TROY UNIVERSITY PROGRAM AT HUST

Chapter 7 – System of linear equations

MTH112, PRE-CALCULUS ALGEBRA

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Outline

- Systems of Linear Equations: Substitution and Elimination
- Systems of Linear Equations: Matrices
- Systems of Linear Equations: Determinants
- Matrix Algebra
- Systems of Nonlinear Equations
- Systems of Inequalities
- Linear Programming

Systems of Linear Equations: Substitution and Elimination

- Solve Systems of Equations by Substitution
- Solve Systems of Equations by Elimination
- Identify Inconsistent Systems of Equations Containing Two Variables
- Express the Solution of a System of Dependent Equations Containing Two Variables
- Solve Systems of Three Equations Containing Three Variables
- Identify Inconsistent Systems of Equations Containing Three Variables
- Express the Solution of a System of Dependent Equations Containing Three Variables

Examples

Example1

A movie theater sells tickets for \$8.00 each, with seniors receiving a discount of \$2.00. One evening the theater took in \$3580 in revenue. If x represents the number of tickets sold at \$8.00 and y the number of tickets sold at the discounted price of \$6.00, write an equation that relates these variables.

In Example 1, suppose that we also know that 525 tickets were sold that evening.

Solve Systems of Equations by Substitution

- Example

$$\text{Solve: } \begin{cases} 2x + y = 5 \\ -4x + 6y = 12 \end{cases}$$

Solve Systems of Equations by Elimination

Rules for Obtaining an Equivalent System of Equations

1. Interchange any two equations of the system.
2. Multiply (or divide) each side of an equation by the same nonzero constant.
3. Replace any equation in the system by the sum (or difference) of that equation and a nonzero multiple of any other equation in the system.

- Example 1

Solve: $\begin{cases} 2x + 3y = 1 \\ -x + y = -3 \end{cases}$

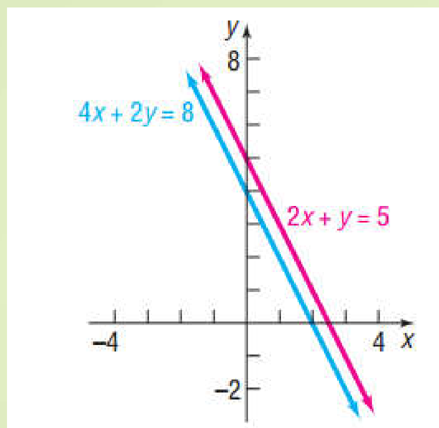
- Example 2

A movie theater sells tickets for \$8.00 each, with seniors receiving a discount of \$2.00. One evening the theater sold 525 tickets and took in \$3580 in revenue. How many of each type of ticket were sold?

Identify Inconsistent Systems of Equations Containing Two Variables

- Example 1:

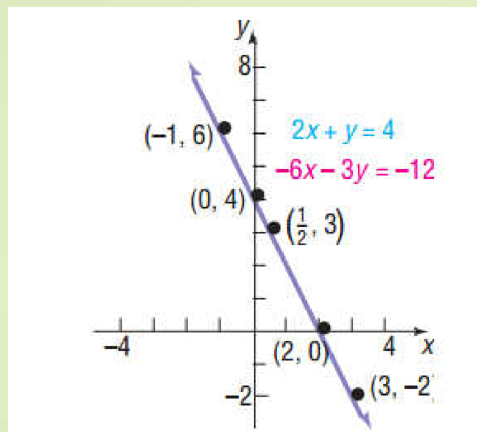
$$\text{Solve: } \begin{cases} 2x + y = 5 \\ 4x + 2y = 8 \end{cases}$$



Express the Solution of a System of Dependent Equations Containing Two Variables

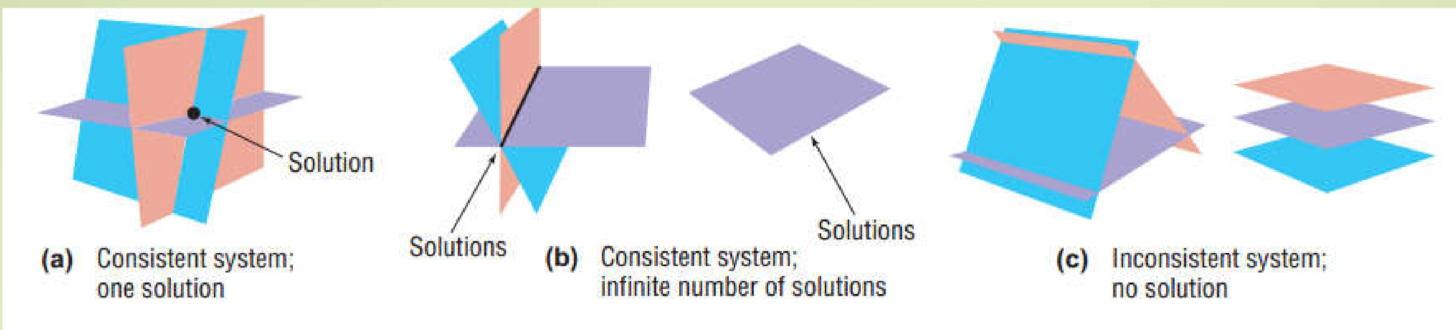
- Example :

$$\text{Solve: } \begin{cases} 2x + y = 4 \\ -6x - 3y = -12 \end{cases}$$



Solve Systems of Three Equations Containing Three Variables

- This system has:
  - Exactly one solution
  - No solution
  - Infinitely many solutions



Solve Systems of Three Equations Containing Three Variables

- Example

Use the method of elimination to solve the system of equations.

$$\begin{cases} x + y - z = -1 & (1) \\ 4x - 3y + 2z = 16 & (2) \\ 2x - 2y - 3z = 5 & (3) \end{cases}$$

Identify Inconsistent Systems of Equations Containing Three Variables

- Example

$$\text{Solve: } \begin{cases} 2x + y - z = -2 & (1) \\ x + 2y - z = -9 & (2) \\ x - 4y + z = 1 & (3) \end{cases}$$

Express the Solution of a System of Dependent Equations Containing Three Variables

- Example 1:

$$\text{Solve: } \begin{cases} x - 2y - z = 8 & (1) \\ 2x - 3y + z = 23 & (2) \\ 4x - 5y + 5z = 53 & (3) \end{cases}$$

- Example 2:

Find real numbers a, b, and c so that the graph of the quadratic function $y = ax^2 + bx + c$ contains the points $(-1, -4)$, $(1, 6)$, and $(3, 0)$.

Systems of Linear Equations: Matrices

- Write the Augmented Matrix of a System of Linear Equations
- Write the System of Equations from the Augmented Matrix
- Perform Row Operations on a Matrix
- Solve a System of Linear Equations Using Matrices

Introduction

Consider the following system of linear equations

$$\begin{cases} x + 4y = 14 \\ 3x - 2y = 0 \end{cases}$$

If we choose not to write the symbols used for the variables, we can represent this system as

$$\left[\begin{array}{cc|c} 1 & 4 & 14 \\ 3 & -2 & 0 \end{array}\right]$$

Matrix

DEFINITION A **matrix** is defined as a rectangular array of numbers,

$$\begin{array}{c} \\ \text{Row } 1 \\ \text{Row } 2 \\ \vdots \\ \text{Row } i \\ \vdots \\ \text{Row } m \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{Column } 1 & \text{Column } 2 & & \text{Column } j & & \text{Column } n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array} \qquad \textbf{(1)}$$

Each number a_{ij} of the matrix has two indexes: the **row index** i and the **column index** j. The matrix shown in display (1) has m rows and n columns. The numbers a_{ij} are usually referred to as the **entries** of the matrix. For example, a_{23} refers to the entry in the second row, third column.

Write the Augmented Matrix of a System of Linear Equations

Example :

Write the augmented matrix of each system of equations.

(a) $\begin{cases} 3x - 4y = -6 & (1) \\ 2x - 3y = -5 & (2) \end{cases}$

(b) $\begin{cases} 2x - y + z = 0 & (1) \\ x + z - 1 = 0 & (2) \\ x + 2y - 8 = 0 & (3) \end{cases}$

Perform Row Operations on a Matrix

Row operations on a matrix are used to solve systems of equations when the system is written as an augmented matrix. There are three basic row operations:

- ✓ Interchange any two rows.
- ✓ Replace a row by a nonzero multiple of that row.
- ✓ Replace a row by the sum of that row and a constant nonzero multiple of some other row.

Perform Row Operations on a Matrix

Example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & -1 & 2 \end{array}\right] \underset{R_2 = -4r_1 + r_2}{\longrightarrow} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ -4(1)+4 & -4(2)+(-1) & -4(3)+2 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -9 & -10 \end{array}\right]$$

Solve a System of Linear Equations Using Matrices

To solve a system of linear equations using matrices, we use row operations on the augmented matrix of the system to obtain a matrix that is in *row echelon form*.

DEFINITION

A matrix is in **row echelon form** when the following conditions are met:

1. The entry in row 1, column 1 is a 1, and only 0's appear below it.
2. The first nonzero entry in each row after the first row is a 1, only 0's appear below it, and the 1 appears to the right of the first nonzero entry in any row above.
3. Any rows that contain all 0's to the left of the vertical bar appear at the bottom.

Solve a System of Linear Equations Using Matrices

For example, for a system of three equations containing three variables x,y and z, with a unique solution, the augmented matrix is in row echelon form if it is of the form

$$\left[\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}\right]$$

Where a, b, c, d, e and f are real numbers.

The methodology used to write a matrix in reduced row echelon form is called **Gauss-Jordan elimination.**

How to Solve a System of Linear Equations Using Matrices

- Example 1

$$\text{Solve: } \begin{cases} 2x + 2y \phantom{{}+z} = 6 & (1) \\ x + y + z = 1 & (2) \\ 3x + 4y - z = 13 & (3) \end{cases}$$

- Example 2

$$\text{Solve: } \begin{cases} 6x - y - z = 4 & (1) \\ -12x + 2y + 2z = -8 & (2) \\ 5x + y - z = 3 & (3) \end{cases}$$

- Example 3

$$\text{Solve: } \begin{cases} x + y + z = 6 \\ 2x - y - z = 3 \\ x + 2y + 2z = 0 \end{cases}$$

Financial Planning

Financial Planning

Adam and Michelle require an additional \$25,000 in annual income (beyond their pension benefits). They are rather risk averse and have narrowed their investment choices down to Treasury notes that yield 3%, Treasury bonds that yield 5%, or corporate bonds that yield 6%. If they have \$600,000 to invest and want the amount invested in Treasury notes to equal the total amount invested in Treasury bonds and corporate bonds, how much should be placed in each investment?

Systems of Linear Equations: Determinants

- Evaluate 2 by 2 Determinants
- Use Cramer's Rule to Solve a System of Two Equations Containing Two Variables
- Evaluate 3 by 3 Determinants
- Use Cramer's Rule to Solve a System of Three Equations Containing Three Variables
- Know Properties of Determinants

Evaluate 2 by 2 Determinants

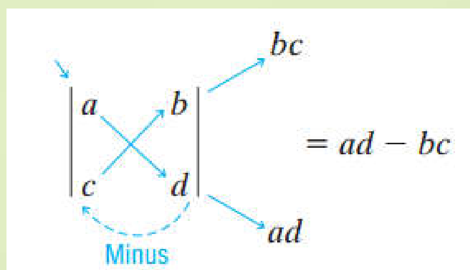
DEFINITION

If a, b, c, and d are four real numbers, the symbol

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is called a **2 by 2 determinant.** Its value is the number $ad - bc$; that is,

$$\boxed{D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \textbf{(1)}}$$



Evaluate 2 by 2 Determinants

Example:

Evaluate: $\begin{vmatrix} 3 & -2 \\ 6 & 1 \end{vmatrix}$

Use Cramer's Rule to Solve a System of Two Equations Containing Two Variables

THEOREM

Cramer's Rule for Two Equations Containing Two Variables

The solution to the system of equations

$$\begin{cases} ax + by = s & (1) \\ cx + dy = t & (2) \end{cases} \qquad \textbf{(5)}$$

is given by

$$x = \frac{\begin{vmatrix} s & b \\ t & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & s \\ c & t \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \textbf{(6)}$$

provided that

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$$

Use Cramer's Rule to Solve a System of Two Equations Containing Two Variables

Let D as follows:

$$\begin{cases} ax + by = s \\ cx + dy = t \end{cases} \qquad D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

And D_x, D_y

$$D_x = \begin{vmatrix} s & b \\ t & d \end{vmatrix} \qquad D_y = \begin{vmatrix} a & s \\ c & t \end{vmatrix}$$

Cramer's Rule then states that if $D \neq 0$

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad \textbf{(7)}$$

Evaluate 3 by 3 Determinants

A 3 by 3 determinant is symbolized by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The value of a 3 by 3 determinant may be defined in terms of 2 by 2 determinants by the following formula:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\text{2 by 2 determinant}} \overset{\text{Minus}}{-} a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{\text{2 by 2 determinant}} \overset{\text{Plus}}{+} a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\text{2 by 2 determinant}}$$

Finding Minors of a 3 by 3 Determinant

The 2 by 2 determinants shown in the previous formula (9) are called **minors** of the 3 by 3 determinant. For an n by n determinant, the **minor** M_{ij} of entry a_{ij} is the determinant resulting from removing the ith row and jth column.

Example

For the determinant $A = \begin{vmatrix} 2 & -1 & 3 \\ -2 & 5 & 1 \\ 0 & 6 & -9 \end{vmatrix}$, find: (a) M_{12} (b) M_{23}

cofactor

DEFINITION

For an n by n determinant A, the **cofactor** of entry a_{ij}, denoted by A_{ij}, is given by

$$\boxed{A_{ij} = (-1)^{i+j} M_{ij}}$$

where M_{ij} is the minor of entry a_{ij}.

Use Cramer's Rule to Solve a System of Three Equations Containing Three Variables

Consider the following system of three equations containing three variables

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases}$$

The determinant D of the coefficients of the variables

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Use Cramer's Rule to Solve a System of Three Equations Containing Three Variables

If $D \neq 0$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

Then the unique solution of system is given by

THEOREM

Cramer's Rule for Three Equations Containing Three Variables

$$\boxed{x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D}}$$

where

$$D_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix} \qquad D_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix} \qquad D_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

Cramer's Rule with Inconsistent or Dependent Systems

Cramer's Rule with Inconsistent or Dependent Systems

- If $D = 0$ and at least one of the determinants D_x, D_y, or D_z is different from 0, then the system is inconsistent and the solution set is $\varnothing$ or $\{\ \}$.
- If $D = 0$ and all the determinants D_x, D_y, and D_z equal 0, then the system is consistent and dependent so that there are infinitely many solutions. The system must be solved using row reduction techniques.

Examples

Example 1

Use Cramer's Rule, if applicable, to solve the following system:

$$\begin{cases} 2x + y - z = 3 & (1) \\ -x + 2y + 4z = -3 & (2) \\ x - 2y - 3z = 4 & (3) \end{cases}$$

Determinant

Let A be a n by n matrix. The determinant of A, denoted by $|A|$, is

$$|A| = \sum_{j=1}^{n} a_{ij} A_{ij}$$

Or

$$|A| = \sum_{i=1}^{n} a_{ij} A_{ij}$$

Example

Example

Evaluating a 3 × 3 Determinant

Find the value of the 3 by 3 determinant: $\begin{vmatrix} 3 & 0 & -1 \\ 4 & 6 & 2 \\ 8 & -2 & 3 \end{vmatrix}$

Know Properties of Determinants

THEOREM The value of a determinant changes sign if any two rows (or any two columns) are interchanged. **(11)**

Proof for 2 by 2 Determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \text{and} \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc)$$

Example

$$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2 \qquad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

Know Properties of Determinants

THEOREM If all the entries in any row (or any column) equal 0, the value of the determinant is 0. **(12)**

THEOREM If any two rows (or any two columns) of a determinant have corresponding entries that are equal, the value of the determinant is 0. **(13)**

Example

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

Know Properties of Determinants

THEOREM If any row (or any column) of a determinant is multiplied by a nonzero number k, the value of the determinant is also changed by a factor of k. **(14)**

Example

$$\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix}$$

$$\begin{vmatrix} k & 2k \\ 4 & 6 \end{vmatrix}$$

Know Properties of Determinants

THEOREM If the entries of any row (or any column) of a determinant are multiplied by a nonzero number k and the result is added to the corresponding entries of another row (or column), the value of the determinant remains unchanged. **(15)**

Example

$$\begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix}$$

Matrix Algebra

- Find the Sum and Difference of Two Matrices
- Find Scalar Multiples of a Matrix
- Find the Product of Two Matrices
- Find the Inverse of a Matrix
- Solve a System of Linear Equations Using an Inverse Matrix

Matrix

DEFINITION

A **matrix** is defined as a rectangular array of numbers:

$$\begin{array}{c|cccccc}
 & \text{Column 1} & \text{Column 2} & & \text{Column } j & & \text{Column } n \\
\hline
\begin{array}{c}\text{Row 1}\\ \text{Row 2}\\ \vdots \\ \text{Row } i \\ \vdots \\ \text{Row } m\end{array} & \multicolumn{6}{c}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}}
\end{array}$$

Each number a_{ij} of the matrix has two indexes: the **row index** i and the **column index** j. The matrix shown here has m rows and n columns. The numbers a_{ij} are usually referred to as the **entries** of the matrix. For example, a_{23} refers to the entry in the second row, third column.

Matrix

Example

In a survey of 900 people, the following information was obtained:

200 males	Thought federal defense spending was too high
150 males	Thought federal defense spending was too low
45 males	Had no opinion
315 females	Thought federal defense spending was too high
125 females	Thought federal defense spending was too low
65 females	Had no opinion

	Too High	**Too Low**	**No Opinion**
Male	200	150	45
Female	315	125	65

Matrix

Example

(a) $\begin{bmatrix} 5 & 0 \\ -6 & 1 \end{bmatrix}$ A 2 by 2 square matrix (b) $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix}$ A 1 by 3 matrix

(c) $\begin{bmatrix} 6 & -2 & 4 \\ 4 & 3 & 5 \\ 8 & 0 & 1 \end{bmatrix}$ A 3 by 3 square matrix

Equality of matrices

DEFINITION Two matrices A and B are said to be **equal,** written as

$$\boxed{A = B}$$

provided that A and B have the same number of rows and the same number of columns and each entry a_{ij} in A is equal to the corresponding entry b_{ij} in B.

Example 1

$$\begin{bmatrix} 2 & 1 \\ 0.5 & -1 \end{bmatrix} = \begin{bmatrix} \sqrt{4} & 1 \\ \frac{1}{2} & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} \sqrt{9} & \sqrt{4} & 1 \\ 0 & 1 & \sqrt[3]{-8} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ 6 & 1 \end{bmatrix} \neq \begin{bmatrix} 4 & 0 \\ 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 2 \\ 6 & 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 4 & 1 & 2 & 3 \\ 6 & 1 & 2 & 4 \end{bmatrix}$$

Equality of matrices

Example 2:Find a, b, c, d such that $\begin{bmatrix} 1 & a & 0 \\ -2 & 3 & b \end{bmatrix} = \begin{bmatrix} c & 5 & d \\ 2 & 6 & 9 \end{bmatrix}$

Find the Sum and Difference of Two Matrices

Suppose that A and B represent two m by n matrices. We define their **sum** $\boldsymbol{A} + \boldsymbol{B}$ to be the m by n matrix formed by adding the corresponding entries a_{ij} of A and b_{ij} of B.

The **difference** $\boldsymbol{A} - \boldsymbol{B}$ is defined as the m by n matrix formed by subtracting the entries b_{ij} in B from the corresponding entries a_{ij} in A

Addition and subtraction of matrices are allowed only for matrices having the same number *m* of rows and the same number *n* of columns.

Adding and Subtracting Matrices

Example

Suppose that

$$A = \begin{bmatrix} 2 & 4 & 8 & -3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 & 0 & 1 \\ 6 & 8 & 2 & 0 \end{bmatrix}$$

Find: (a) $A + B$ (b) $A - B$

Properties

Many of the algebraic properties of sums of real numbers are also true for sums of matrices. Suppose that *A*, *B*, and *C* are *m* by *n* matrices.

Then matrix addition is **commutative.** That is,

Commutative Property of Matrix Addition

$$\boxed{A + B = B + A}$$

Matrix addition is also **associative.** That is,

Associative Property of Matrix Addition

$$\boxed{(A + B) + C = A + (B + C)}$$

Properties

A matrix whose entries are all equal to 0 is called a **zero matrix.** Each of the following matrices is a zero matrix.

$$\boxed{A + 0 = 0 + A = A}$$

Find Scalar Multiples of a Matrix

If k is a real number and A is an m by n matrix, the matrix kA is the m by n matrix formed by multiplying each entry in A by k. The number k is sometimes referred to as a **scalar,** and the matrix kA is called a **scalar multiple** of A.

Properties

Properties of Scalar Multiplication

$$k(hA) = (kh)A$$

$$(k + h)A = kA + hA$$

$$k(A + B) = kA + kB$$

Find Scalar Multiples of a Matrix

Examples

Suppose that

$$A = \begin{bmatrix} 3 & 1 & 5 \\ -2 & 0 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 1 & 0 \\ 8 & 1 & -3 \end{bmatrix} \quad C = \begin{bmatrix} 9 & 0 \\ -3 & 6 \end{bmatrix}$$

Find: (a) $4A$ (b) $\frac{1}{3}C$ (c) $3A - 2B$

Find the Product of Two Matrices

DEFINITION

A **row vector** $\boldsymbol{R}$ is a 1 by n matrix

$$R = [r_1 \quad r_2 \quad \cdots \quad r_n]$$

A **column vector** $\boldsymbol{C}$ is an n by 1 matrix

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The **product** $\boldsymbol{RC}$ of R times C is defined as the number

$$\boxed{RC = [r_1 \quad r_2 \cdots r_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = r_1c_1 + r_2c_2 + \cdots + r_nc_n}$$

Find the Product of Two Matrices

Example 1: The Product of a Row Vector and a Column Vector

If $R = [3 \quad -5 \quad 2]$ and $C = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$, then RC

Example 2: Using Matrices to compute Revenue

A clothing store sells men's shirts for $40, silk ties for $20, and wool suits for $400. Last month, the store had sales consisting of 100 shirts, 200 ties, and 50 suits. What was the total revenue due to these sales?

Find the product of two matrices

DEFINITION

Let A denote an m by r matrix and let B denote an r by n matrix. The **product** AB is defined as the m by n matrix whose entry in row i, column j is the product of the ith row of A and the jth column of B.

A		B
m by r		r by n

Must be same for AB to be defined

AB is m by n.

Find the product of two matrices

Example:

Find the product AB if

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 5 & 8 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 5 & 1 & 4 \\ 4 & 8 & 0 & 6 \\ -3 & 1 & -2 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 4 & -1 \\ 5 & 8 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 & 4 \\ 4 & 8 & 0 & 6 \\ -3 & 1 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \text{Row 1 of } A & \text{Row 1 of } A & \text{Row 1 of } A & \text{Row 1 of } A \\ \text{times} & \text{times} & \text{times} & \text{times} \\ \text{column 1 of } B & \text{column 2 of } B & \text{column 3 of } B & \text{column 4 of } B \\ & & & \\ \text{Row 2 of } A & \text{Row 2 of } A & \text{Row 2 of } A & \text{Row 2 of } A \\ \text{times} & \text{times} & \text{times} & \text{times} \\ \text{column 1 of } B & \text{column 2 of } B & \text{column 3 of } B & \text{column 4 of } B \end{bmatrix}$$

Find the product of two matrices

Example 1:

If

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$$

find: (a) AB (b) BA

Example 2:

If

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}$$

find: (a) AB (b) BA

Properties of the product of two Matrices

THEOREM Matrix multiplication is not commutative.

Associative Property of Matrix Multiplication

$$A(BC) = (AB)C$$

Distributive Property

$$A(B + C) = AB + AC$$

Identity matrix

For an *n* by *n* square matrix, the entries located in row *i*, column *i*, $1 \leq i \leq n$, are called the **diagonal entries** or **the main diagonal.** The *n* by *n* square matrix whose diagonal entries are 1's, while all other entries are 0's, is called the **identity matrix** $\boldsymbol{I_n}$

Examples

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplication with an Identity Matrix

Properties of an Identity Matrix

Identity Property

If A is an m by n matrix, then

$$\boxed{I_m A = A \quad \text{and} \quad A I_n = A}$$

If A is an n by n square matrix,

$$\boxed{A I_n = I_n A = A}$$

Multiplication with an Identity Matrix

Example

Let

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 \\ 4 & 6 \\ 5 & 2 \end{bmatrix}$$

Find: (a) AI_3 (b) I_2A (c) BI_2

Find the Inverse of a Matrix

DEFINITION

Let A be a square n by n matrix. If there exists an n by n matrix A^{-1}, read "A inverse," for which

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

then A^{-1} is called the **inverse** of the matrix A.

Example

Show that the inverse of

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{is} \quad A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Finding the Inverse of a Nonsingular Matrix

Procedure for Finding the Inverse of a Nonsingular Matrix*

To find the inverse of an n by n nonsingular matrix A, proceed as follows:

Step 1: Form the matrix $[A|I_n]$.

Step 2: Transform the matrix $[A|I_n]$ into reduced row echelon form.

Step 3: The reduced row echelon form of $[A|I_n]$ will contain the identity matrix I_n on the left of the vertical bar; the n by n matrix on the right of the vertical bar is the inverse of A.

Example

The matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix}$$

is nonsingular. Find its inverse.

Finding the Inverse of a singular Matrix

If transforming the matrix into reduced row echelon form does not result in the identity matrix I_n to the left of the vertical bar, A is singular and has no inverse.

Example

Show that the matrix $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ has no inverse.

Singular matrix

If the determinant of A is zero, A is singular. That means there does not exists the inverse of A, say A^{-1}.

Systems of Nonlinear Equations

- Solve a System of Nonlinear Equations Using Substitution
- Solve a System of Nonlinear Equations Using Elimination

Solve a System of Nonlinear Equations Using Substitution

Before we begin, two comments are in order.

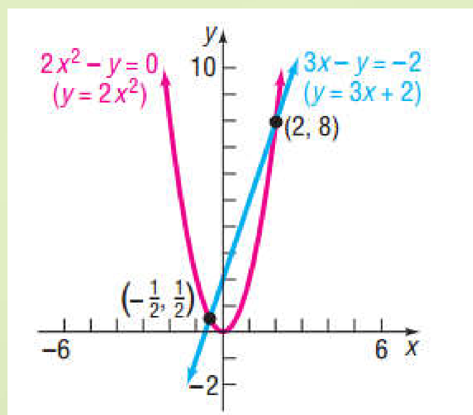
- If the system contains two variables and if the equations in the system are easy to graph, then graph them. By graphing each equation in the system, you can get an idea of how many solutions a system has and approximately where they are located.
- Extraneous solutions can creep in when solving nonlinear systems, so it is imperative that all apparent solutions be checked.

Solving a System of Nonlinear Equations Using Substitution

Example 1

Solve the following system of equations:

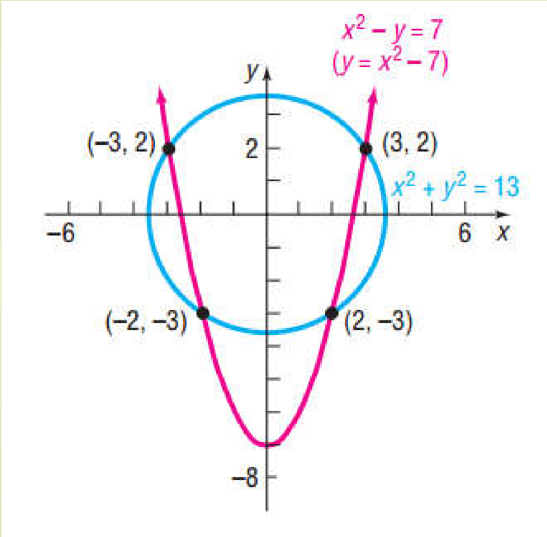
$$\begin{cases} 3x - y = -2 & (1) \\ 2x^2 - y = 0 & (2) \end{cases}$$



Solve a System of Nonlinear Equations Using Elimination

Example 2

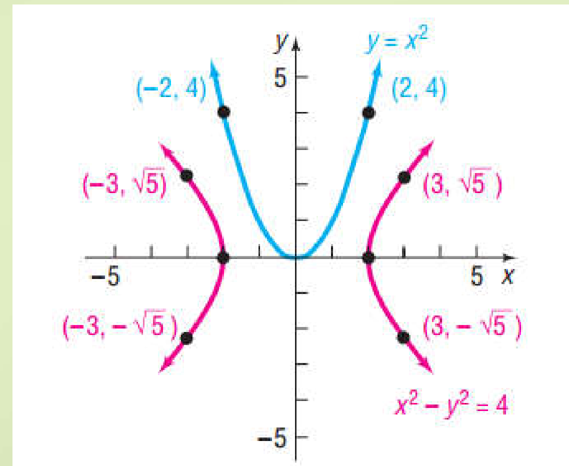
Solve: $\begin{cases} x^2 + y^2 = 13 & (1) \\ x^2 - y = 7 & (2) \end{cases}$



Solve a System of Nonlinear Equations Using Elimination

Example 3:

$$\text{Solve: } \begin{cases} x^2 - y^2 = 4 & (1) \\ y = x^2 & (2) \end{cases}$$



Solve a System of Nonlinear Equations Using Elimination

Example 4

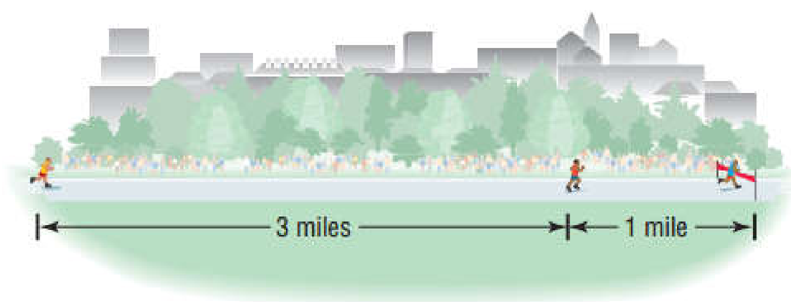
Solve: $\begin{cases} x^2 + x + y^2 - 3y + 2 = 0 & (1) \\ x + 1 + \dfrac{y^2 - y}{x} = 0 & (2) \end{cases}$

Example 5

Solve: $\begin{cases} 3xy - 2y^2 = -2 & (1) \\ 9x^2 + 4y^2 = 10 & (2) \end{cases}$

Solve a System of Nonlinear Equations Using Elimination

Running a long distance



In a 50-mile race, the winner crosses the finish line 1 mile ahead of the second-place runner and 4 miles ahead of the third-place runner. Assuming that each runner maintains a constant speed throughout the race, by how many miles does the second-place runner beat the third-place runner?

Systems of Inequalities

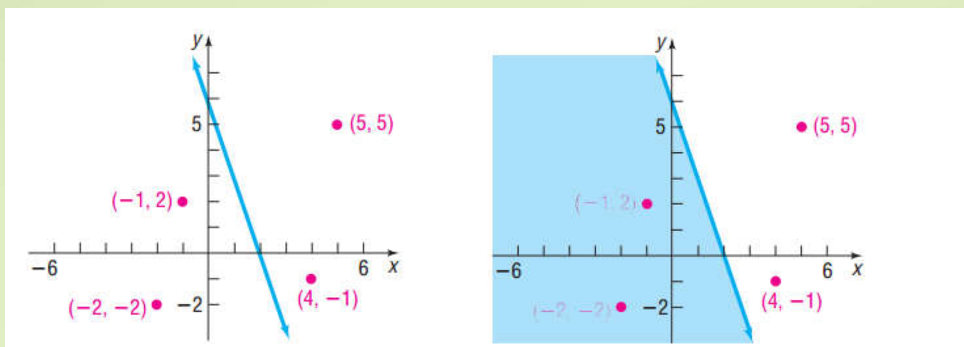
- Graph an Inequality
- Graph a System of Inequalities

Graph an Inequality

An inequality in two variables x and y is **satisfied** by an ordered pair (a, b) if, when x is replaced by *a* and *y* by *b*, a true statement results. The **graph of an inequality in two variables** x and *y* consists of all points (x, y) whose coordinates satisfy the inequality.

Example

Graph the linear inequality: $3x + y \leq 6$



(a) $3x + y = 6$

(b) Graph of $3x + y \leq 6$

Graphing an inequality

Steps for Graphing an Inequality

STEP 1: Replace the inequality symbol by an equal sign and graph the resulting equation. If the inequality is strict, use dashes; if it is nonstrict, use a solid mark. This graph separates the xy-plane into two or more regions.

STEP 2: In each region, select a test point P.

(a) If the coordinates of P satisfy the inequality, so do all the points in that region. Indicate this by shading the region.

(b) If the coordinates of P do not satisfy the inequality, none of the points in that region do.

Example

Graph: $x^2 + y^2 \leq 4$

Linear Inequalities

Linear Inequalities

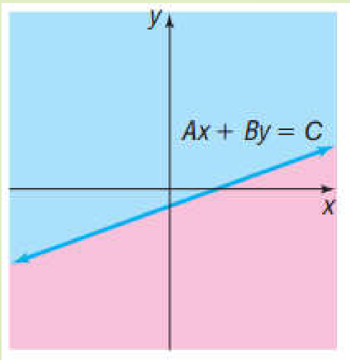
A linear inequality is an inequality in one of the forms

$$Ax + By < C \qquad Ax + By > C \qquad Ax + By \leq C \qquad Ax + By \geq C$$

where A and B are not both zero.

The graph of the corresponding equation of a linear inequality is a line that separates the xy-plane into two regions, called **half-planes.** See Figure

As shown, $Ax + By = C$ is the equation of the boundary line, and it divides the plane into two half-planes: one for which $Ax + By < C$ and the other for which $Ax + By > C$. Because of this, for linear inequalities, only one test point is required.



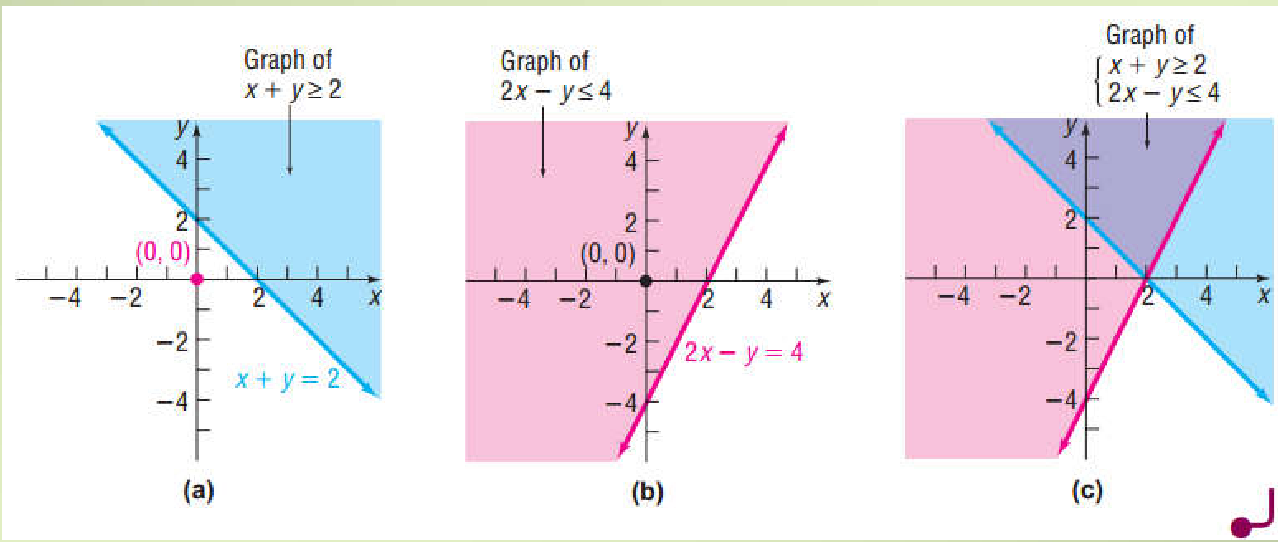
Graph a System of Inequalities

The **graph of a system of inequalities** in two variables x and y is the set of all points (x, y) that simultaneously satisfy *each* inequality in the system. The graph of a system of inequalities can be obtained by graphing each inequality individually and then determining where, if at all, they intersect.

Graph a System of Inequalities

Example

Graph the system: $\begin{cases} x + y \geq 2 \\ 2x - y \leq 4 \end{cases}$



Graph a System of Inequalities

Examples

Graph the systems:

(a) $\begin{cases} 2x - y \geq 0 \\ 2x - y \geq 2 \end{cases}$

(b) $\begin{cases} x + 2y \leq 2 \\ x + 2y \geq 6 \end{cases}$

Linear Programming

- Set up a Linear Programming Problem
- Solve a Linear Programming Problem

Set up a Linear Programming Problem

DEFINITION

A **linear programming problem** in two variables x and y consists of maximizing (or minimizing) a linear objective function

$$z = Ax + By \qquad A \text{ and } B \text{ are real numbers, not both } 0$$

subject to certain conditions, or constraints, expressible as linear inequalities in x and y.

In general, every linear programming problem has two components:

- A linear objective function that is to be maximized or minimized
- A collection of linear inequalities that must be satisfied simultaneously

Set up a Linear Programming Problem

Financial Planning

A retired couple has up to \$25,000 to invest. As their financial adviser, you recommend that they place at least \$15,000 in Treasury bills yielding 2% and at most \$5000 in corporate bonds yielding 3%. Develop a model that can be used to determine how much money should be placed in each investment so that income is maximized.

If I represents income, then $I = 0.02x + 0.03y$. This linear expression is called the objective function.

This linear programming problem may be modeled as "Maximize $I = 0.02x + 0.03y$ " subject to the conditions that

$$\begin{cases} x \geq 0, y \geq 0 \\ x + y \leq 25 \\ x \geq 15 \\ y \leq 5 \end{cases}$$

Solve a Linear Programming Problem

To maximize (or minimize) the quantity $z = Ax + By$, we need to identify points (x, y) that make the expression for z the largest (or smallest) possible. But not all points (x, y) are eligible; only those that also satisfy each linear inequality (constraint) can be used. We refer to each point (x, y) that satisfies the system of linear inequalities (the constraints) as a **feasible point.** In a linear programming problem, we seek the feasible point(s) that maximizes (or minimizes) the objective function.

DEFINITION A **solution** to a linear programming problem consists of a feasible point that maximizes (or minimizes) the objective function, together with the corresponding value of the objective function.

Solve a Linear Programming Problem

THEOREM

Location of the Solution of a Linear Programming Problem

If a linear programming problem has a solution, it is located at a corner point of the graph of the feasible points.

If a linear programming problem has multiple solutions, at least one of them is located at a corner point of the graph of the feasible points.

In either case, the corresponding value of the objective function is unique.

Procedure for Solving a Linear Programming Problem

STEP 1: Write an expression for the quantity to be maximized (or minimized). This expression is the objective function.

STEP 2: Write all the constraints as a system of linear inequalities and graph the system.

STEP 3: List the corner points of the graph of the feasible points.

STEP 4: List the corresponding values of the objective function at each corner point. The largest (or smallest) of these is the solution.

Solve a Linear Programming Problem

Example

Minimize the expression

$$z = 2x + 3y$$

subject to the constraints

$$y \le 5 \qquad x \le 6 \qquad x + y \ge 2 \qquad x \ge 0 \qquad y \ge 0$$

Solve a Linear Programming Problem

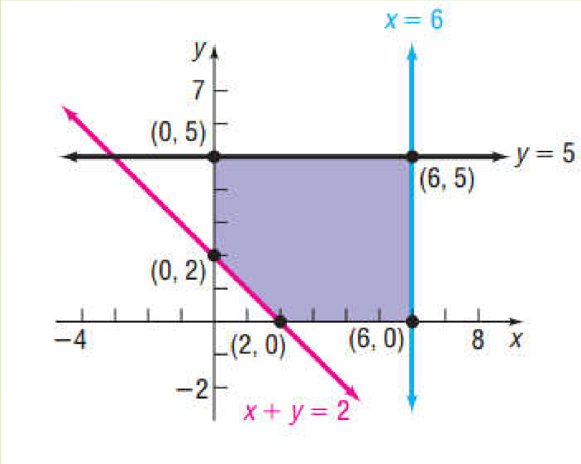
STEP 1: The objective function is $z = 2x + 3y$

STEP 2: We seek the smallest value of z that can occur if x and y are solutions of the system of linear inequalities

$$\begin{gathered} x \leq 6, \\ y \leq 5, \\ x + y \geq 2, \\ x \geq 0, \\ y \geq 0 \end{gathered}$$

Solve a Linear Programming Problem

- STEP 3: The graph of this system (the set of feasible points) is shown as the shaded region in Figure . We have also plotted the corner points.



Solve a Linear Programming Problem

STEP 4: Table lists the corner points and the corresponding values of the objective function. From the table, we can see that the minimum value of z is 4, and it occurs at the point $(2,0)$

Corner Point (x, y)	Value of the Objective Function $z = 2x + 3y$
(0, 2)	$z = 2(0) + 3(2) = 6$
(0, 5)	$z = 2(0) + 3(5) = 15$
(6, 5)	$z = 2(6) + 3(5) = 27$
(6, 0)	$z = 2(6) + 3(0) = 12$
(2, 0)	$z = 2(2) + 3(0) = 4$