TROY UNIVERSITY PROGRAM AT HUST

Chapter 8 – Sequences; Induction; the Binomial Theorem

MTH112, PRE-CALCULUS ALGEBRA

DR. DOAN DUY TRUNG

Outline

- Sequences
- Arithmetic Sequences
- Geometric Sequences; Geometric Series
- Mathematical Induction
- The Binomial Theorem

Sequences

- Write the First Several Terms of a Sequence
- Write the Terms of a Sequence Defined by a Recursive Formula
- Use Summation Notation
- Find the Sum of a Sequence

Sequence

DEFINITION A **sequence** is a function whose domain is the set of positive integers.

Example 1:

For the sequence $f(n) = \dfrac{1}{n}$, we write

$$\underbrace{a_1 = f(1) = 1,}_{\text{first term}} \quad \underbrace{a_2 = f(2) = \frac{1}{2},}_{\text{second term}} \quad \underbrace{a_3 = f(3) = \frac{1}{3},}_{\text{third term}} \quad \underbrace{a_4 = f(4) = \frac{1}{4},\ldots}_{\text{fourth term}} \quad \underbrace{a_n = f(n) = \frac{1}{n},\ldots}_{\text{n}^{\text{th}}\text{ term}}$$

Example 2:

Write down the first six terms of the following sequence and graph it.

$$\{a_n\} = \left\{\frac{n-1}{n}\right\}$$

The Factorial Symbol

Some sequences in mathematics involve a special product called a *factorial*.

DEFINITION

If $n \geq 0$ is an integer, the **factorial symbol $n!$** is defined as follows:

$$\boxed{\begin{aligned} 0! &= 1 \qquad 1! = 1 \\ n! &= n(n-1)\cdot \cdots \cdot 3 \cdot 2 \cdot 1 \qquad \text{if } n \geq 2 \end{aligned}}$$

Example

Write down the first five terms of the following recursively defined sequence.

$$s_1 = 1, \qquad s_n = n s_{n-1}$$

Use Summation Notation

Summation Notation

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

Example 1

Write out each sum.

(a) $\sum_{k=1}^{n} \frac{1}{k}$

(b) $\sum_{k=1}^{n} k!$

Example 2

Express each sum using summation notation.

(a) $1^2 + 2^2 + 3^2 + \cdots + 9^2$

(b) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}$

Properties of Sequences

THEOREM

Properties of Sequences

If $\{a_n\}$ and $\{b_n\}$ are two sequences and c is a real number, then:

$$\sum_{k=1}^{n} (ca_k) = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c\sum_{k=1}^{n} a_k \quad \textbf{(1)}$$

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \quad \textbf{(2)}$$

$$\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k \quad \textbf{(3)}$$

$$\sum_{k=j+1}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{j} a_k \quad \text{where } 0 < j < n \quad \textbf{(4)}$$

Formula for Sum of Sequences

THEOREM

Formulas for Sums of Sequences

$$\sum_{k=1}^{n} c = \underbrace{c + c + \cdots + c}_{n \text{ terms}} = cn \qquad c \text{ is a real number} \tag{5}$$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \tag{8}$$

Finding the Sum of Sequences

Example

Find the sum of each sequence.

(a) $\sum_{k=1}^{5} (3k)$

(b) $\sum_{k=1}^{10} (k^3 + 1)$

(c) $\sum_{k=1}^{24} (k^2 - 7k + 2)$

(d) $\sum_{k=6}^{20} (4k^2)$

Arithmetic Sequences

- Determine If a Sequence Is Arithmetic
- Find a Formula for an Arithmetic Sequence
- Find the Sum of an Arithmetic Sequence

Determine If a Sequence Is Arithmetic

DEFINITION An **arithmetic sequence*** may be defined recursively as $a_1 = a$, $a_n - a_{n-1} = d$, or as

$$a_1 = a, \qquad a_n = a_{n-1} + d \tag{1}$$

where $a_1 = a$ and d are real numbers. The number a is the first term, and the number d is called the **common difference.**

Example 1

Show that the following sequence is arithmetic. Find the first term and the common difference.

$$\{s_n\} = \{3n + 5\}$$

Example 2

Show that the sequence $\{t_n\} = \{4 - n\}$ is arithmetic. Find the first term and the common difference.

Find a Formula for an Arithmetic Sequence

THEOREM **nth Term of an Arithmetic Sequence**

For an arithmetic sequence $\{a_n\}$ whose first term is a_1 and whose common difference is d, the nth term is determined by the formula

$$\boxed{a_n = a_1 + (n - 1)d \qquad \textbf{(2)}}$$

Example 1

Find the forty-first term of the arithmetic sequence: 2, 6, 10, 14, 18, . . .

Example 2

The eighth term of an arithmetic sequence is 75, and the twentieth term is 39.

(a) Find the first term and the common difference.

(b) Give a recursive formula for the sequence.

(c) What is the nth term of the sequence?

Find the Sum of an Arithmetic Sequence

THEOREM

Sum of the First *n* Terms of an Arithmetic Sequence

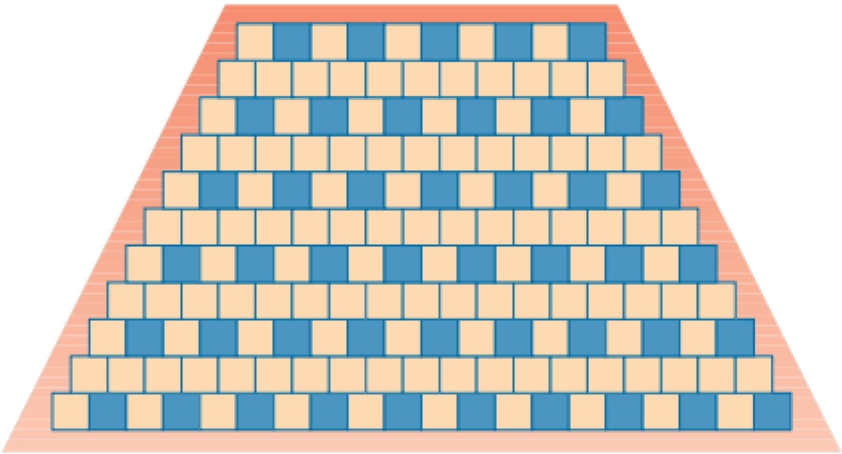
Let $\{a_n\}$ be an arithmetic sequence with first term a_1 and common difference d. The sum S_n of the first n terms of $\{a_n\}$ may be found in two ways:

$$
\boxed{
\begin{aligned}
S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\
&= \sum_{k=1}^{n} [a_1 + (k-1)d] = \frac{n}{2}[2a_1 + (n-1)d] && \textbf{(3)} \\
&= \frac{n}{2}(a_1 + a_n) && \textbf{(4)}
\end{aligned}
}
$$

Find the Sum of an Arithmetic Sequence

Example

A ceramic tile floor is designed in the shape of a trapezoid 20 feet wide at the base and 10 feet wide at the top. See Figure 7. The tiles, 12 inches by 12 inches, are to be placed so that each successive row contains one less tile than the preceding row. How many tiles will be required?



Geometric Sequences; Geometric Series

- Determine If a Sequence Is Geometric
- Find a Formula for a Geometric Sequence
- Find the Sum of a Geometric Sequence
- Determine Whether a Geometric Series Converges or Diverges
- Solve Annuity Problems

Determine If a Sequence Is Geometric

DEFINITION A **geometric sequence*** may be defined recursively as $a_1 = a$, $\frac{a_n}{a_{n-1}} = r$, or as

$$\boxed{a_1 = a, \qquad a_n = ra_{n-1}} \quad \textbf{(1)}$$

where $a_1 = a$ and $r \neq 0$ are real numbers. The number a_1 is the first term, and the nonzero number r is called the **common ratio.**

Example 1

Show that the following sequence is geometric.

$$\{s_n\} = 2^{-n}$$

Example 2

Show that the following sequence is geometric.

$$\{t_n\} = \{3 \cdot 4^n\}$$

Find the first term and the common ratio.

Find a Formula for a Geometric Sequence

THEOREM

nth Term of a Geometric Sequence

For a geometric sequence $\{a_n\}$ whose first term is a_1 and whose common ratio is r, the nth term is determined by the formula

$$\boxed{a_n = a_1 r^{n-1} \qquad r \neq 0 \qquad \textbf{(2)}}$$

Example

(a) Find the nth term of the geometric sequence: $10, 9, \frac{81}{10}, \frac{729}{100}, \ldots$

(b) Find the ninth term of this sequence.

(c) Find a recursive formula for this sequence.

Find the Sum of a Geometric S

THEOREM

Sum of the First *n* Terms of a Geometric Sequence

Let $\{a_n\}$ be a geometric sequence with first term a_1 and common ratio r, where $r \neq 0, r \neq 1$. The sum S_n of the first n terms of $\{a_n\}$ is

$$\boxed{\begin{aligned} S_n &= a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} = \sum_{k=1}^{n} a_1r^{k-1} \\ &= a_1 \cdot \frac{1 - r^n}{1 - r} \qquad r \neq 0, 1 \end{aligned} \qquad \textbf{(3)}}$$

Example

Find the sum S_n of the first n terms of the sequence $\left\{\left(\frac{1}{2}\right)^n\right\}$;

Determine Whether a Geometric Series Converges or Diverges

DEFINITION

An infinite sum of the form

$$a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + \cdots$$

with first term a_1 and common ratio r, is called an **infinite geometric series** and is denoted by

$$\sum_{k=1}^{\infty} a_1 r^{k-1}$$

Determine Whether a Geometric Series Converges or Diverges

If this finite sum S_n approaches a number L as $n \to \infty$, we say the infinite geometric series $\sum_{k=1}^{\infty} a_1 r^{k-1}$ **converges.** We call L the **sum of the infinite geometric series** and we write

$$L = \sum_{k=1}^{\infty} a_1 r^{k-1}$$

If a series does not converge, it is called a **divergent series.**

THEOREM

Convergence of an Infinite Geometric Series

If $|r| < 1$, the infinite geometric series $\sum_{k=1}^{\infty} a_1 r^{k-1}$ converges. Its sum is

$$\boxed{\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1 - r}} \quad \textbf{(7)}$$

Determine Whether a Geometric Series Converges or Diverges

Example

Determine if the geometric series

$$\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^{k-1} = 2 + \frac{4}{3} + \frac{8}{9} + \cdots$$

converges or diverges. If it converges, find its sum.

Mathematical Induction

THEOREM **The Principle of Mathematical Induction**

Suppose that the following two conditions are satisfied with regard to a statement about natural numbers:

CONDITION I: The statement is true for the natural number 1.

CONDITION II: If the statement is true for some natural number k, it is also true for the next natural number $k + 1$.

Then the statement is true for all natural numbers.

Example 1

Using Mathematical Induction

Show that the following statement is true for all natural numbers n.

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Example 2

Show that $3^n - 1$ is divisible by 2 for all natural numbers n.

The Binomial Theorem

- Evaluate $\binom{n}{j}$
- Use the Binomial Theorem (p. 972)

Evaluate $\binom{n}{j}$

DEFINITION

If j and n are integers with $0 \le j \le n$, the symbol $\binom{n}{j}$ is defined as

$$\boxed{\binom{n}{j} = \frac{n!}{j!(n-j)!} \qquad \textbf{(1)}}$$

$$\boxed{\binom{n}{0} = 1 \quad \binom{n}{1} = n \quad \binom{n}{n-1} = n \quad \binom{n}{n} = 1}$$

Example 1

Find:

(a) $\binom{3}{1}$ (b) $\binom{4}{2}$ (c) $\binom{8}{7}$

Use the Binomial Theorem

THEOREM

Binomial Theorem

Let x and a be real numbers. For any positive integer n, we have

$$\boxed{\begin{aligned}(x+a)^n &= \binom{n}{0}x^n + \binom{n}{1}ax^{n-1} + \cdots + \binom{n}{j}a^j x^{n-j} + \cdots + \binom{n}{n}a^n \\ &= \sum_{j=0}^{n}\binom{n}{j}x^{n-j}a^j \qquad \textbf{(2)}\end{aligned}}$$

Example

Use the Binomial Theorem to expand $(x+2)^5$.

Use the Binomial Theorem

Based on the expansion of $(x + a)^n$, the term containing x^j is

$$\boxed{\binom{n}{n-j} a^{n-j} x^j \qquad (3)}$$

Example 1

Find the coefficient of y^8 in the expansion of $(2y + 3)^{10}$.

Example 2

Find the sixth term in the expansion of $(x + 2)^9$.

Use the Binomial Theorem

THEOREM If n and j are integers with $1 \le j \le n$, then

$$\boxed{\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j} \qquad \textbf{(4)}}$$