TROY UNIVERSITY PROGRAM AT HUST

Chapter 3 – Functions and Their Graphs

MTH112, PRE-CALCULUS ALGEBRA

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Outline

- Functions
- The Graph of a Function
- Properties of Functions
- Library of Functions; Piecewise-defined Functions
- Graphing Techniques: Transformations
- Mathematical Models: Building Functions

Functions

- Determine Whether a Relation Represents a Function
- Find the Value of a Function
- Find the Domain of a Function Defined by an Equation
- Form the Sum, Difference, Product, and Quotient of Two Functions

Determine Whether a Relation Represents a Function

- When the value of one variable is related to the value of a second variable, we have a *relation*.
- A **relation** is a correspondence between two sets. If x and y are two elements in these sets and if a relation exists between x and y, then we say that x **corresponds** to y or that y **depends on** x, and we write $x \to y$.
- There are a number of ways to express relations between two sets. For example, the equation $y = 3x - 1$ shows a relation between x and y.

Determine Whether a Relation Represents a Function

- Not only can a relation be expressed through an equation or graph, but we can also express a relation through a technique called *mapping*. A **map** illustrates a relation by using a set of inputs and drawing arrows to the corresponding element in the set of outputs. **Ordered pairs** can be used to represent $x \rightarrow y$ as (x, y)
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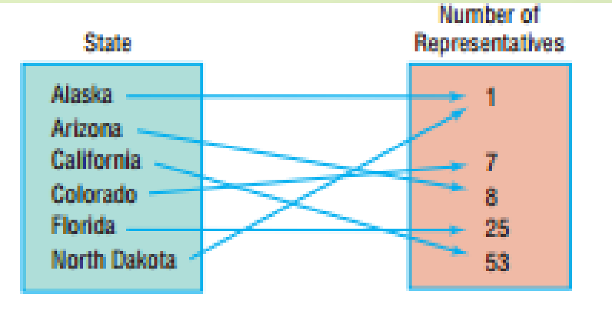
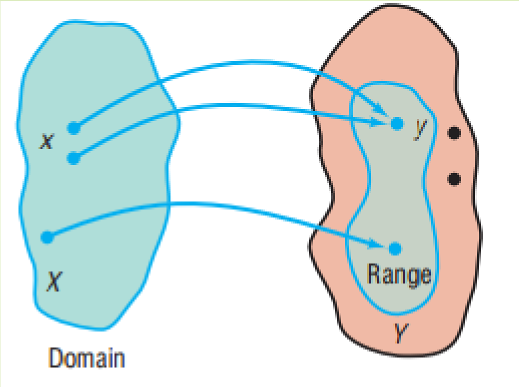


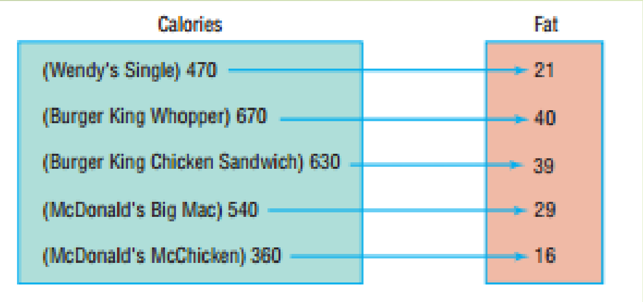
Determine Whether a Relation Represents a Function

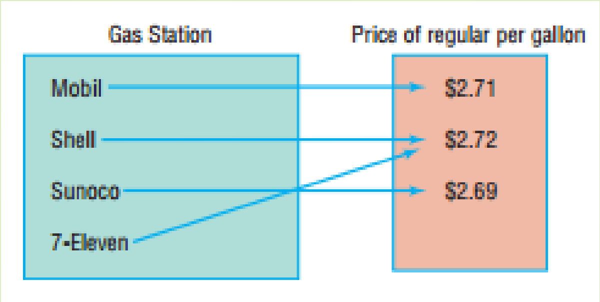
- Let *X* and *Y* be two nonempty sets.* A **function** from *X* into *Y* is a relation that associates with each element of *X* exactly one element of *Y*.

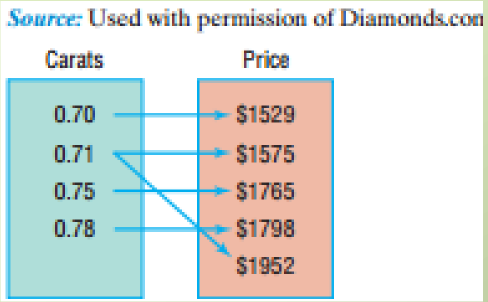


Determine Whether a Relation Represents a Function

- Determine which of the following relations represent a function. If the relation is a function, then state its domain and range.







Determine Whether a Relation Represents a Function

- Determine whether each relation represent a function. If it is a function, state the domain and range.

(a) $\{(1, 4), (2, 5), (3, 6), (4, 7)\}$

(b) $\{(1, 4), (2, 4), (3, 5), (6, 10)\}$

(c) $\{(-3, 9), (-2, 4), (0, 0), (1, 1), (-3, 8)\}$

- Determine if the equation $y = 2x - 5$ define y as a function of x
- Determine if the equation $x^2 + y^2 = 1$ define y as a function of x.

Find the Value of a Function

- Functions are often denoted by letters such as *F*, *g*, *G*, and others.
- We refer to $f(x)$ as the **value of at the number x;** $f(x)$ is the number that results when x is given and the function is applied; $f(x)$ is the output corresponding to x or the image of x.
- For a function $y = f(x)$ the variable x is called the **independent variable**, because it can be assigned any of the permissible numbers from the domain. The variable y is called the **dependent variable**, because its value depends on x

Find the Value of a Function

- For the function f defined by $f(x) = 2x^2 - 3x$, evaluate

(a) $f(3)$	(b) $f(x) + f(3)$	(c) $3f(x)$	(d) $f(-x)$
(e) $-f(x)$	(f) $f(3x)$	(g) $f(x + 3)$	(h) $\dfrac{f(x + h) - f(x)}{h}$

- In general, when a function is defined by an equation in x and y, we say that the function is given **implicitly.** If it is possible to solve the equation for y in terms of x, then we write $y = f(x)$ and say that the function is given **explicitly.**

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Implicit Form	**Explicit Form**
$3x + y = 5$	$y = f(x) = -3x + 5$
$x^2 - y = 6$	$y = f(x) = x^2 - 6$
$xy = 4$	$y = f(x) = \dfrac{4}{x}$

Find the Domain of a Function Defined by an Equation

- **Finding the Domain of a Function Defined by an Equation**
  - Start with the domain as the set of real numbers.
  - If the equation has a denominator, exclude any numbers that give a zero denominator.
  - If the equation has a radical of even index, exclude any numbers that cause the expression inside the radical to be negative.
- Find the domain of each of the following functions

(a) $f(x) = x^2 + 5x$ (b) $g(x) = \dfrac{3x}{x^2 - 4}$

(c) $h(t) = \sqrt{4 - 3t}$ (d) $F(x) = \dfrac{\sqrt{3x + 12}}{x - 5}$

Form the Sum, Difference, Product, and Quotient of Two Functions

If f and g are functions.

- The **sum** $\boldsymbol{f+g}$ is the function defined by $(f+g)(x)=f(x)+g(x)$
- The **difference** $\boldsymbol{f-g}$ is the function defined by $(f-g)(x)=f(x)-g(x)$
- The **product** $\boldsymbol{f.g}$ is the function defined by $(f.g)(x)=f(x).g(x)$
- The **quotient** $\boldsymbol{\frac{f}{g}}$ is the function defined by $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)} \quad g(x)\neq 0$

The domain of $\frac{f}{g}$ consists of the numbers x for which $g(x)\neq 0$ and that are in the domains of both f and g.

Domain of $\frac{f}{g}=\{x|g(x)\neq 0\}$ ∩ domain of f ∩ domain of g

Form the Sum, Difference, Product, and Quotient of Two Functions

- Operations on Functions

Let f and g be two functions defined as

$$f(x) = \frac{1}{x+2} \quad \text{and} \quad g(x) = \frac{x}{x-1}$$

Find the following, and determine the domain in each case.

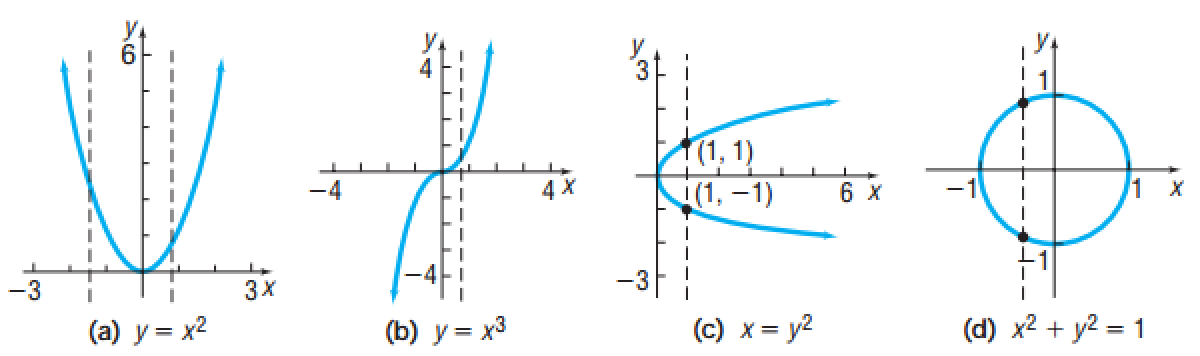
(a) $(f+g)(x)$ (b) $(f-g)(x)$ (c) $(f \cdot g)(x)$ (d) $\left(\frac{f}{g}\right)(x)$

The Graph of a Function

- Identify the Graph of a Function
- Obtain Information from or about the Graph of a Function

Identify the Graph of a Function

- Vertical-line Test: A set of points in the xy-plane is the graph of a function if and only if every vertical line intersects the graph in at most one point.
- Example: Which of the graphs are graphs of functions?



Obtain Information from or about the Graph of a Function

Consider the function: $f(x) = \dfrac{x+1}{x+2}$

(a) Find the domain of f.

(b) Is the point $\left(1, \frac{1}{2}\right)$ on the graph of f?

(c) If $x = 2$, what is $f(x)$? What point is on the graph of f?

(d) If $f(x) = 2$, what is x? What point is on the graph of f?

(e) What are the x-intercepts of the graph of f (if any)? What point(s) are on the graph of f?

Properties of Functions

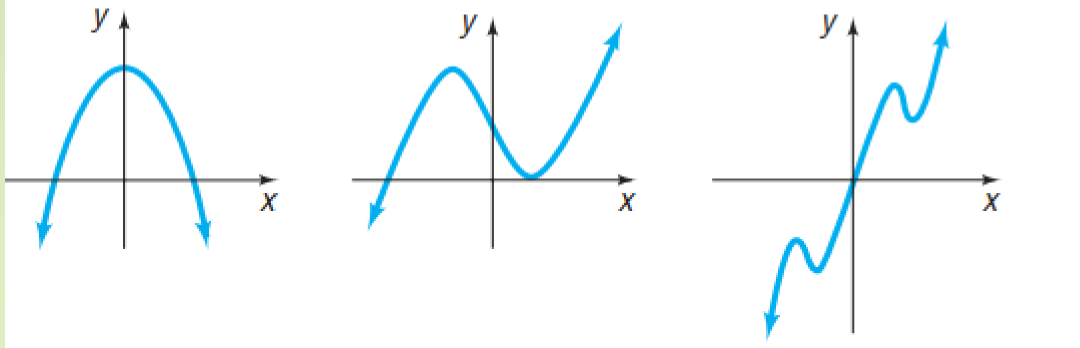
- Determine Even and Odd Functions from a Graph
- Identify Even and Odd Functions from the Equation
- Use a Graph to Determine Where a Function Is Increasing, Decreasing, or Constant
- Use a Graph to Locate Local Maxima and Local Minima
- Use a Graph to Locate the Absolute Maximum and the Absolute Minimum

Determine Even and Odd Functions from a Graph

- A function is **even** if, for every number x in its domain, the number $-x$ is also in the domain and $f(-x) = f(x)$
- A function is **odd** if, for every number x in its domain, the number $-x$ is also in the domain and $f(-x) = -f(x)$
- A function is even if and only if its graph is symmetric with respect to the *y*-axis.A function is odd if and only if its graph is symmetric with respect to the origin.

Determine Even and Odd Functions from a Graph

- Determine whether each graph given in Figure is the graph of an even function, an odd function, or a function that is neither even nor odd



(a) (b) (c)

Identify Even and Odd Functions from the Equation

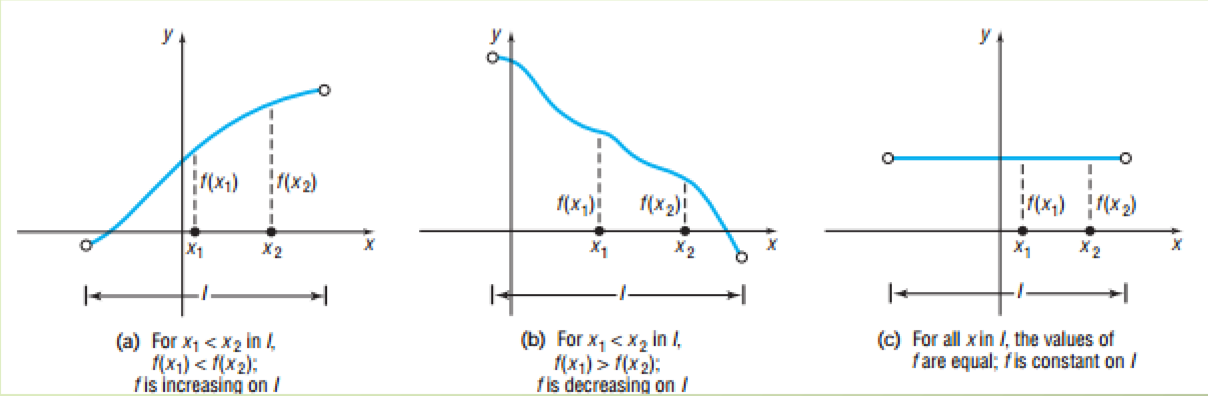
- Determine whether each of the following functions is even, odd, or neither. Then determine whether the graph is symmetric with respect to the *y*-axis, or with respect to the origin.

(a) $f(x) = x^2 - 5$ (b) $g(x) = x^3 - 1$

(c) $h(x) = 5x^3 - x$ (d) $F(x) = |x|$

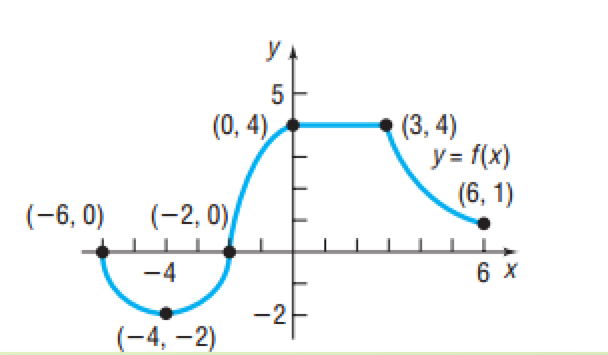
Use a Graph to Determine Where a Function Is Increasing, Decreasing, or Constant

- A function is **increasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$ we have $f(x_1) < f(x_2)$
- A function is **decreasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$ we have $f(x_1) > f(x_2)$
- A function f is **constant** on an open interval I if, for all choices of x in I, the values $f(x)$ are equal.



Use a Graph to Determine Where a Function Is Increasing, Decreasing, or Constant

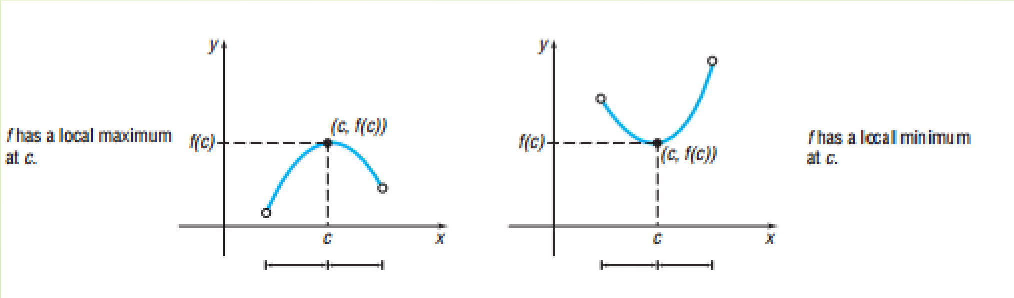
- Where is the function in Figure increasing? Where is it decreasing? Where is it constant



Use a Graph to Locate Local Maxima and Local Minima

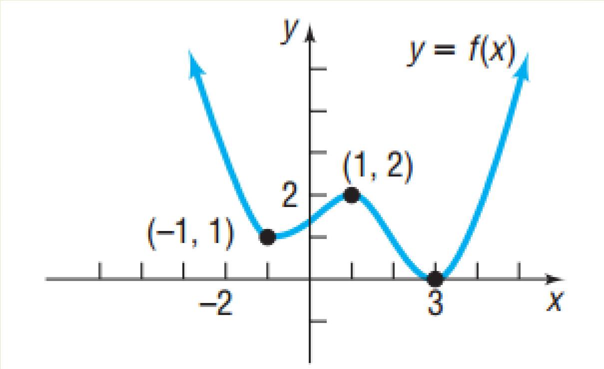
Suppose *f* is a function defined on an open interval containing c

- A function f has a **local maximum** at c if there is an open interval I containing c so that for all x in I, $f(x) \leq f(c)$. We call a **local maximum value of f.**
- A function f has a **local minimum** at c if there is an open interval I containing c so that for all x in I, $f(x) \geq f(c)$. We call a **local minimum value of f.**



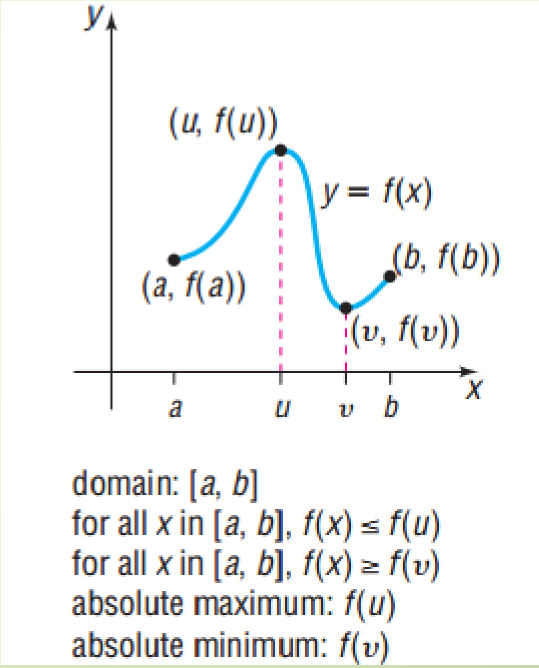
Use a Graph to Locate Local Maxima and Local Minima

- At what value(s) of x, if any, does f have a local maximum? List the local maximum values.
- At what value(s) of x, if any, does f have a local minimum? List the local minimum values.
- Find the intervals on which f is increasing. Find the intervals on which is decreasing.



Use a Graph to Locate the Absolute Maximum and the Absolute Minimum

- Look at the graph of the function f given in Figure. The domain of f is the closed interval $[a, b]$. Also, the largest value of f is $f(u)$ and the smallest value of f is $f(v)$.

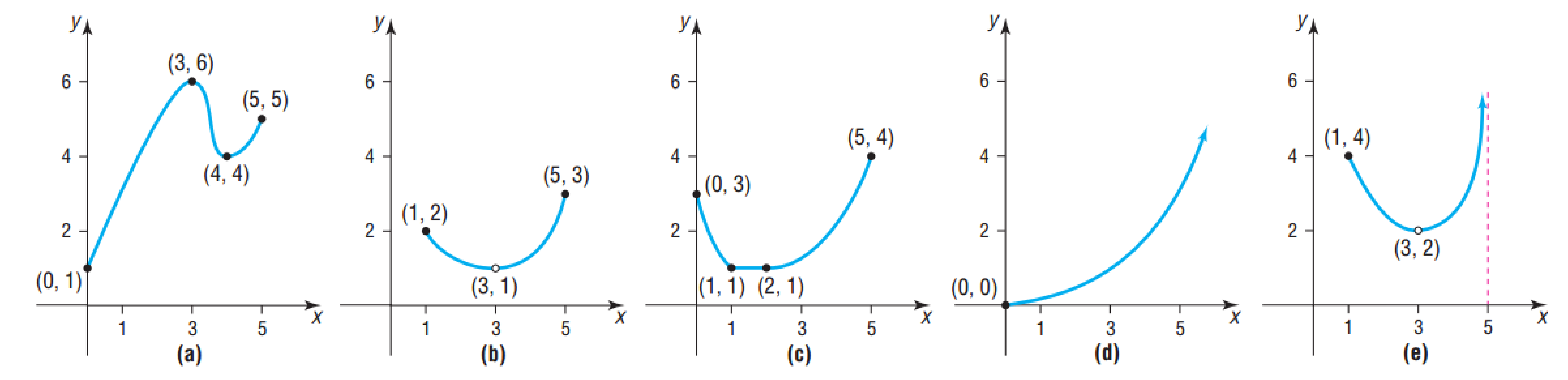


Use a Graph to Locate the Absolute Maximum and the Absolute Minimum

- Let *f* denote a function defined on some interval *I*. If there is a number *u* in *I* for which $f(x) \leq f(u)$ for all x in *I*, then $f(u)$ is the **absolute maximum of *f*** on *I* and we say **the absolute maximum of *f* occurs at *u***.
- If there is a number v in I for which $f(x) \geq f(v)$ for all x in *I*, then $f(v)$ is the **absolute minimum of *f*** on *I* and we say **the absolute minimum of *f* occurs at v**

Use a Graph to Locate the Absolute Maximum and the Absolute Minimum

- For each graph of a function $y = f(x)$ in Figure on the following page, find the absolute maximum and the absolute minimum, if they exist.



Find the Average Rate of Change of a Function

- If a and b, $a \neq b$, are in the domain of a function $y = f(x)$, the average rate of change of f from a to b is defined as

$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Example

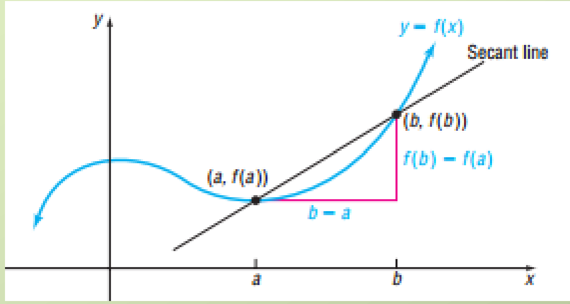
Find the average rate of change of $f(x) = 3x^2$:

(a) From 1 to 3 (b) From 1 to 5 (c) From 1 to 7

The Secant Line

- The average rate of change of a function has an important geometric interpretation. Look at the graph of $y = f(x)$ in Figure. We have labeled two points on the graph: $(a, f(a))$ and $(b, f(b))$. The line containing these two points is called the **secant line**; its slope is

$$m_{sec} = \frac{f(b) - f(a)}{b - a}$$



- The average rate of change of a function from a to b equals the slope of the secant line containing the two point $(a, f(a))$ and $(b, f(b))$ on its graph.

The Secant Line

Suppose that $g(x) = 3x^2 - 2x + 3$.

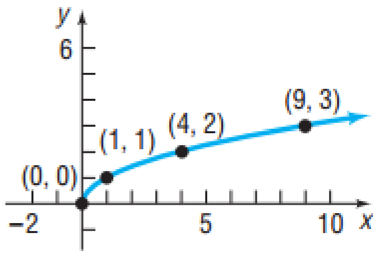
(a) Find the average rate of change of g from -2 to 1.

(b) Find an equation of the secant line containing $(-2, g(-2))$ and $(1, g(1))$.

(c) Using a graphing utility, draw the graph of g and the secant line obtained in part (b) on the same screen.

Library of Functions

Figure 28



Properties of $f(x) = \sqrt{x}$

1. The domain and the range are the set of nonnegative real numbers.
2. The x-intercept of the graph of $f(x) = \sqrt{x}$ is 0. The y-intercept of the graph of $f(x) = \sqrt{x}$ is also 0.
3. The function is neither even nor odd.
4. The function is increasing on the interval $(0, \infty)$.
5. The function has an absolute minimum of 0 at $x = 0$.

Library of Functions

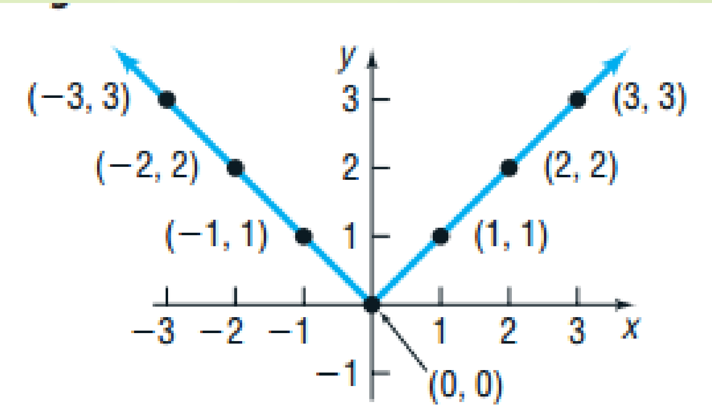
Properties of $f(x) = \sqrt[3]{x}$

1. The domain and the range are the set of all real numbers.
2. The x-intercept of the graph of $f(x) = \sqrt[3]{x}$ is 0. The y-intercept of the graph of $f(x) = \sqrt[3]{x}$ is also 0.
3. The graph is symmetric with respect to the origin. The function is odd.
4. The function is increasing on the interval $(-\infty, \infty)$.
5. The function does not have any local minima or any local maxima.

Library of Functions

Properties of $f(x) = |x|$

1. The domain is the set of all real numbers. The range of f is $\{y|y \geq 0\}$.
2. The x-intercept of the graph of $f(x) = |x|$ is 0. The y-intercept of the graph of $f(x) = |x|$ is also 0.
3. The graph is symmetric with respect to the y-axis. The function is even.
4. The function is decreasing on the interval $(-\infty, 0)$. It is increasing on the interval $(0, \infty)$.
5. The function has an absolute minimum of 0 at $x = 0$.



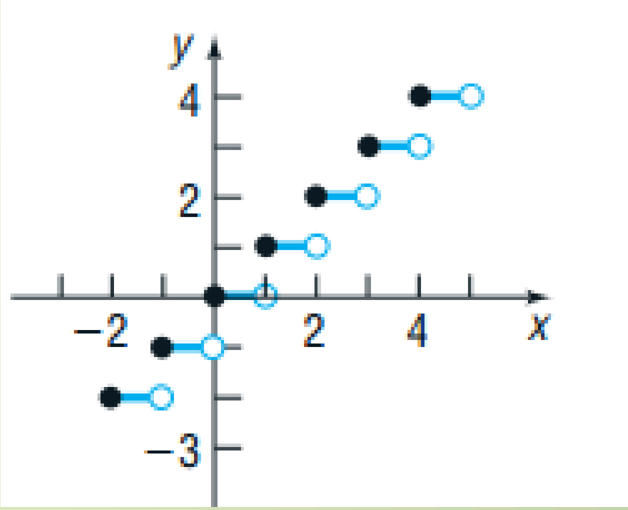
Library of Functions

- Identity Function: $f(x) = x$
- Square Function: $f(x) = x^2$
- Cube Function: $f(x) = x^3$
- Square Root Function: $f(x) = \sqrt{x}$
- Cube root Function $f(x) = \sqrt[3]{x}$

Library Functions

- Greatest Integer Function: $f(x) = int(x)^*$ =greatest integer less than or equal to x

x	$y = f(x) = \text{int}(x)$	(x, y)
-1	-1	$(-1, -1)$
$-\frac{1}{2}$	-1	$\left(-\frac{1}{2}, -1\right)$
$-\frac{1}{4}$	-1	$\left(-\frac{1}{4}, -1\right)$
0	0	$(0, 0)$
$\frac{1}{4}$	0	$\left(\frac{1}{4}, 0\right)$
$\frac{1}{2}$	0	$\left(\frac{1}{2}, 0\right)$
$\frac{3}{4}$	0	$\left(\frac{3}{4}, 0\right)$



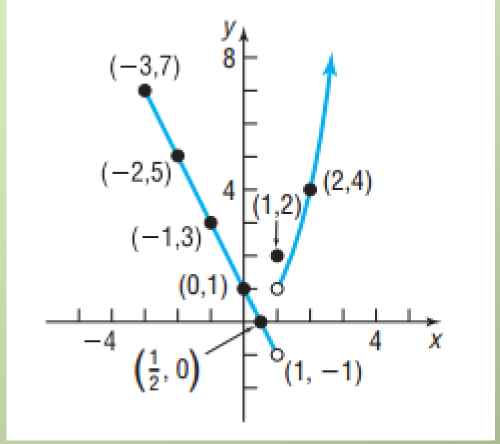
Graph Piecewise-defined Functions

- Sometimes a function is defined using different equations on different parts of its domain. When a function is defined by different equations on different parts of its domain, it is called a **piecewise-defined function**.
- Example

The function f is defined as

$$f(x) = \begin{cases} -2x + 1 & \text{if } -3 \le x < 1 \\ 2 & \text{if } x = 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

(a) Find $f(-2)$, $f(1)$, and $f(2)$.
(b) Determine the domain of f.
(c) Locate any intercepts.
(d) Graph f.
(e) Use the graph to find the range of f.
(f) Is f continuous on its domain?



Graph Piecewise-defined Functions

- Example on Cost of Electricity: In the summer of 2009, Duke Energy supplied electricity to residences of Ohio for a monthly customer charge of $4.50 plus 4.2345¢ per kilowatt-hour (kWhr) for the first 1000 kWhr supplied in the month and 5.3622¢ per kWhr for all usage over 1000 kWhr in the month
  - What is the charge for using 300 kWhr in a month?
  - What is the charge for using 1500 kWhr in a month?
  - If C is the monthly charge for x kWhr, develop a model relating the monthly charge and kilowatt-hours used. That is, express C as a function of x.

Graphing Techniques: Transformations

- Graph Functions Using Vertical and Horizontal Shifts
- Graph Functions Using Compressions and Stretches
- Graph Functions Using Reflections about the x-Axis and the y-Axis

Graph Functions Using Vertical Shifts

- If a positive real number k is added to the output of a function $y = f(x)$, the graph of the new function $y = f(x) + k$ is the graph of **shifted vertically up** k units.
- If a positive real number k is subtracted to the output of a function $y = f(x)$, the graph of the new function $y = f(x) - k$ is the graph of **shifted vertically down** k units.
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Graph Functions Using Vertical Shifts

- Example on Vertical Shift Up: Use the graph of $f(x) = x^2$ to obtain the graph of $g(x) = x^2 + 3$.

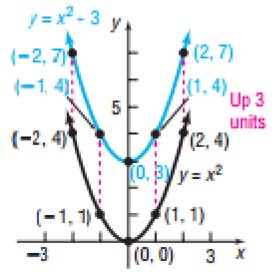


Table 7

x	$y = f(x) = x^2$	$y = g(x) = x^2 + 3$
−2	4	7
−1	1	4
0	0	3
1	1	4
2	4	7

Graph Functions Using Vertical Shifts

- Example on Vertical Shift Down: Use the graph of $f(x) = x^2$ to obtain the graph of $g(x) = x^2 - 4$

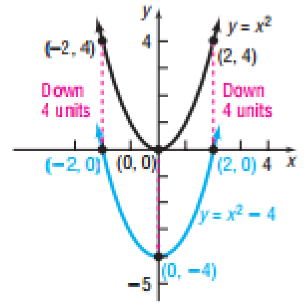


Table 8

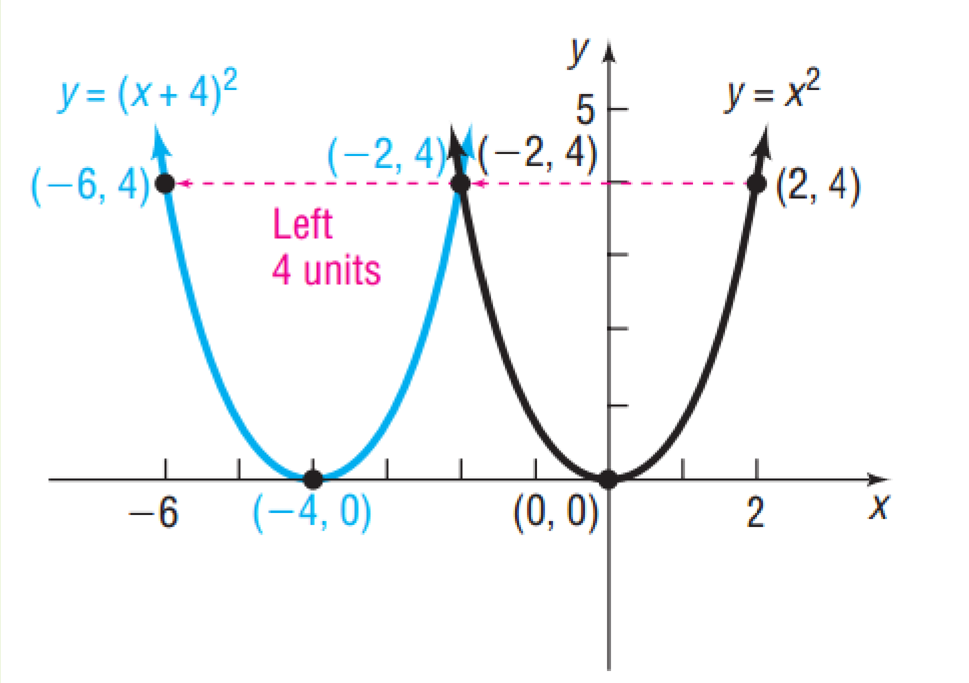
x	$y = f(x) = x^2$	$y = g(x) = x^2 - 4$
−2	4	0
−1	1	−3
0	0	−4
1	1	−3
2	4	0

Graph Functions Using Horizontal Shifts

- If the argument x of a function f is replaced by $x - h, h > 0$, the graph of the new function $y = f(x - h)$ is the graph of f **shifted horizontally right** units.
- If the argument x of a function f is replaced by $x + h, h > 0$, the graph of the new function $y = f(x + h)$ is the graph of f **shifted horizontally left** units.

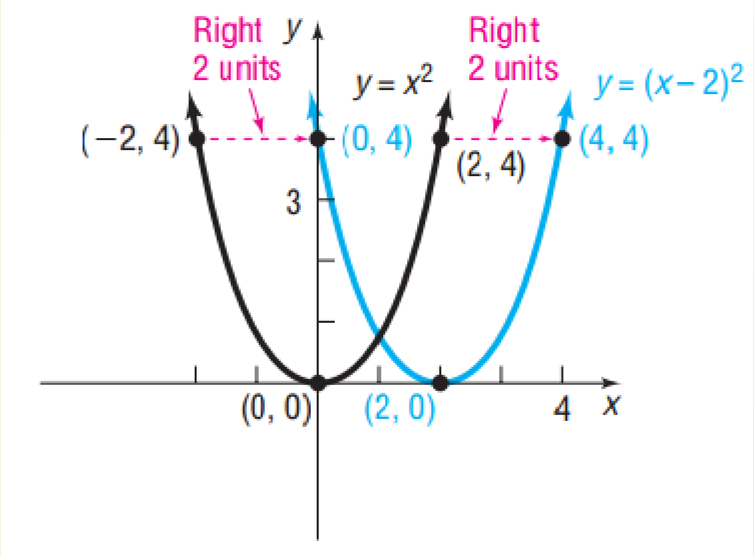
Horizontal Shift to the Left

- Example: Use the graph of $f(x) = x^2$ to obtain the graph of $g(x) = (x + 4)^2$



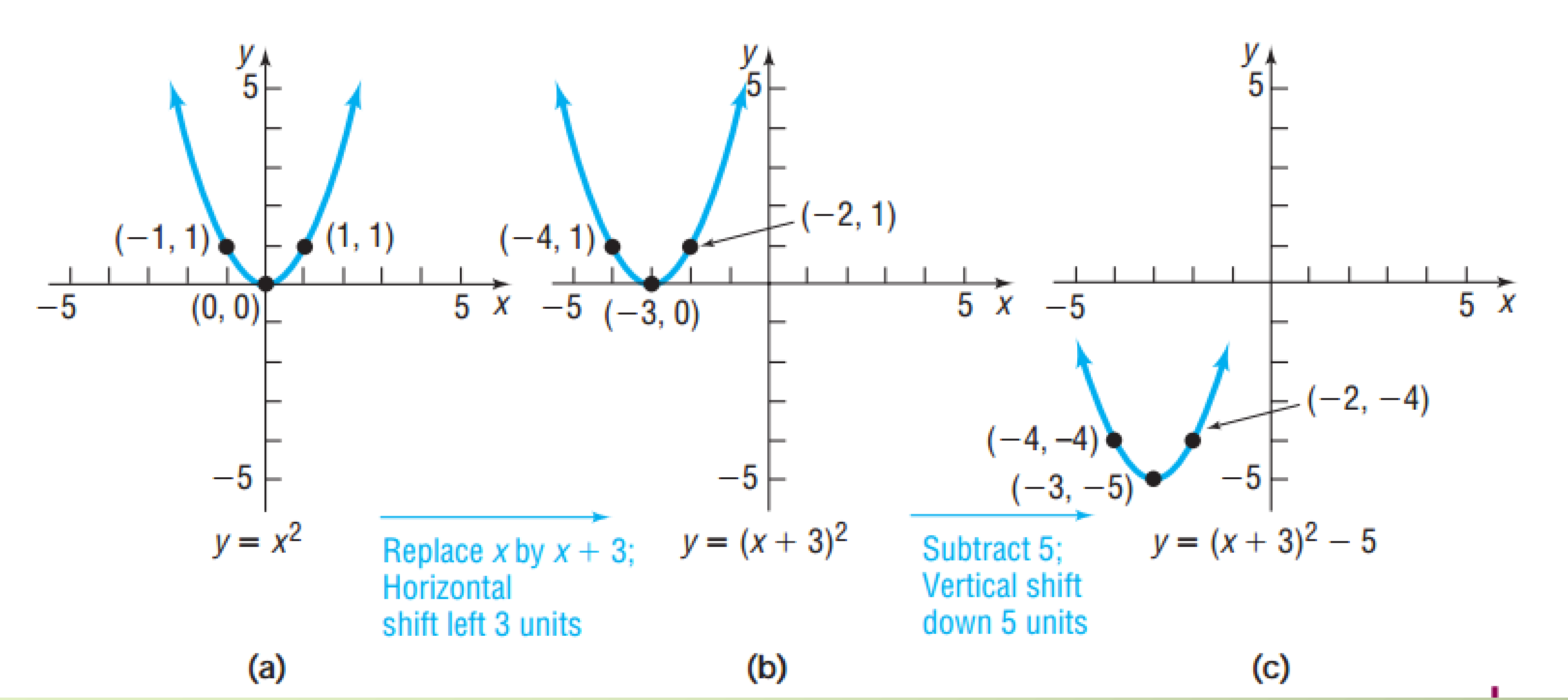
Horizontal Shift to the Right

- Example: Use the graph of $f(x) = x^2$ to obtain the graph of $g(x) = (x - 2)^2$



Combining Vertical and Horizontal Shifts

- Example: Graph the function $f(x) = (x+3)^2 - 5$



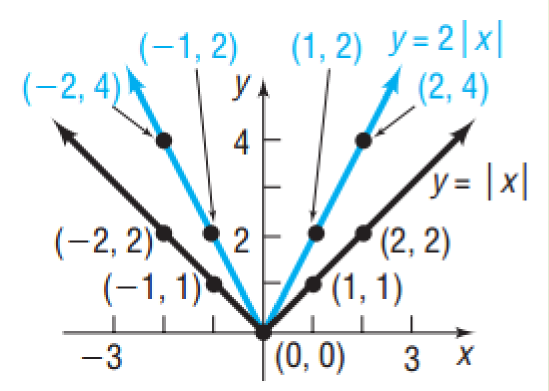
Graph Functions Using Compressions and Stretches

- When the right side of a function $y = f(x)$ is multiplied by a positive number a, the graph of the new function $y = af(x)$ is obtained by multiplying each y-coordinate on the graph of $y = f(x)$ by a. The new graph is a **vertically compressed** (if $0 < a < 1$) or a **vertically stretched** (if $a > 1$) version of the graph of $y = f(x)$.

Vertical Stretch

- Example: Use the graph of $f(x) = |x|$ to obtain the graph of $g(x) = 2|x|$

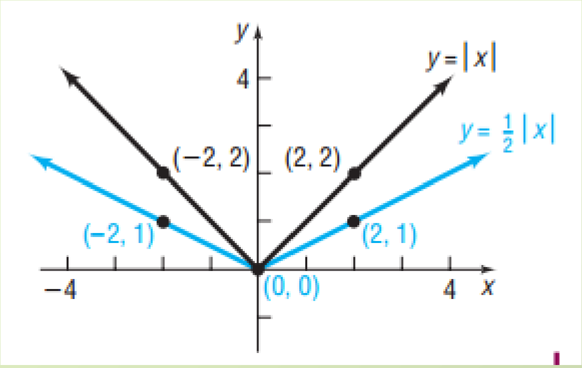
x	$y = f(x) = \|x\|$	$y = g(x) = 2\|x\|$
−2	2	4
−1	1	2
0	0	0
1	1	2
2	2	4



Vertical Compression

- Example: Use the graph of $f(x) = |x|$ to obtain the graph of $g(x) = \frac{1}{2}|x|$

x	$y = f(x) = \|x\|$	$y = g(x) = \frac{1}{2}\|x\|$
-2	2	1
-1	1	$\frac{1}{2}$
0	0	0
1	1	$\frac{1}{2}$
2	2	1



Graph Functions Using Compressions and Stretches

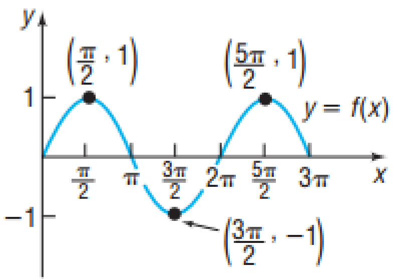
- If the argument x of a function $y = f(x)$ is multiplied by a positive number a, the graph of the new function $y = f(ax)$ is obtained by multiplying each x-coordinate of $y = f(x)$ by $\frac{1}{a}$. A **horizontal compression** results if $a > 1$, and a **horizontal stretch** occurs if $0 < a < 1$

Graphing Using Stretches and Compressions

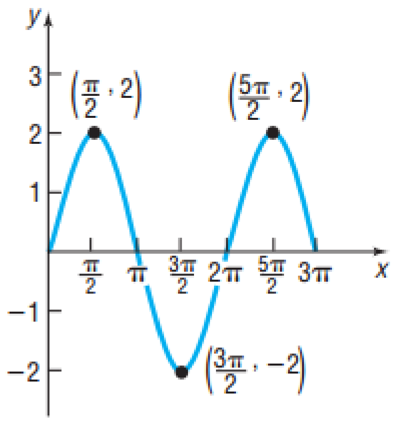
- Example: The graph of $y = f(x)$ is given in Figure. Use this graph to find the graphs of

(a) $y = 2f(x)$ (b) $y = f(3x)$

Figure

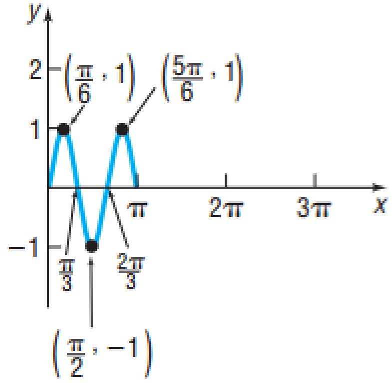


Figure



$y = 2f(x)$

Figure

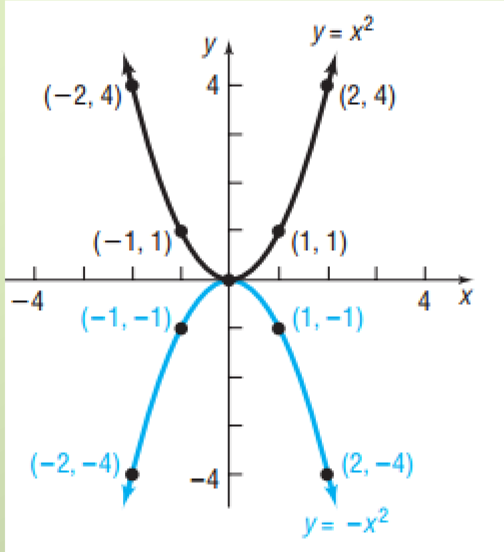


$y = f(3x)$

Reflection about the x-Axis

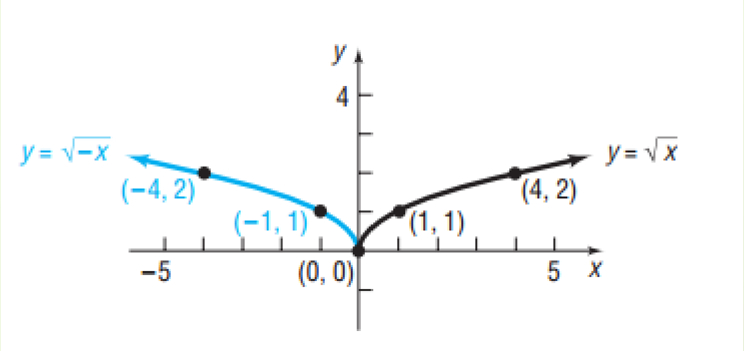
- When the right side of the function $y = f(x)$ is multiplied by -1 , the graph of the new function is the **reflection about the x-axis** of the graph of the function $y = f(x)$.
- Example: Graph the function $f(x) = -x^2$

x	$y = x^2$	$y = -x^2$
−2	4	−4
−1	1	−1
0	0	0
1	1	−1
2	4	−4



Reflection about the y-Axis

- When the graph of the function $y = f(x)$ is known, the graph of the new function $y = f(-x)$ is the **reflection about the *y*-axis** of the graph of the function $y = f(x)$
- Example: Graph the function $f(x) = \sqrt{-x}$



Summary of graphing techniques

SUMMARY OF GRAPHING TECHNIQUES

To Graph:	Draw the Graph of f and:	Functional Change to $f(x)$
Vertical shifts		
$y = f(x) + k, \quad k > 0$	Raise the graph of f by k units.	Add k to $f(x)$.
$y = f(x) - k, \quad k > 0$	Lower the graph of f by k units.	Subtract k from $f(x)$.
Horizontal shifts		
$y = f(x + h), \quad h > 0$	Shift the graph of f to the left h units.	Replace x by $x + h$.
$y = f(x - h), \quad h > 0$	Shift the graph of f to the right h units.	Replace x by $x - h$.
Compressing or stretching		
$y = af(x), \quad a > 0$	Multiply each y-coordinate of $y = f(x)$ by a. Stretch the graph of f vertically if $a > 1$. Compress the graph of f vertically if $0 < a < 1$.	Multiply $f(x)$ by a.
$y = f(ax), \quad a > 0$	Multiply each x-coordinate of $y = f(x)$ by $\frac{1}{a}$. Stretch the graph of f horizontally if $0 < a < 1$. Compress the graph of f horizontally if $a > 1$.	Replace x by ax.
Reflection about the x-axis		
$y = -f(x)$	Reflect the graph of f about the x-axis.	Multiply $f(x)$ by -1.
Reflection about the y-axis		
$y = f(-x)$	Reflect the graph of f about the y-axis.	Replace x by $-x$.