TROY UNIVERSITY PROGRAM AT HUST

Chapter 1- Equations and Inequalities

MTH112, PRE-CALCULUS ALGEBRA

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Outline

- Linear Equations
- Quadratic Equations
- Complex numbers, quadratic equations in the complex number system
- Radical equations; equations Quadratic in form; Factorable Equations
- Solving Inequalities
- Equations and Inequalities Involving absolute value
- Problem solving: Interest, Mixture, Uniform Motion, constant rate, Job applications

Linear Equations

- Solve a linear equation
- Solve equations that lead to Linear Equation
- Solve problems that can be modeled by linear equation

Solve a Linear Equation

- A linear equation in one variable is equivalent to an equation of the form $ax + b = 0$, where a, b are real numbers and $a \neq 0$.

$$ax + b = 0 \qquad a \neq 0$$

$$ax = -b \qquad \text{Subtract } b \text{ from both sides.}$$

$$x = \frac{-b}{a} \qquad \text{Divide both sides by } a,\ a \neq 0.$$

- Example: Solve the equation: $\frac{1}{2}(x+5) - 4 = \frac{1}{3}(2x-1)$

Solve Equations That Lead to Linear Equations

- Example: Solve the equation:

$$(2y + 1)(y - 1) = (y + 5)(2y - 5)$$

$$\frac{3}{x-2} = \frac{1}{x-1} + \frac{7}{(x-1)(x-2)}$$

- In the United States we measure temperature in both degrees Fahrenheit (°F) and degrees Celsius (°C), which are related by the formula $C = 5/9(F - 32)$. What are the Fahrenheit temperatures corresponding to Celsius temperatures of 0°, 10°, 20°, and 30°C?

Solve Problems That Can Be Modeled by Linear Equations

- **Steps for Solving Applied Problems**
  - **STEP 1:** Read the problem carefully, perhaps two or three times. Pay particular attention to the question being asked in order to identify what you are looking for. If you can, determine realistic possibilities for the answer.
  - **STEP 2:** Assign a letter (variable) to represent what you are looking for, and, if necessary, express any remaining unknown quantities in terms of this variable.
  - **STEP 3:** Make a list of all the known facts, and translate them into mathematical expressions. These may take the form of an equation (or, later, an inequality) involving the variable.The equation (or inequality) is called the **model.** If possible, draw an appropriately labeled diagram to assist you. Sometimes a table or chart helps.
  - **STEP 4:** Solve the equation for the variable, and then answer the question, usually using a complete sentence.
  - **STEP 5:** Check the answer with the facts in the problem. If it agrees, congratulations! If it does not agree, try again.

Solve Problems That Can Be Modeled by Linear Equations

- Example 1 – Investments: A total of $18,000 is invested, some in stocks and some in bonds. If the amount invested in bonds is half that invested in stocks, how much is invested in each category?
- Example 2 – Determining an Hourly Wage: Shannon grossed $435 one week by working 52 hours. Her employer pays time-and -a-half for all hours worked in excess of 40 hours. With this information, can you determine Shannon's regular hourly wage?

Quadratic Equations

- Solve a Quadratic Equation by Factoring
- Solve a Quadratic Equation by Completing the Square
- Solve a Quadratic Equation Using the Quadratic Formula
- Solve Problems That Can Be Modeled by Quadratic Equations

Quadratic equation

- A **quadratic equation** is an equation equivalent to one of the form $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$. This form is said to be standard form.

Solve a Quadratic Equation by Factoring

- When a quadratic equation is written in standard form $ax^2 + bx + c = 0$ it may be possible to factor the expression on the left side into the product of two first-degree polynomials. Then, by using the Zero-Product Property.

- Example: Solve the equations:

(a) $x^2 + 6x = 0$ (b) $2x^2 = x + 3$

- When the left side factors into two linear equations with the same solution, the quadratic equation is said to have a **repeated solution.** This solution is also called a **root of multiplicity 2**, or a **double root.**

The Square Root Method

- If $x^2 = p$ and $p \geq 0$, then $x = \sqrt{p}$ or $x = -\sqrt{p}$
- Example: $x^2 = 4$, then $x = \pm\sqrt{4} = \pm 2$
- Example: Solve each equation using the square root method

(a) $x^2 = 5$ (b) $(x - 2)^2 = 16$

Solving a Quadratic Equation by Completing the Square

- Example: Solve by completing the square:

(a) $x^2 + 5x + 4 = 0$

(b) $2x^2 - 8x - 5 = 0$

Solve a Quadratic Equation Using the Quadratic Formula

- Consider the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$.
  - If $b^2 - 4ac < 0$, this equation has no real solution
  - If $b^2 - 4ac \geq 0$, the real solution(s) of this equation is (are) given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quantity $b^2 - 4ac$ is called the **discriminant**

Solve a Quadratic Equation Using the Quadratic Formula

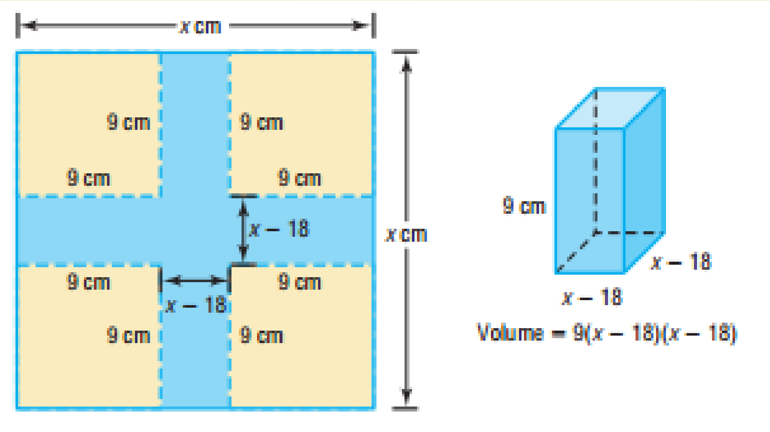
For a quadratic equation $ax^2 + bx + c = 0$

1. If $b^2 - 4ac > 0$, there are two unequal real solutions
2. If $b^2 - 4ac = 0$, there is a repeated solution, a root of multiplicity 2
3. If $b^2 - 4ac < 0$, there is no real solution.

Example: Solve the equation: $\frac{25}{2}x^2 - 30x + 18 = 0$

Solve Problems That Can Be Modeled by Quadratic Equations

- Example: From each corner of a square piece of sheet metal, remove a square of side 9 centimeters. Turn up the edges to form an open box. If the box is to hold 144 cubic centimeters what should be the dimensions of the piece of sheet metal?



Complex Numbers; Quadratic Equations in the Complex Number System*

- Add, Subtract, Multiply, and Divide Complex Numbers
- Solve Quadratic Equations in the Complex Number System

Complex Numbers

- The **imaginary unit,** which we denote by ***i***, is the number whose square is -1. That is, $i^2 = -1$.
- **Complex numbers** are numbers of the form $a + bi$, where *a* and *b* are real numbers. The real number *a* is called the **real part** of the number $a + bi$, the real number *b* is called the **imaginary part** of $a + bi$ and i is the imaginary unit, so $i^2 = -1$.
- Example: the complex number $-5 + 6i$ has the real part -5 and the imaginary part 6

Add, Subtract Complex Numbers

- Equality of Complex Numbers:

$a + bi = c + di$ if and only if $a = c$ and $b = d$.

- Sum of Complex Numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

- Difference of Complex Numbers

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Example: Adding and Subtracting Complex Numbers

(a) $(3 + 5i) + (-2 + 3i)$

(b) $(2 + 5i) - (7 + 4i)$

Multiplying Complex Numbers

- Product of Complex Numbers

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

- Example: $(2+4i)(1-5i)$

- If $z = a + bi$ is a complex number , then its **conjugate,** denoted by $\bar{z}$ is defined as

$$\bar{z} = \overline{a+bi} = a - bi$$

- The product of a complex number and its conjugate is a nonnegative real number. That is, if $z = a + bi$, then

$$z\bar{z} = a^2 + b^2$$

Multiplying Complex Number

- To express the reciprocal of a nonzero complex number z in standard form, multiply the numerator and denominator of $\frac{1}{z}$ by $\bar{z}$. That is, if $z = a + bi$ is a nonzero complex number, then

$$\frac{1}{a+bi} = \frac{1}{z} = \frac{1}{z} . \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z.\bar{z}} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$$

- Write $\frac{1}{3+4i}$ in standard form
- Write each of the following in standard form

(a) $\frac{1+4i}{5-12i}$ (b) $\frac{2-3i}{4-3i}$

Multiplying Complex Number

- If $z = 2 - 3i$ and $w = 5 + 2i$, write each of the following in standard form

(a) $\frac{z}{w}$ (b) $\overline{z + w}$ (c) $z + \bar{z}$

Multiplying Complex Numbers

- The conjugate of a real number is the real number itself
- The conjugate of the conjugate of a complex number is the complex number itself

$$(\bar{\bar{z}}) = z$$

- The conjugate of the sum of two complex numbers equals the sum of their conjugates

$$\overline{z + w} = \bar{z} + \bar{w}$$

- The conjugate of the product of two complex numbers equals the product of their conjugates

$$\overline{z.w} = \bar{z}.\bar{w}$$

Power of i

- The **powers of *i*** follow a pattern that is useful to know.

$$i^1 = i \qquad\qquad i^5 = i^4 \cdot i = 1 \cdot i = i$$
$$i^2 = -1 \qquad\qquad i^6 = i^4 \cdot i^2 = -1$$
$$i^3 = i^2 \cdot i = -1 \cdot i = -i \qquad\qquad i^7 = i^4 \cdot i^3 = -i$$
$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1 \qquad\qquad i^8 = i^4 \cdot i^4 = 1$$

- Write $(2 + i)^3$ in standard form

Solve Quadratic Equations in the Complex Number System

- If N is a positive real number, we define the **principal square root of $-N$** ,denoted by $\sqrt{-N}$, as

$$\sqrt{-N} = \sqrt{N}i$$

Where i is the imaginary unit and $i^2 = -1$

- Evaluating the Square root of a Negative Number

(a) $\sqrt{-1}$ (b) $\sqrt{-4}$ (c)$\sqrt{-8}$

- Solving Equations

(a) $x^2 = 4$ (b) $x^2 = -9$

Quadratic Equation

- In the complex number system, the solutions of the quadratic equation $ax^2 + bx + c = 0$, where *a*, *b*, and c are real numbers and $a \neq 0$, are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Solve the equation $x^2 - 4x + 8 = 0$ in the complex number system

Quadratic Equation

- In the complex number system, consider a quadratic equation $ax^2 + bx + c = 0$ with real coefficients
  1. If $b^2 - 4ac > 0$, the equation has two unequal real solutions
  2. If $b^2 - 4ac = 0$, the equation has a repeated real solution, a double root.
  3. If $b^2 - 4ac < 0$, the equation has two complex solutions that are not real. The solutions are conjugates of each other.

Radical Equations; Equations Quadratic in Form; Factorable Equations

- Solve Radical Equations
- Solve Equations Quadratic in Form
- Solve Equations by Factoring

Solve Radical Equations

- Find the real solutions of the equation:

(a) $\sqrt[3]{2x-4} - 2 = 0$

(b) $\sqrt{x-1} = x - 7$

(c) $\sqrt{2x+3} - \sqrt{x+2} = 2$

Solve Equations Quadratic Form

- The equation $x^4 + x^2 - 12 = 0$ is not quadratic in x, but it is quadratic in x^2. That is, if we let $u = x^2$ we get $u^2 + u - 12 = 0$, a quadratic equation
- In general, if an appropriate substitution u transforms an equation into one of the form $au^2 + bu + c = 0$ $(a \neq 0)$, then the original equation is called an equation of the quadratic type or an equation quadratic in form
- Example: Find the real solutions of the equation:

(a) $(x+2)^2 + 11(x+2) - 12 = 0$ (b) $(x^2-1)^2 + (x^2-1) - 12 = 0$

(c) $x + 2\sqrt{x} - 3 = 0$

Solve Equations by Factoring

- Example: Find the real solutions of the equation:

(a) $x^4 = 4x^2$ (b) $x^3 - x^2 - 4x + 4 = 0$

Solving Inequalities

- Use Interval Notation
- Use Properties of Inequalities
- Solve Inequalities
- Solve Combined Inequalities

Use Interval Notation

- An **open interval**, denoted by **(*a*, *b*)**, consists of all real numbers x for which $a < x < b$.
- A **closed interval**, denoted by **[*a*, *b*]**, consists of all real numbers x for which $a \leq x \leq b$
- The **half-open**, or **half-closed**, **intervals** are **(*a*, *b*]**, consisting of all real numbers x for which $a < x \leq b$ and **[*a*, *b*)**, consisting of all real numbers x for which $a \leq x < b$.
- In each of these definitions, a is called the left endpoint and b the right endpoint of the interval.
- ∞

Use Interval Notation

Interval	Inequality	Graph
The open interval (a, b)	$a < x < b$	a b
The closed interval $[a, b]$	$a \leq x \leq b$	a b
The half-open interval $[a, b)$	$a \leq x < b$	a b
The half-open interval $(a, b]$	$a < x \leq b$	a b
The interval $[a, \infty)$	$x \geq a$	a
The interval (a, ∞)	$x > a$	a
The interval $(-\infty, a]$	$x \leq a$	a
The interval $(-\infty, a)$	$x < a$	a
The interval $(-\infty, \infty)$	All real numbers	

Use Properties of Inequalities

- Nonnegative Property: For any real number

$$a^2 \geq 0 \qquad \textbf{(1)}$$

- Addition Property of Inequalities: For real numbers a, b and c

$$\text{If } a < b, \text{ then } a + c < b + c. \qquad \textbf{(2a)}$$
$$\text{If } a > b, \text{ then } a + c > b + c. \qquad \textbf{(2b)}$$

- Multiplication Properties for Inequalities: For real numbers a, b , c

$$\left.\begin{array}{l}\text{If } a < b \text{ and if } c > 0, \text{ then } ac < bc.\\ \text{If } a < b \text{ and if } c < 0, \text{ then } ac > bc.\end{array}\right. \qquad \textbf{(3a)}$$

$$\left.\begin{array}{l}\text{If } a > b \text{ and if } c > 0, \text{ then } ac > bc.\\ \text{If } a > b \text{ and if } c < 0, \text{ then } ac < bc.\end{array}\right. \qquad \textbf{(3b)}$$

Solve Inequalities

- Solve the inequality:

(a) $3 - 2x < 5$ (b) $4x + 7 \geq 2x - 3$

- Solve the inequality: $-5 < 3x - 2 < 1$
- Solve the inequality: $(4x - 1)^{-1} > 0$

Equations and Inequalities Involving Absolute Value

- Solve Equations Involving Absolute Value
- Solve Inequalities Involving Absolute Value

Solve Equations Involving Absolute Value

- If a is a positive real number and if u is any algebraic expression, then

$$|u| = a \quad \text{is equivalent to} \quad u = a \quad \text{or} \quad u = -a \qquad \textbf{(1)}$$

- If a is a positive number and if u is an algebraic expression, then

$$|u| < a \quad \text{is equivalent to} \quad -a < u < a \qquad \textbf{(2)}$$

$$|u| \le a \quad \text{is equivalent to} \quad -a \le u \le a \qquad \textbf{(3)}$$

- Solve the inequality: $|2x - 5| > 3$

Problem Solving: Interest, Mixture, Uniform Motion, Constant Rate Job Applications

- Translate Verbal Descriptions into Mathematical Expressions
- Solve Interest Problems
- Solve Mixture Problems
- Solve Uniform Motion Problems
- Solve Constant Rate Job Problems

Translating Verbal Descriptions into Mathematical Expressions

- For uniform motion, the constant speed of an object equals the distance traveled divided by the time required.
  - *Translation:* If is the speed, *d* the distance, and *t* the time, then $r = \frac{d}{t}$
- Let x denote a number
  - The number 5 times as large as x is $5x$
  - The number 3 less than x is $x - 3$
  - The number that exceeds x by 4 is $x + 4$
  - The number that, when added to x, gives 5 is $5 - x$

Solve Interest Problems

- **Interest** is money paid for the use of money.
- The total amount borrowed is called the principal
- The rate of interest, expressed as a percent, is the amount charged for the use of the principal for a given period of time, usually on a yearly basis.
- Simple Interest Formula

If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged is

$$I = Prt \qquad (1)$$

Solve Interest Problems

- Example on Finance- Computing Interest on a Loan: Suppose that Juanita borrows $500 for 6 months at the simple interest rate of 9% per annum. What is the interest that Juanita will be charged on the loan? How much does Juanita owe after 6 months?
- Example on Financial Planning: Candy has $70,000 to invest and wants an annual return of $2800, which requires an overall rate of return of 4%. She can invest in a safe, government-insured certificate of deposit, but it only pays 2%. To obtain 4%, she agrees to invest some of her money in noninsured corporate bonds paying 7%. How much should be placed in each investment to achieve her goal?

Solve Mixture Problems

- Example on Blending Coffees: The manager of a Starbucks store decides to experiment with a new blend of coffee. She will mix some B grade Colombian coffee that sells for $5 per pound with some A grade Arabica coffee that sells for $10 per pound to get 100 pounds of the new blend. The selling price of the new blend is to be $7 per pound, and there is to be no difference in revenue from selling the new blend versus selling the other types. How many pounds of the B grade Colombian and A grade Arabica coffees are required?

Solve Uniform Motion Problems

- Uniform Motion Formula:

If an object moves at an average speed (rate) r, the distance d covered in time t is given by the formula

$$d = rt \qquad \textbf{(2)}$$

That is, Distance = Rate · Time.

Solve Uniform Motion Problems

- Example on Physics Uniform Motion: Tanya, who is a long-distance runner, runs at an average speed of 8 miles per hour (mi/hr). Two hours after Tanya leaves your house, you leave in your Honda and follow the same route. If your average speed is how long will it be before you catch up to Tanya? How far will each of you be from your home?