TROY UNIVERSITY PROGRAM AT HUST

Chapter 5 – Polynomial and Rational Functions

MTH112, PRE-CALCULUS ALGEBRA

DR. DOAN DUY TRUNG

Outline

- Polynomial Functions and Models
- Properties of Rational Functions
- The Graph of a Rational Function
- Polynomial and Rational Inequalities
- The Real Zeros of a Polynomial Function
- Complex Zeros; Fundamental Theorem of Algebra

Polynomial Functions and Models

- Identify Polynomial Functions and Their Degree
- Graph Polynomial Functions Using Transformations
- Identify the Real Zeros of a Polynomial Function and Their Multiplicity
- Analyze the Graph of a Polynomial Function
- Build Cubic Models from Data

Identify Polynomial Functions and Their Degree

A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad \textbf{(1)}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers and n is a nonnegative integer. The domain of a polynomial function is the set of all real numbers.

- The **degree** of a polynomial function is the largest power of x that appears.
- The zero polynomial function $f(x) = 0 + 0x + 0x^2 + \cdots + 0x^n$

Identify Polynomial Functions and Their Degree

- Determine which of the following are polynomial functions. For those that are, state the degree for those that are not, tell why not

(a) $f(x) = 2 - 3x^4$ (b) $g(x) = \sqrt{x}$ (c) $h(x) = \dfrac{x^2 - 2}{x^3 - 1}$

(d) $F(x) = 0$ (e) $G(x) = 8$ (f) $H(x) = -2x^3(x - 1)^2$

Summary of the properties of some polynomial functions

Degree	Form	Name	Graph
No degree	$f(x) = 0$	Zero function	The x-axis
0	$f(x) = a_0, \quad a_0 \neq 0$	Constant function	Horizontal line with y-intercept a_0
1	$f(x) = a_1x + a_0, \quad a_1 \neq 0$	Linear function	Nonvertical, nonhorizontal line with slope a_1 and y-intercept a_0
2	$f(x) = a_2x^2 + a_1x + a_0, \quad a_2 \neq 0$	Quadratic function	Parabola: graph opens up if $a_2 > 0$; graph opens down if $a_2 < 0$

Power functions

A power function of degree n is a monomial function of the form

$$f(x) = ax^n$$

Where a is a real number, $a \neq 0$, and $n > 0$ is an integer.

Properties of Power functions, $f(x) = x^n$

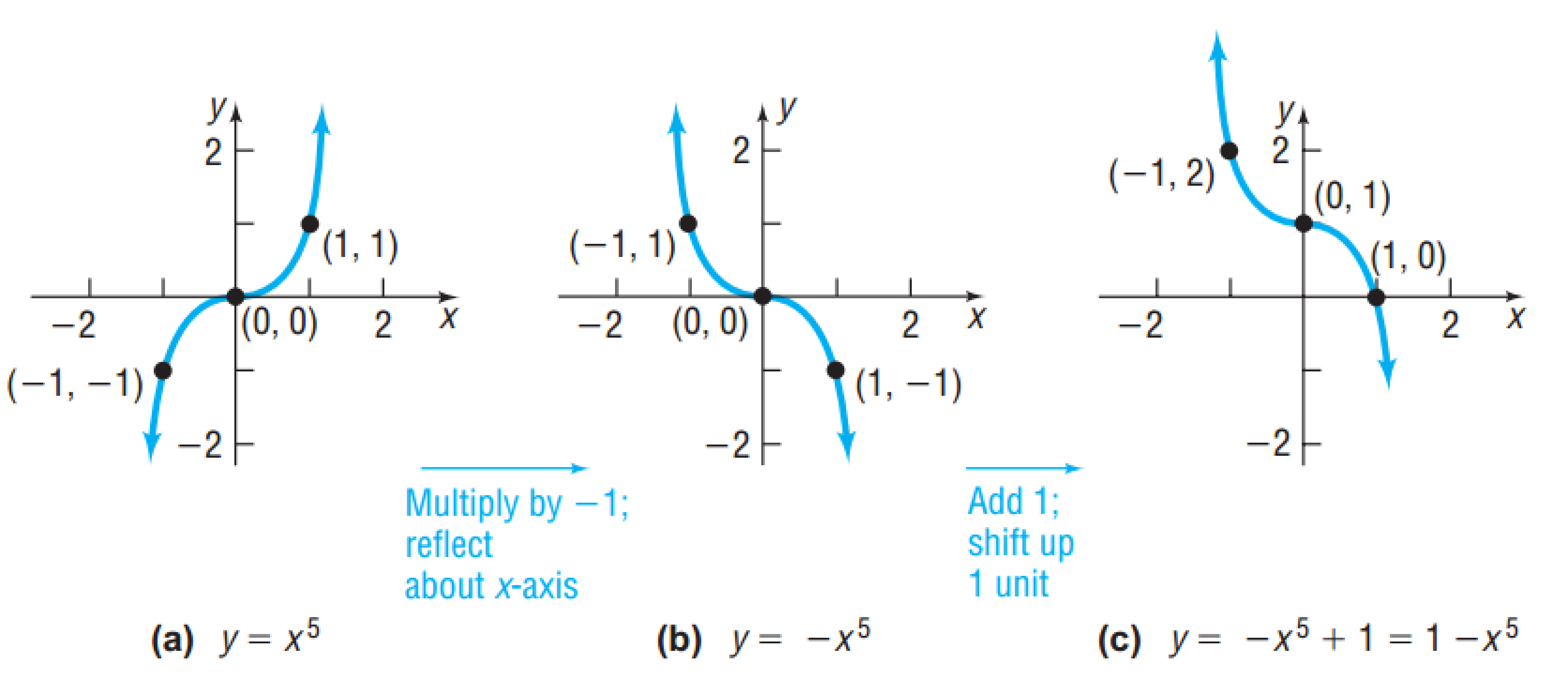
Properties of Power Functions, $f(x) = x^n$, n Is a Positive Odd Integer

1. f is an odd function, so its graph is symmetric with respect to the origin.
2. The domain and the range are the set of all real numbers.
3. The graph always contains the points $(-1, -1)$, $(0, 0)$, and $(1, 1)$.
4. As the exponent n increases in magnitude, the function increases more rapidly when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

Graph Polynomial Functions Using Transformations

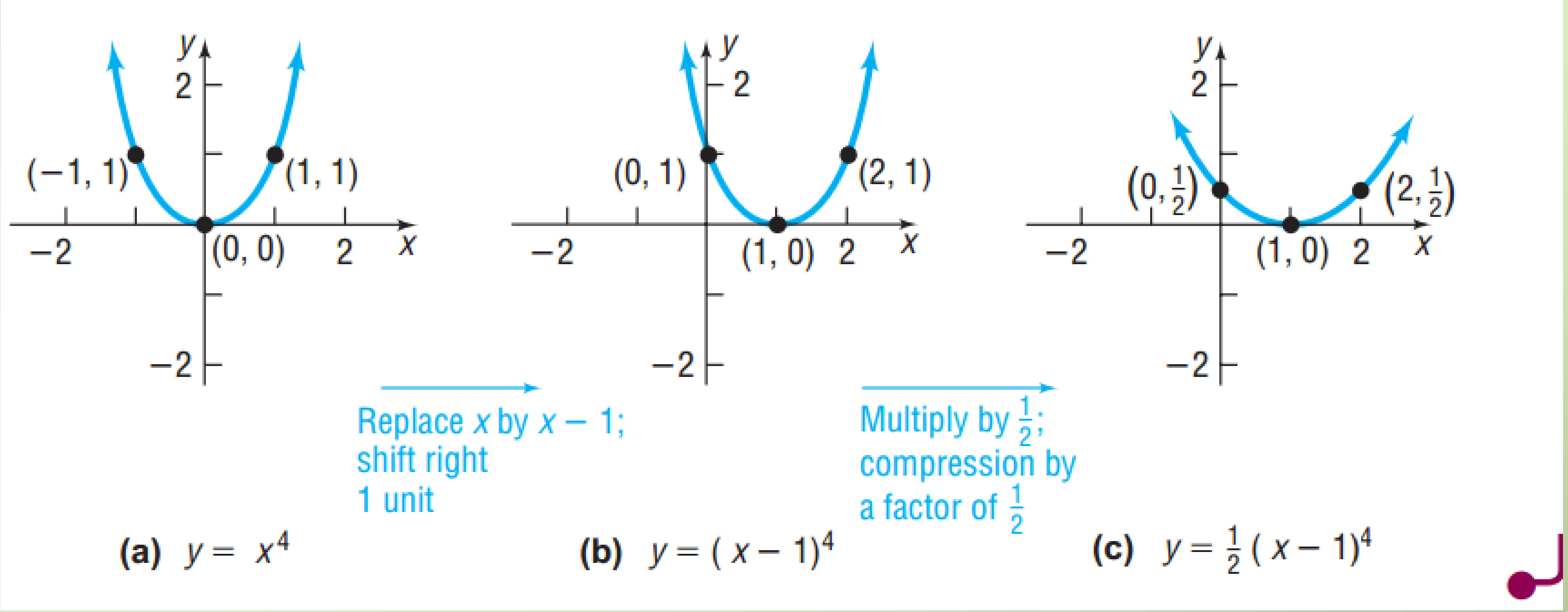
Graph $f(x) = 1 - x^5$

It is helpful to rewrite f as $f(x) = -x^5 + 1$



Graph Polynomial Functions Using Transformations

- Graph $f(x) = \frac{1}{2}(x-1)^4$



Identify the Real Zeros of a Polynomial Function and Their Multiplicity

If f is a function and r is a real number for which $f(r) = 0$, then r is called a **real zero** of f.

The following statements are equivalent:

- ✓ r is a real zero of a polynomial function f
- ✓ r is an x-intercept of the graph of f
- ✓ $x - r$ is a factor of f
- ✓ r is a solution to the equation $f(x) = 0$

Identify the Real Zeros of a Polynomial Function and Their Multiplicity

- If $(x - r)^m$ is a factor of a polynomial f and $(x - r)^{m+1}$ is not a factor of f, then r is called a zero of multiplicity m of f.

For the polynomial: $f(x) = x^2(x - 2)$

(a) Find the x- and y-intercepts of the graph of f.

(b) Use the x-intercepts to find the intervals on which the graph of f is above the x-axis and the intervals on which the graph of f is below the x-axis.

(c) Locate other points on the graph and connect all the points plotted with a smooth, continuous curve.

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Number chosen	-1	1	3
Value of *f*	$f(-1) = -3$	$f(1) = -1$	$f(3) = 9$
Location of graph	Below x-axis	Below x-axis	Above x-axis
Point on graph	$(-1, -3)$	$(1, -1)$	$(3, 9)$

Turning Points

- The points at which a graph changes direction are called **turning points.**
- If *f* is a polynomial function of degree *n*, then the graph of *f* has at most turning points
- If the graph of a polynomial function *f* has turning points, the degree of *f* is at least *n*.

End Behavior

- The behavior of the graph of a function for large values of x, either positive or negative, is referred to as its **end behavior.**

End Behavior

For large values of x, either positive or negative, the graph of the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

resembles the graph of the power function

$$y = a_n x^n$$

Analyze the Graph of a Polynomial Function

Analyze the graph of the polynomial function

$$f(x) = x^2(x-4)(x+1)$$

End behavior $y = x^4$

x-intercepts, $\{-1,0,4\}$

y-intercepts $\{0\}$

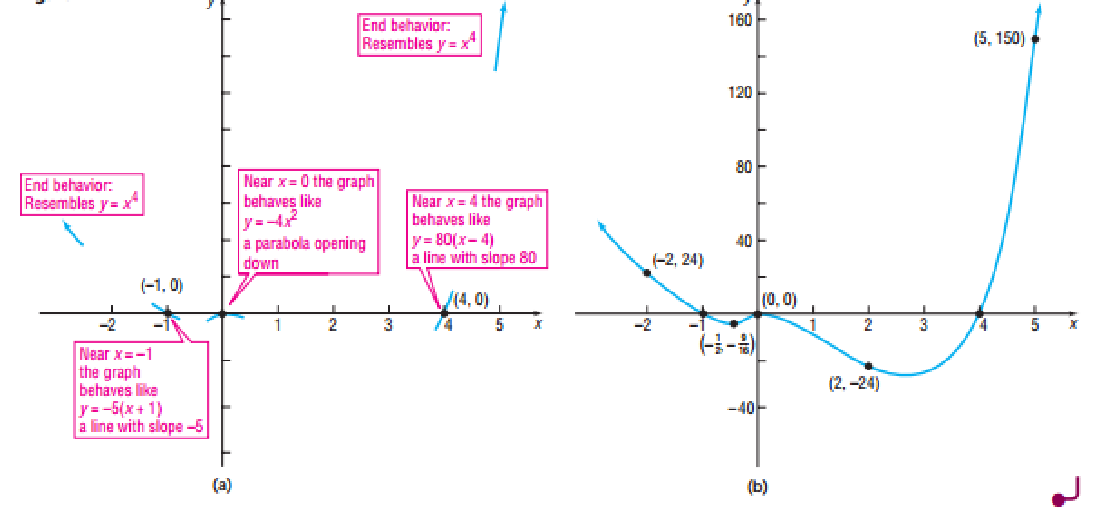
At most 3Turning points

Near -1: $f(x) = x^2(x-4)(x+1) \approx (-1)^2(-1-4)(x+1) = -5(x+1)$ A line with slope -5

Near 0: $f(x) = x^2(x-4)(x+1) \approx x^2(0-4)(0+1) = -4x^2$ A parabola opening down

Near 4: $f(x) = x^2(x-4)(x+1) \approx 4^2(x-4)(4+1) = 80(x-4)$ A line with slope 80

Figure 21



Properties of Rational Functions

- Find the Domain of a Rational Function
- Find the Vertical Asymptotes of a Rational Function
- Find the Horizontal or Oblique Asymptote of a Rational Function

$$R(x) = \frac{x^2 - 4}{x^2 + x + 1} \qquad F(x) = \frac{x^3}{x^2 - 4} \qquad G(x) = \frac{3x^2}{x^4 - 1}$$

Rational Function

A **rational function** is a function of the form

$$\boxed{R(x) = \frac{p(x)}{q(x)}}$$

where p and q are polynomial functions and q is not the zero polynomial. The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Rational Function

- Examples: Find the Domain of a Rational Function

a) $R(x) = \frac{2x^2-4}{x+5}$

b) $R(x) = \frac{1}{x^2-4}$

c) $R(x) = \frac{x^3}{x^2+1}$

d) $R(x) = \frac{x^2-1}{x-1}$

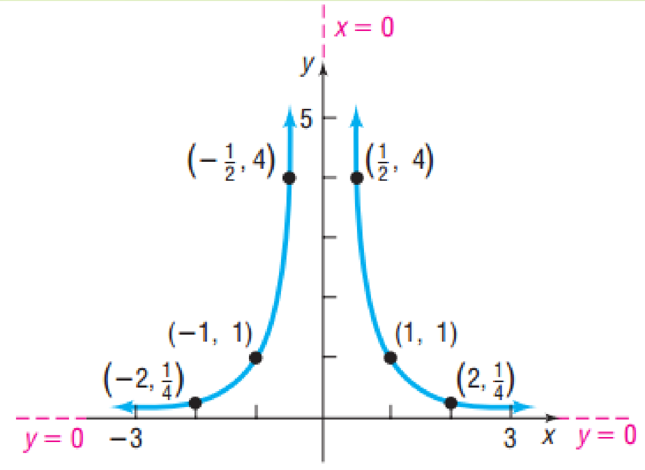
Rational Function

Analyze the graph of $H(x) = 1/x^2$

Domain: $x \neq 0$

$$H(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = H(x)$$

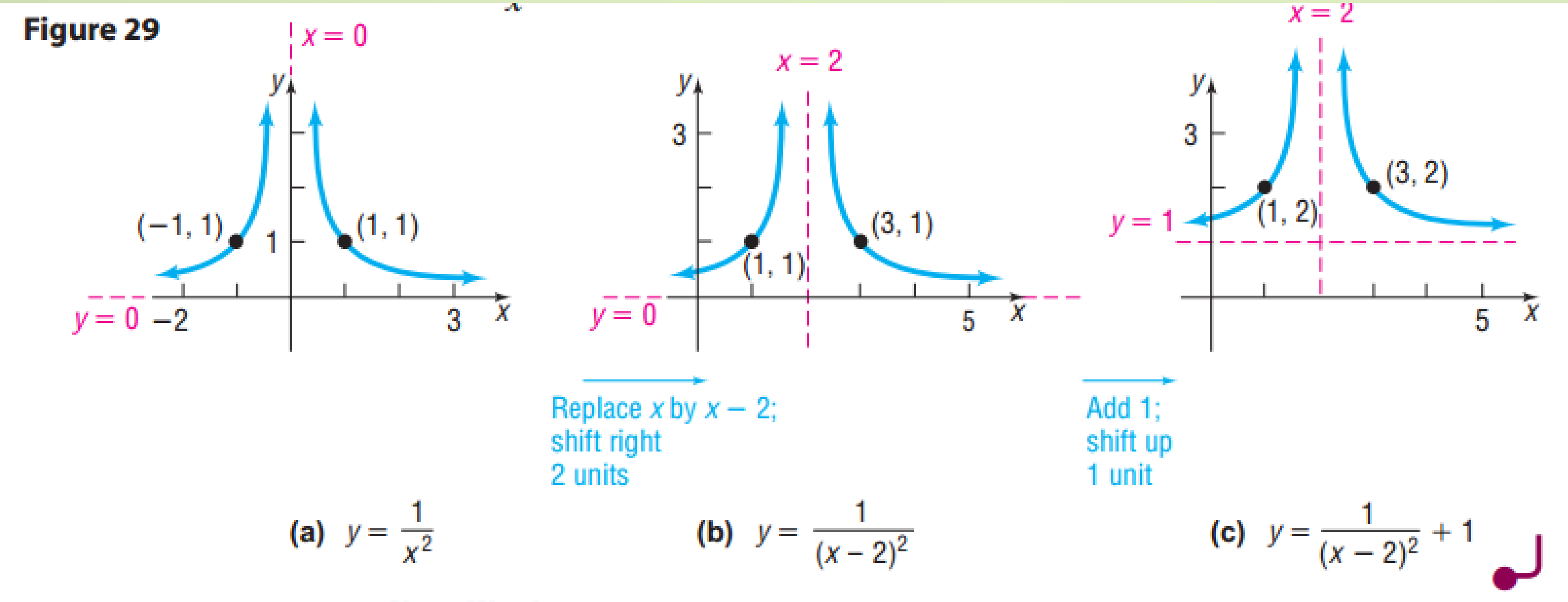
H is an even function, so its graph is symmetric with respect to the *y*-axis.



Using Transformations to Graph a Rational Function

Graph the rational function: $R(x) = \frac{1}{(x-2)^2} + 1$

Domain: $x \neq 2$



Asymptotes

Let $R(x) = \frac{1}{(x-2)^2} + 1$ be a function

x	$R(x)$
10	1.0156
100	1.0001
1000	1.000001
10,000	1.00000001

(a)

x	$R(x)$
−10	1.0069
−100	1.0001
−1000	1.000001
−10,000	1.00000001

(b)

x	$R(x)$
1.5	5
1.9	101
1.99	10,001
1.999	1,000,001
1.9999	100,000,001

(c)

x	$R(x)$
2.5	5
2.1	101
2.01	10,001
2.001	1,000,001
2.0001	100,000,001

(d)

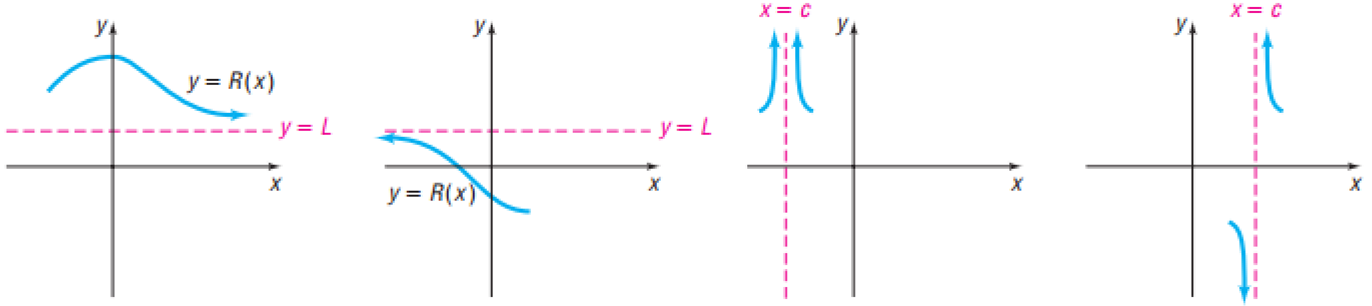
Asymptotes

Let R denote a function.

If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R. [Refer to Figures 30(a) and (b).]

If, as x approaches some number c, the values $|R(x)| \to \infty\,[R(x) \to -\infty$ or $R(x) \to \infty]$, then the line $x = c$ is a **vertical asymptote** of the graph of R. [Refer to Figures 30(c) and (d).]

Figure 30



(a) End behavior: As $x \to \infty$, the values of $R(x)$ approach L [$\lim_{x\to\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

(b) End behavior: As $x \to -\infty$, the values of $R(x)$ approach L [$\lim_{x\to-\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

(c) As x approaches c, the values of $|R(x)| \to \infty$ [$\lim_{x\to c^-} R(x) = \infty$; $\lim_{x\to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

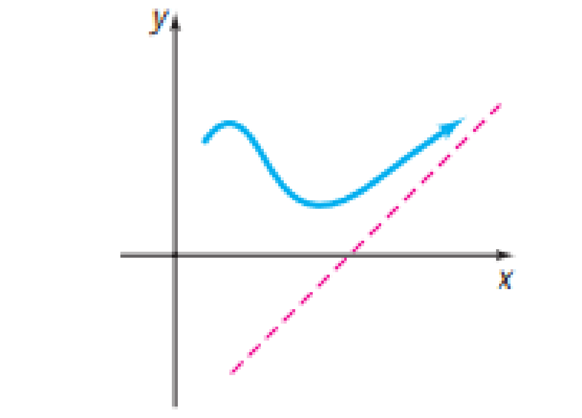
(d) As x approaches c, the values of $|R(x)| \to \infty$ [$\lim_{x\to c^-} R(x) = -\infty$; $\lim_{x\to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

Asymptote

- A horizontal asymptote, when it occurs, describes the **end behavior** of the graph as $x \to \infty$ or as $x \to -\infty$. **The graph of a function may intersect a horizontal asymptote.**
- A vertical asymptote, when it occurs, describes the behavior of the graph when x is close to some number c. **The graph of a rational function will never intersect a vertical asymptote.**

Oblique Asymptote

- If, as $x \to -\infty$ or as $x \to \infty$, the value of a rational function $R(x)$ approaches a liner expression $ax + b$, $a \neq 0$, then the line $y = ax + b$, $a \neq 0$, is an oblique asymptote of R.
- An oblique asymptote, when it occurs, describes the end behavior of the graph. **The graph of a function may intersect an oblique asymptote.**



Find the Vertical Asymptotes of a Rational Function

Locating Vertical Asymptotes

A rational function $R(x) = \dfrac{p(x)}{q(x)}$, *in lowest terms*, will have a vertical asymptote $x = r$ if r is a real zero of the *denominator* q. That is, if $x - r$ is a factor of the denominator q of a rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, R will have the vertical asymptote $x = r$.

- Find the vertical asymptote, if any, of the graph of each rational function

(a) $F(x) = \dfrac{x+3}{x-1}$ (b) $R(x) = \dfrac{x}{x^2-4}$

(c) $H(x) = \dfrac{x^2}{x^2+1}$ (d) $G(x) = \dfrac{x^2-9}{x^2+4x-21}$

Find the Horizontal or Oblique Asymptote of a Rational Function

- If a rational function $R(x)$ is **proper,** that is, if the degree of the numerator is less than the degree of the denominator, then as or as the value of approaches 0.
- Consequently, the line $y = 0$ is a horizontal asymptote of the graph
- If a rational function is proper, the line $y = 0$ is a horizontal asymptote of its graph.

Finding a Horizontal Asymptote

- Find the horizontal asymptote, if one exists, of the graph of

$$R(x) = \frac{x - 12}{4x^2 + x + 1}$$

- If a rational function $R(x) = \frac{p(x)}{q(x)}$ is improper, that is, if the degree of the numerator is greater than or equal to the degree of the denominator, we use long division to write the rational function as the sum of a polynomial $f(x)$ (the quotient) plus a proper rational function.

$$R(x) = \frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$$

Finding a Horizontal Asymptote

- Since $\frac{r(x)}{q(x)}$ is proper, $\frac{r(x)}{q(x)} \to 0$ as $x \to -\infty$ or as $x \to \infty$. As a result

$$R(x) = \frac{p(x)}{q(x)} \to f(x) \qquad \text{as } x \to -\infty \text{ or as } x \to \infty$$

- If $f(x) = b$, a constant, the line $y = b$ is a horizontal asymptote of the graph of R
- If $f(x) = ax + b, a \neq 0$, the line $y = ax + b$ is an oblique asymptote of the graph of R
- In all other cases, the graph of R approaches the graph of f, and there are no horizontal or oblique asymptotes.

Finding a Horizontal or Oblique Asymptote

- Find the horizontal or oblique asymptote, if one exists, of the graph of

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1}$$

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1}$$

The Graph of a Rational Function

- Analyze the Graph of a Rational Function
- Solve Applied Problems Involving Rational Functions

Analyze the Graph of a Rational Function

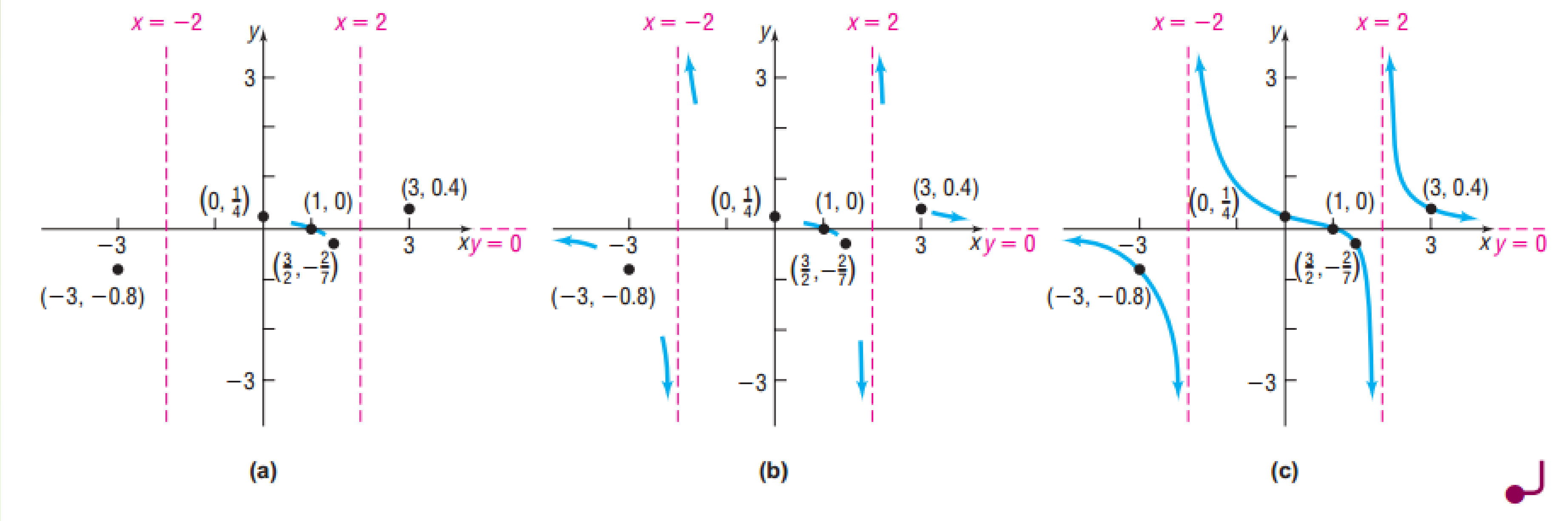
- Analyze the graph of the rational function: $R(x) = \frac{x-1}{x^2-4}$

$$\xrightarrow{\quad -2 \qquad 1 \qquad 2 \quad} x$$

Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, 2)$	$(2, \infty)$
Number chosen	-3	0	$\frac{3}{2}$	3
Value of *R*	$R(-3) = -0.8$	$R(0) = \frac{1}{4}$	$R\left(\frac{3}{2}\right) = -\frac{2}{7}$	$R(3) = 0.4$
Location of graph	Below *x*-axis	Above *x*-axis	Below *x*-axis	Above *x*-axis
Point on graph	$(-3, -0.8)$	$\left(0, \frac{1}{4}\right)$	$\left(\frac{3}{2}, -\frac{2}{7}\right)$	$(3, 0.4)$

Analyze the Graph of a Rational Function

- Analyze the graph of the rational function: $R(x) = \frac{x-1}{x^2-4}$



Analyzing the Graph of a Rational Function R

- **STEP 1:** Factor the numerator and denominator of *R*. Find the domain of the rational function.
- **STEP 2:** Write *R* in lowest terms.
- **STEP 3:** Locate the intercepts of the graph. The x-intercepts are the zeros of the numerator of *R* that are in the domain of *R*. Determine the behavior of the graph of *R* near each x-intercept.
- **STEP 4:** Determine the vertical asymptotes. Graph each vertical asymptote using a dashed line
- **STEP 5:** Determine the horizontal or oblique asymptote, if one exists. Determine points, if any, at which the graph of *R* intersects this asymptote. Graph the asymptote using a dashed line. Plot any points at which the graph of *R* intersects the asymptote

(continue)

Analyzing the Graph of a Rational Function R

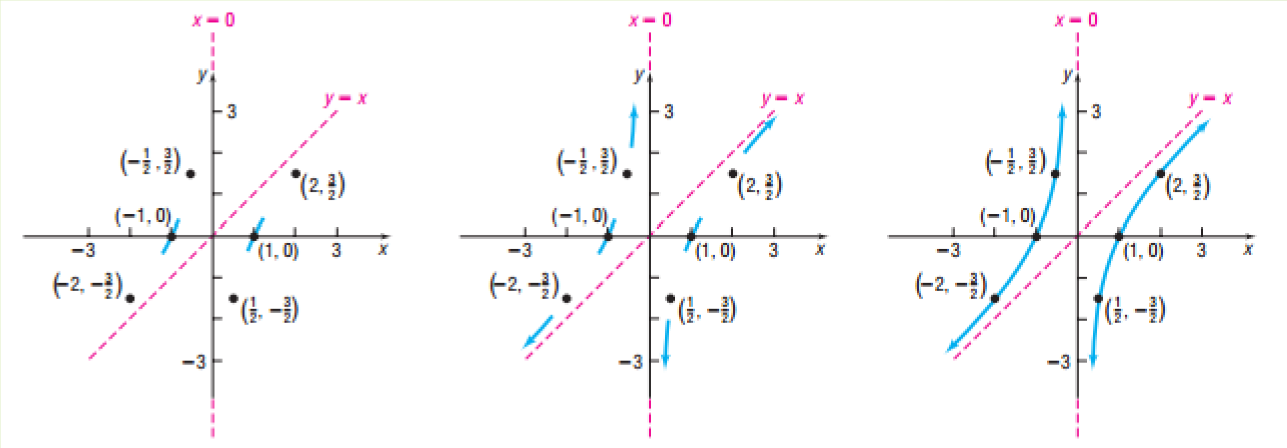
- **STEP 6:** Use the zeros of the numerator and denominator of *R* to divide the x-axis into intervals. Determine where the graph of *R* is above or below the x-axis by choosing a number in each interval and evaluating *R* there. Plot the points found.
- **STEP 7:** Analyze the behavior of the graph of *R* near each asymptote and indicate this behavior on the graph.
- **STEP 8:** Use the results obtained in Steps 1 through 7 to graph *R*.

Analyzing the Graph of a Rational Function R

- Analyze the graph of the rational function:

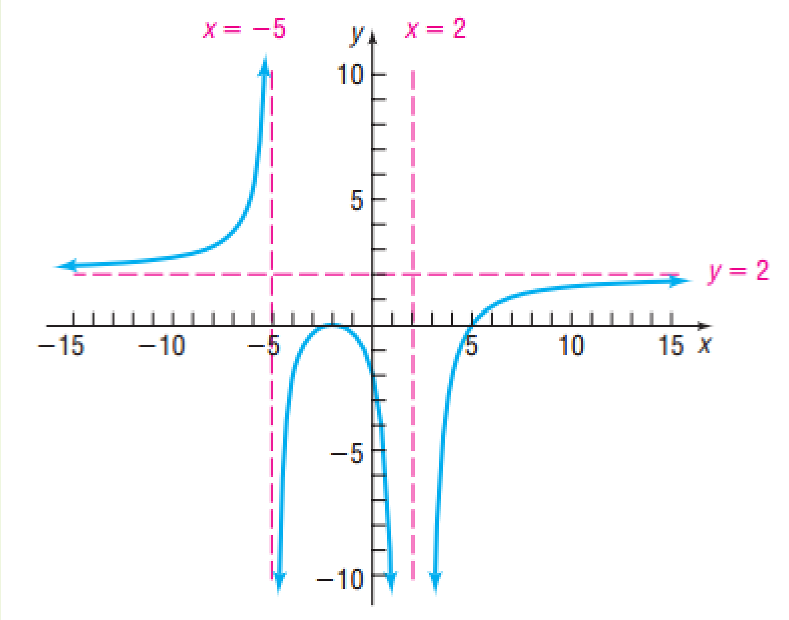
$$R(x) = \frac{x^2 - 1}{x}$$

	-1	0	1	x
Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Number chosen	-2	$-\frac{1}{2}$	$\frac{1}{2}$	2
Value of R	$R(-2) = -\frac{3}{2}$	$R\left(-\frac{1}{2}\right) = \frac{3}{2}$	$R\left(\frac{1}{2}\right) = -\frac{3}{2}$	$R(2) = \frac{3}{2}$
Location of graph	Below x-axis	Above x-axis	Below x-axis	Above x-axis
Point on graph	$\left(-2, -\frac{3}{2}\right)$	$\left(-\frac{1}{2}, \frac{3}{2}\right)$	$\left(\frac{1}{2}, -\frac{3}{2}\right)$	$\left(2, \frac{3}{2}\right)$



Constructing a Rational Function from Its Graph

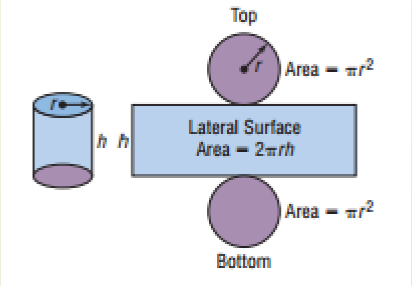
- Find a rational function that might have the graph shown in the following Figure.



Solve Applied Problems Involving Rational Functions

Reynolds Metal Company manufactures aluminum cans in the shape of a cylinder with a capacity of 500 cubic centimeters ($\frac{1}{2}$ liter). The top and bottom of the can are made of a special aluminum alloy that costs 0.05 cent per square centimeter. The sides of the can are made of material that cost 0.02 cent per square centimeter.

Express the cost of material for the can as a function of the radius r of the can?

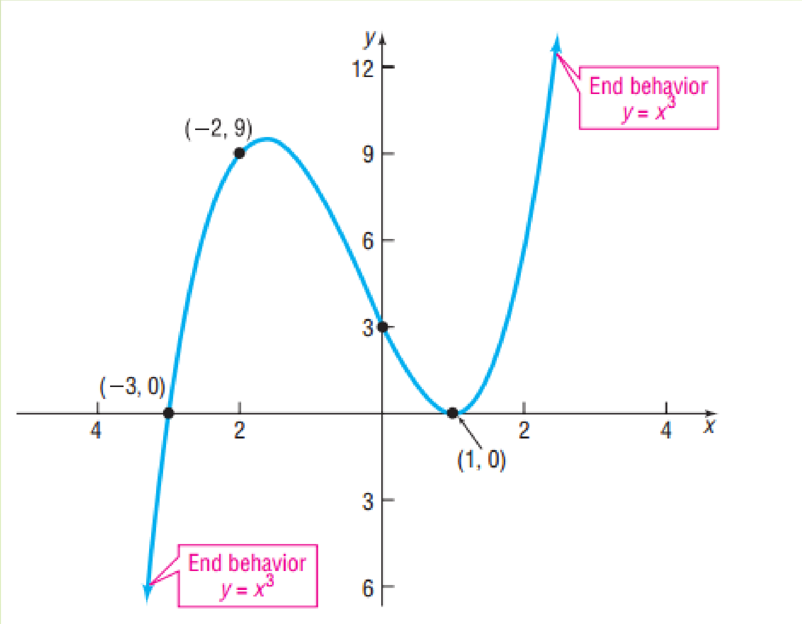


Polynomial and Rational Inequalities

- Solve Polynomial Inequalities
- Solve Rational Inequalities

Solve Polynomial Inequalities

- Solve $(x+3)(x-1)^2 > 0$ by graphing $f(x) = (x+3)(x-1)^2$



$$\mathrm{f}(x) > 0 \Leftrightarrow x \in (-3,1) \cup (1,\infty)$$

How to Solve a Polynomial Inequality Algebraically

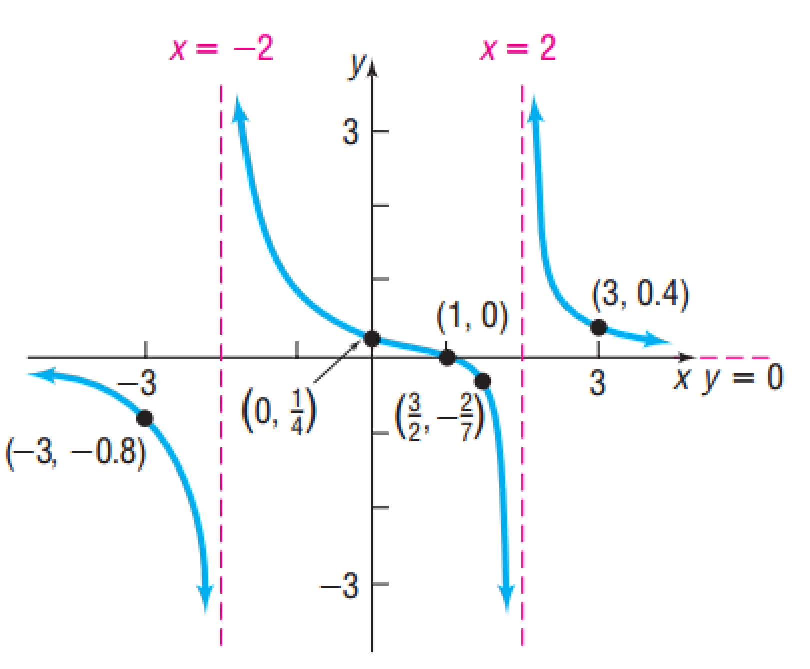
Solve the inequality $x^4 > x$

Rearrange the inequality so that: $x^4 - x > 0$

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Number chosen	-1	$\frac{1}{2}$	2
Value of *f*	$f(-1) = 2$	$f\left(\frac{1}{2}\right) = -\frac{7}{16}$	$f(2) = 14$
Conclusion	Positive	Negative	Positive

Solve Rational Inequalities

Solve $\frac{x-1}{x^2-4} \geq 0$ by graphing $R(x) = \frac{x-1}{x^2-4}$



Solutions: $(-2,1] \cup (2,\infty)$

How to Solve a Rational Inequality Algebraically

Solve the inequality $\frac{4x+5}{x+2} \geq 3$

Rearrange the inequality so that 0 is on the right side $\frac{x-1}{x+2} \geq 0$

Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, \infty)$
Number chosen	-3	0	2
Value of *f*	$f(-3) = 4$	$f(0) = -\frac{1}{2}$	$f(2) = \frac{1}{4}$
Conclusion	Positive	Negative	Positive

Solutions: $x \in (-\infty, -2) \cup [1, \infty)$

The Real Zeros of a Polynomial Function

- Use the Remainder and Factor Theorems
- Use the Rational Zeros Theorem to List the Potential Rational Zeros of a Polynomial Function
- Find the Real Zeros of a Polynomial Function
- Solve Polynomial Equations
- Use the Theorem for Bounds on Zeros
- Use the Intermediate Value Theorem

Use the Remainder and Factor Theorems

Division Algorithm for Polynomials

If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is a polynomial whose degree is greater than zero, then there are unique polynomial functions $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \text{or} \quad \underset{\substack{\uparrow \\ \text{dividend}}}{f(x)} = \underset{\substack{\uparrow \\ \text{quotient}}}{q(x)}\underset{\substack{\uparrow \\ \text{divisor}}}{g(x)} + \underset{\substack{\uparrow \\ \text{remainder}}}{r(x)} \qquad \textbf{(1)}$$

where $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$.

- $f(x)$ is the **dividend**, $g(x)$ is the **divisor**, $q(x)$ is the **quotient**, and $r(x)$ is the **remainder**.

Remainder Theorem

If $f(x) = (x - c)g(x) + R$, then $f(c) = R$

Remainder Theorem: Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$

Factor Theorem: Let f be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$

Example: Use the Factor Theorem to determine whether the function $f(x) = 2x^3 - x^2 + 2x - 3$ has factor

(a) $x - 1$ (b) $x + 3$

Number of Real Zeros

Number of Real Zeros: A polynomial function cannot have more real zeros than its degree.

Use the Rational Zeros Theorem to List the Potential Rational Zeros of a Polynomial Function

Rational Zeros Theorem

Let f be a polynomial function of degree 1 or higher of the form

$$\boxed{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad a_n \neq 0, \quad a_0 \neq 0}$$

where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0 and q must be a factor of a_n.

- Example: List the potential rational zeros of $f(x) = 2x^3 + 11x^2 - 7x - 6$
- Example: Find the real zeros of the polynomial function $f(x) = 2x^3 + 11x^2 - 7x - 6$. Write f in factored form.

Solve Polynomial Equations

THEOREM Every polynomial function with real coefficients can be uniquely factored into a product of linear factors and/or irreducible (prime) quadratic factors.

THEOREM A polynomial function of odd degree that has real coefficients has at least one real zero.

Example: Find the real solutions of the equation: $x^5 - 5x^4 + 12x^3 - 23x^2 + 32x - 16 = 0$

Use the Theorem for Bounds on Zeros

THEOREM

Bounds on Zeros

Let f denote a polynomial function whose leading coefficient is 1.

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

A bound M on the real zeros of f is the smaller of the two numbers

$$\boxed{\text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\}, 1 + \text{Max}\{|a_0|, |a_1|, \cdots, |a_{n-1}|\} \quad \textbf{(4)}}$$

where Max { } means "choose the largest entry in { }."

Find a bound on the real zeros of each polynomial function.

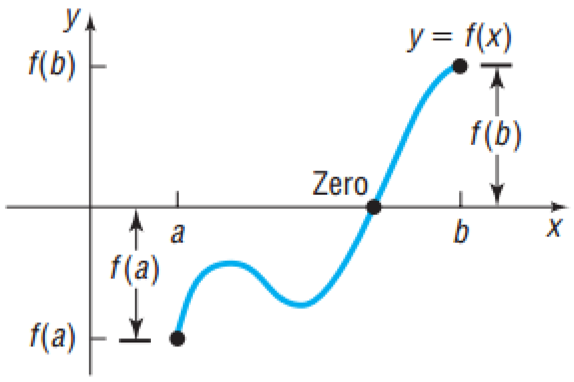
(a) $f(x) = x^5 + 3x^3 - 9x^2 + 5$ (b) $g(x) = 4x^5 - 2x^3 + 2x^2 + 1$

Use the Intermediate Value Theorem

THEOREM **Intermediate Value Theorem**

Let f denote a polynomial function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, there is at least one real zero of f between a and b.

Figure 49 If $f(a) < 0$ and $f(b) > 0$, there is a zero between a and b.



Use the Intermediate Value Theorem

Example: Show that $f(x) = x^5 - x^3 - 1$ has a zero between 1 and 2

Complex Zeros; Fundamental Theorem of Algebra

- Use the Conjugate Pairs Theorem
- Find a Polynomial Function with Specified Zeros
- Find the Complex Zeros of a Polynomial Function

Use the Conjugate Pairs Theorem

CONJUGATE PAIRS THEOREM

Let $f(x)$ be a polynomial function whose coefficients are real numbers. If $r = a + bi$ is a zero of f, the complex conjugate $\bar{r} = a - bi$ is also a zero of f.

COROLLARY

A polynomial function f of odd degree with real coefficients has at least one real zero.

Find a polynomial function f of degree 4 whose coefficients are real numbers that has the zeros $1, 1$, and $-4 + i$.

Use the Conjugate Pairs Theorem

THEOREM Every polynomial function with real coefficients can be uniquely factored over the real numbers into a product of linear factors and/or irreducible quadratic factors.

Find the complex zeros of the polynomial function

$$f(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$$

Write f in factored form.