y)\times r_{u_y i_6}}{\sum_{y\in\{2,3\}} sim(u_1,u_y)} \\
&= \frac{sim(u_1,u_2)r_{u_2,i_6}+sim(u_1,u_3)r_{u_3,i_6}}{sim(u_1,u_2)+sim(u_1,u_3)} \\
&= \frac{0.977\times 2+0.979\times 3}{0.977+0.979} \\
&\approx 2.5 \\
r_{u_1,i_7} &= \frac{\sum_{y\in\{2,3\}} sim(u_1,u_y)\times r_{u_y i_7}}{\sum_{y\in\{2,3\}} sim(u_1,u_y)} \\
&= \frac{sim(u_1,u_2)r_{u_2,i_7}+sim(u_1,u_3)r_{u_3,i_7}}{sim(u_1,u_2)+sim(u_1,u_3)} \\
&= \frac{0.977\times 5+0.979\times 0}{0.977+0.979} \\
&\approx 2.5
\end{aligned}
$$

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Question 4. What is the play prediction for new data: (Sunny, Cool, High, Strong)? Use the Naive Bayes classifier. The training data is as below. Show all your calculations.

Day	Outlook	Temperature	Humidity	Wind	Play Tennis
Day1	Sunny	Hot	High	Weak	No
Day2	Sunny	Hot	High	Strong	No
Day3	Overcast	Hot	High	Weak	Yes
Day4	Rain	Mild	High	Weak	Yes
Day5	Rain	Cool	Normal	Weak	Yes
Day6	Rain	Cool	Normal	Strong	No
Day7	Overcast	Cool	Normal	Strong	Yes
Day8	Sunny	Mild	High	Weak	No
Day9	Sunny	Cool	Normal	Weak	Yes
Day10	Rain	Mild	Normal	Weak	Yes
Day11	Sunny	Mild	Normal	Strong	Yes
Day12	Overcast	Mild	High	Strong	Yes
Day13	Overcast	Hot	Normal	Weak	Yes
Day14	Rain	Mild	High	Strong	No

Solution. To predict whether he plays tennis or not if given that outlook is Sunny, temperature is Cool, humidity is High, wind is Strong, we must answer the two following questions:

- Given conditions that outlook is Sunny, temperature is Cool, humidity is High, wind is Strong, what is the probability that he would like to play tennis?
- Given conditions that outlook is Sunny, temperature is Cool, humidity is High, wind is Strong, what is the probability that he would *not* like to play tennis?

If given those conditions, the probability that he would like to play is higher than the probability that we would not like to play, then we would predict that "Yes, he would like to play With those conditions", and say "No" otherwise. Now, let's denote Y be the event that he plays, and $\overline{Y}$ be the event that he does not play. Then, mathematically, we can say that

$$\begin{aligned}
&\text{if } P(Y|Sunny, Cool, High, Strong) \geq P(\overline{Y}|Sunny, Cool, High, Strong), \text{ predict target is True} \\
&\text{if } P(Y|Sunny, Cool, High, Strong) < P(\overline{Y}|Sunny, Cool, High, Strong), \text{ predict target is False}
\end{aligned}$$

Now, let's compute those probabilities. First, the probability that he plays given that outlook is Sunny, temperature is Cool, humidity is High, wind is Strong is

$$\begin{aligned}
P(Y|Sunny, Cool, High, Strong) &= \frac{P(Sunny, Cool, High, Strong)|Y)P(Y)}{P(Sunny, Cool, High, Strong)} && \text{(Bayes rule)} \\
&= \frac{P(Sunny|Y)P(Cool|Y)P(High|Y)P(Strong|Y)P(Y)}{P(Sunny)P(Cool)P(High)P(Strong)} \\
& && \text{(since outlook, temperature, humidity, wind are independent)} \\
&= \frac{\frac{2}{9} \times \frac{3}{9} \times \frac{3}{9} \times \frac{3}{9} \times \frac{9}{14}}{\frac{5}{14} \times \frac{4}{14} \times \frac{7}{14} \times \frac{6}{14}} && \text{(please look at the data given in table)} \\
&\approx 0.242 && (1)
\end{aligned}$$

Next, the probability that he plays given that outlook is Sunny, temperature is Cool, humidity is High, wind is Strong is

$$
\begin{aligned}
P(\overline{Y}|Sunny, Cool, High, Strong) &= \frac{P(Sunny, Cool, High, Strong)|\overline{Y})P(\overline{Y})}{P(Sunny, Cool, High, Strong)} && \text{(Bayes rule)} \\
&= \frac{P(Sunny|\overline{Y})P(Cool|\overline{Y})P(High|\overline{Y})P(Strong|\overline{Y})P(\overline{Y})}{P(Sunny)P(Cool)P(High)P(Strong)} \\
& \qquad \text{(since outlook, temperature, humidity, wind are independent)} \\
&= \frac{\frac{3}{5} \times \frac{1}{5} \times \frac{4}{5} \times \frac{3}{5} \times \frac{5}{14}}{\frac{5}{14} \times \frac{4}{14} \times \frac{7}{14} \times \frac{6}{14}} && \text{(please look at the data given in table)} \\
&= 0.9408 && (2)
\end{aligned}
$$

Comparing (1) and (2), we obtain

$$P(Y|Sunny, Cool, High, Strong) < P(\overline{Y}|Sunny, Cool, High, Strong)$$

Therefore, we predict that he will not play tennis if outlook is Sunny, temperature is Cool, humidity is High, wind is Strong.

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Question 5. The table below is a distance matrix for six items. Show the Dendrogram using single-link hierarchical clustering.

	A	**B**	**C**	**D**	**E**	**F**
A	0					
B	0.12	0				
C	0.51	0.25	0			
D	0.84	0.16	0.14	0		
E	0.28	0.77	0.70	0.45	0	
F	0.34	0.61	0.93	0.20	0.67	0

Solution. Let's consider it step by step.

	A	**B**	**C**	**D**	**E**	**F**
A	0					
B	**0.12**	0				
C	0.51	0.25	0			
D	0.84	0.16	0.14	0		
E	0.28	0.77	0.70	0.45	0	
F	0.34	0.61	0.93	0.20	0.67	0

Table 1: Distance matrix at the beginning

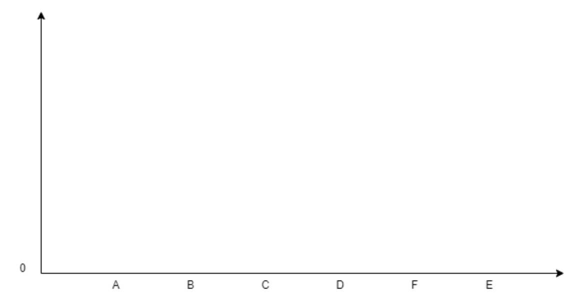

Figure 1: Dendrogram at the beginning

	A,B	**C**	**D**	**E**	**F**
A,B	0				
C	0.25	0			
D	0.16	**0.14**	0		
E	0.28	0.70	0.45	0	
F	0.34	0.93	0.20	0.67	0

Table 2: Distance matrix after first iteration

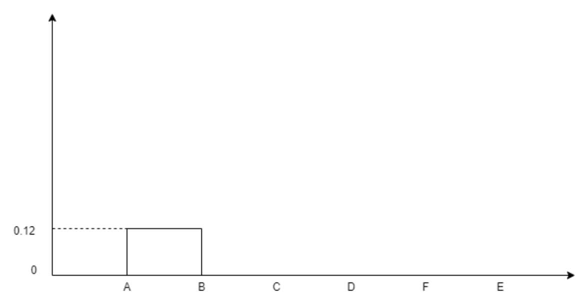

Figure 2: Dendrogram after first iteration

	A,B	**C,D**	**E**	**F**
A,B	0			
C,D	**0.16**	0		
E	0.28	0.45	0	
F	0.34	0.20	0.67	0

Table 3: Distance matrix after second iteration

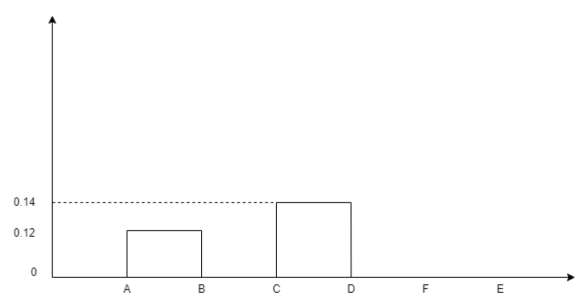

Figure 3: Dendrogram after second iteration

	A,B,C,D	E	F
A,B,C,D	0		
E	0.28	0	
F	**0.20**	0.67	0

Table 4: Distance matrix after third iteration

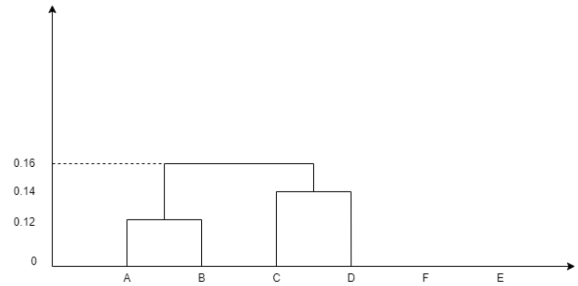

Figure 4: Dendrogram after third iteration

	A,B,C,D,F	E
A,B,C,D,F	0	
E	**0.28**	0

Table 5: Distance matrix after fourth iteration

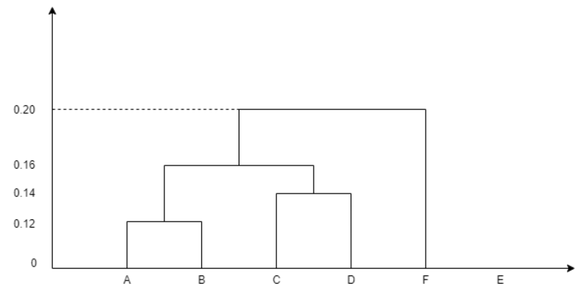

Figure 5: Dendrogram after fourth iteration

	A,B,C,D,F,E
A,B,C,D,F,E	0

Table 6: Distance matrix after fifth iteration

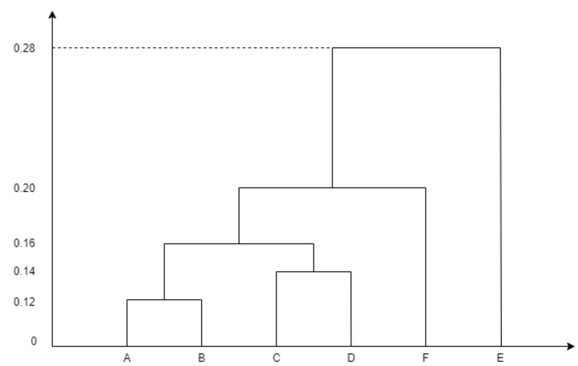

Figure 6: Dendrogram after fifth iteration

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