Hanoi University of Science and Technology
Faculty of Mathematics and Informatics
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EXERCISES ON MTH112

PART I

Sets – Maps – Complex numbers

Exercise 1. Let $f(x), g(x)$ be a function defined on $\mathbb{R}$. We denote by $A = \{x \in \mathbb{R} | f(x) = 0\}$, $B = \{x \in \mathbb{R} | g(x) = 0\}$. Show the solutions the following equations through A, B

a) $f(x)g(x) = 0$ b) $[f(x)]^2 + [g(x)]^2 = 0$

Exercise 2. Let A, B be two sets such that $A = [3; 6), B = (1; 5), C = [2; 4]$. Determine the following set $(A \cap B) \backslash C$.

Exercise 3. Let A, B, C, D be four sets. Prove that

a) $A \cap (B \backslash C) = (A \cap B) \backslash (A \cap C)$.

b) $A \cup (B \backslash A) = A \cup B$.

c) $(A \backslash B) \cap (C \backslash D) = (A \cap C) \backslash (B \cup D)$

Exercise 4. Let f, g be two maps such that

f: $\mathbb{R}\backslash\{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$ $\quad$ g: $\mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{2x}{1+x^2}$

a) Which of the maps are injective, surjective. Determine $g(\mathbb{R})$.

b) Determine the following map $h = g \circ f$.

Exercise 5. Let $f: X \to Y$ be a map. Prove that

a) $f(A \cup B) = f(A) \cup f(B);\ A, B \subset X$.

a) $f(A \cap B) \subset f(A) \cap f(B);\ A, B \subset X$. Give the examples that prove the opposite is false?

b) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B);\ A, B \subset Y$

Exercise 6. Let $f: \mathbb{R} \to \mathbb{R}$ be a map defined by $f(x) = x^2 + 4x - 5, \quad \forall x \in \mathbb{R}$, and the set $A = \{x \in \mathbb{R} \mid -3 \leq x \leq 3\}$. Determine the following sets $f(A), f^{-1}(A)$.

Exercise 7. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a map defined by $f(x, y) = (x + y, x - y)$ and the set $A = \{(x, y) \in \mathrm{R}^2 | x^2 + y^2 = 9\}$. Determine the following sets $f(A)$ and $f^{-1}(A)$.

Exercise 8. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a map defined by $f(x; y) = (x^2 - y;\ x + y)$. Is the map f injective, surjective? Why?

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Exercise 9. Determine that the following sets under the normal addition and multiplication to form the rings or the fields?

a) The odd integers

b) The even integers

c) The rational numbers

d) $X = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$

e) $Y = \{a + b\sqrt{3} | a, b \in \mathbb{Q}\}$

Exercise 10. Show the canonical form of the following complex numbers

a) $(1 + i\sqrt{3})^9$ b) $\frac{(1+i)^{21}}{(1-i)^{13}}$ c) $(2 + i\sqrt{12})^5(\sqrt{3} - i)^{11}$

Exercise 11. Find complex solutions of the following equations

a) $z^2 + z + 1 = 0$ b) $z^2 + 2iz - 5 = 0$ c) $z^4 - 3iz^2 + 4 = 0$

d) $z^6 - 7z^3 - 8 = 0$ e) $\overline{z^7} = \frac{1024}{z^3}$ f)$z^8(\sqrt{3} + i) = 1 - i$ g) $\boldsymbol{iz^2 - (1 + 8i)z + 7 + 17i = 0}$

Exercise 12. Let $\epsilon_1, \dots, \epsilon_{2014}$ be the different 2014-roots of the complex number 1. Compute $A = \sum_{i=1}^{2014} \epsilon_i^2$

Exercise 13. Given the equation $\frac{(x+1)^9 - 1}{x} = 0$

a) Solve the above equation

b) Find the modulus of the solutions

c) Compute the product of its solutions and $\prod_{k=1}^{8} sin\frac{k\pi}{9}$

Exercise 14. Let $f: \mathbb{C} \to \mathbb{C}$ be a map defined by $f(z) = iz^2 + (4 - i)z - 9i$, where i is the imaginary unit. Determine $f^{-1}(\{7\})$

Exercise 15. Let z_1, z_2 be two complex solutions of the equation $z^2 - z + ai = 0$, where a is the real and i is the imaginary unit. Find a that satisfy $|z_1^2 - z_2^2| = 1$

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PART II

Matrix – Determinant – System of linear equations

Exercise 1. Given the following matrices $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -1 \\ 0 & 3 & -2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 4 & 1 \\ 2 & 0 & 2 \end{bmatrix}$. Compute the following matrices : $A + BC, A^T B - C, A(BC), (A + 3B)(B - C)$.

Exercise 2. Given the following matrices $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ and I is the identity matrix of size 2.

a) Compute $F = A^2 - 3A$

b) Find matrix X that satisfy $(A^2 + 5E)X = B^T(3A - A^2)$

Exercise 3. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 1 \\ 3 & -5 & 3 \end{bmatrix}$ be a matrix and $f(x) = 3x^2 - 2x + 5$ be a function. Compute $f(A)$

Exercise 4. Compute A^n, where

a) $A = \begin{bmatrix} \text{cosa} & \text{-sina} \\ \text{sina} & \text{cosa} \end{bmatrix}$.

b) $A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$.

Exercise 5. Find all matrices of size 2 that satisfy

a) $X^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ b) $X^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise 6.

a) Prove that the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that satisfy the following equation: $x^2 - (a + d)x + ad - bc = 0$

b) Prove that if A is a square matrix of size 2 then $A^k = 0, (k > 2) \Leftrightarrow A^2 = 0$.

Exercise 7. Prove the following equalities by using the properties of determinant

a) $\begin{vmatrix} a_1 + b_1 x & a_1 - b_1 x & c_1 \\ a_2 + b_2 x & a_2 - b_2 x & c_2 \\ a_3 + b_3 x & a_3 - b_3 x & c_3 \end{vmatrix} = -2x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

b) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

Exercise 8. Compute the following determinants

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a) $A = \begin{vmatrix} 1 & 3 & 5 & -1 \\ 2 & -1 & -1 & 4 \\ 5 & 1 & -1 & 7 \\ 7 & 7 & 9 & 1 \end{vmatrix}$ b) $B = \begin{vmatrix} a+b & ab & a^2+b^2 \\ b+c & bc & b^2+c^2 \\ c+a & ca & a^2+c^2 \end{vmatrix}$ c) $D = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 9-x^2 \end{vmatrix}$

Exercise 9.

a) Let A be an antisymmetric matrix of an odd size. Prove that $\det(A) = 0$

b) Let A be a square matrix of size 2019. Prove that $\det(A - A^T) = 0$

Exercise 10. Find the rank of the following matrices

a) $A = \begin{bmatrix} 1 & 3 & 5 & -1 \\ 2 & -1 & -1 & 4 \\ 5 & 1 & -1 & 7 \\ 7 & 7 & 9 & 1 \end{bmatrix}$ b) $B = \begin{bmatrix} 4 & 3 & -5 & 2 & 3 \\ 8 & 6 & -7 & 4 & 2 \\ 4 & 3 & -8 & 2 & 7 \\ 4 & 3 & 1 & 2 & -5 \\ 8 & 6 & -1 & 4 & -6 \end{bmatrix}$

Exercise 11. Find m that the rank of matrix $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ -1 & 2 & 2 & 1 \\ 1 & 0 & 4 & m \end{bmatrix}$ is 2

Exercise 12. Find the inverse matrix of the following matrices

a) $A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$ b) $B = \begin{bmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & 1 \end{bmatrix}$ c) $C = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -a & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Exercise 13. Find a that matrix $A = \begin{bmatrix} a+1 & -1 & a \\ 3 & a+1 & 3 \\ a-1 & 0 & a-1 \end{bmatrix}$ is an inverse matrix

Exercise 14. Prove that if a sqare matrix A of size n satisfies $a_k A^k + a_{k-1}A^{k-1} + \cdots + a_1 A + a_0 E = 0, (a_0 \neq 0)$ then A is an inverse matrix.

Exercise 15. Given $A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 3 & 4 \\ 3 & 1 & -1 \end{bmatrix}; B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \\ 0 & 3 \end{bmatrix}; C = \begin{bmatrix} 2 & 12 & 10 \\ 6 & 16 & 7 \end{bmatrix}$. Find matrix X that satisfies $AX + B = C^T$

Exercise 16. Solve the following systems of linear equations

a) $\begin{cases} 3x_1 - 5x_2 + 2x_3 + 4x_4 = 2 \\ 7x_1 - 4x_2 + x_3 + 3x_4 = 5 \\ 5x_1 + 7x_2 - 4x_3 - 6x_4 = 3 \end{cases}$

b) $\begin{cases} 3x_1 - x_2 + 3x_3 = 1 \\ -4x_1 + 2x_2 + x_3 = 3 \\ -2x_1 + x_2 + 4x_3 = 4 \\ 10x_1 - 5x_2 - 6x_3 = -10 \end{cases}$

c) $\begin{cases} 2x_1 + 3x_2 + 4x_3 = 1 \\ 3x_1 - x_2 + x_3 = 2 \\ 5x_1 + 2x_2 + 5x_3 = 3 \\ x_1 - 4x_2 - 3x_3 = 1 \end{cases}$

Exercise 17. Solve the following systems of linear equations by using Gauss' method

a) $\begin{cases} x + 2y - z + 3t = 12 \\ 2x + 5y - z + 11t = 49 \\ 3x + 6y - 4z + 13t = 49 \\ x + 2y - 2z + 9t = 33 \end{cases}$

b) $\begin{cases} x+2y+3z+4t=-4 \\ 3x+7y+10z+11t=-11 \\ x+2y+4z+2t=-3 \\ x+2y+2z+7=-6 \end{cases}$

Exercise 18. Find a that systems of linear equations $\begin{cases} (a+5)x + 3y + (2a+1)z = 0 \\ ax + (a-1)y + 4z = 0 \\ (a+5)x + (a+2)y + 5z = 0 \end{cases}$ has nontrivial solutions.

Exercise 19. Find m that the system of linear equations $\begin{cases} mx_1 + 2x_2 - x_3 = 3 \\ x_1 + mx_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 + x_3 = -m \end{cases}$ has a unique solution

Exercise 20. Given the system of linear equations $\begin{cases} x_1 + 2x_2 - x_3 + mx_4 = 4 \\ -x_1 - x_2 + 3x_3 + 2x_4 = k \\ 2x_1 - x_2 - 3x_3 + (m-1)x_4 = 3 \\ x_1 + x_2 + x_3 + 2mx_4 = 5 \end{cases}$

a) Solve the system of linear equations when $m = 2, k = 5$

b) Find m, k that the system has a unique solution

c) Find m, k that the system has many infinitely many solutions

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