Linear Algebra

Chapter 1 - Sets, Maps and Complex Numbers

Dr. Trung Duy Doan

Faculty of Mathematics and Informatics
Hanoi University of Science and Technology

trung.doanduy@hust.edu.vn

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Overview

1. Sets
2. Maps
3. Algebraic Structures
4. Field of Complex Numbers

Concepts of sets

1) A set is a well-defined collection of distinct objects. The objects that make up a set (also known as the set's elements or members) can be anything: numbers, people, letters of the alphabet, other sets, and so on.
2) Sets are conventionally denoted with capital letters.

Example

- A set of students in an university
- A set of Vietnamese football club
- The set of real number, denoted by $\mathbb{R}$

Basic notations

1) Let E be a set. If a is an element of the set E, then we denote by $a \in E$. Otherwise, we denote by $a \notin E$.
2) The empty set is a set containing no objects. It is written as pair of curly braces with nothing inside $\{\}$ or by using the symbol $\emptyset$.
3) There are two ways of describing, or specifying the members of a set.

i) The first way is by by intensional definition, using a rule or semantic description:

1. A is the set whose members are the first four positive integers.
2. B is the set of colors of the French flag.

ii) The second way is by extension, that is, listing each member of the set.

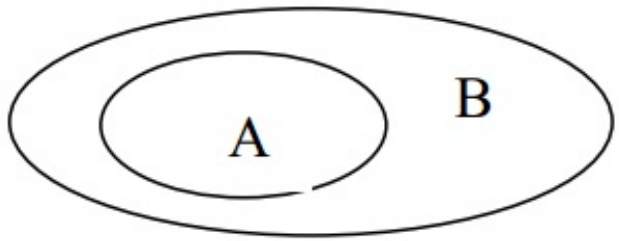
1. $C = \{4, 2, 1, 3\}$
2. $D = \{blue, white, red\}$.

Subset, two equal sets

1) Subset

By logical expression: $\boxed{A \subset B \Leftrightarrow (\forall x, (x \in A) \rightarrow (x \in B))}$

By Venn diagram:



Note that $\emptyset \subset A, \forall A$

2) Two equal sets

By logical expression: $\boxed{A = B \Leftrightarrow A \subset B \wedge B \subset A}$

Example: $A = \{1, 2, 3, 4\}, B = \{1, 2, 3, 4, 5, 6\}, C = \{x \in \mathbb{N} | 0 < x < 5\}$

Hence, $A \subset B$ and $A = C$

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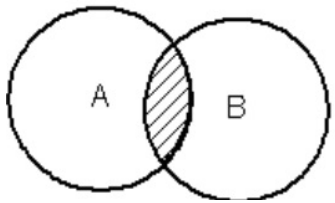
Operations on sets

Let A, B be two sets.

1. Intersection $A \cap B = \{x | x \in A \text{ and } x \in B\}$

$$\begin{cases} x \in A \cap B \Leftrightarrow x \in A \text{ and } x \in B \\ x \notin A \cap B \Leftrightarrow x \notin A \text{ or } x \notin B \end{cases}$$

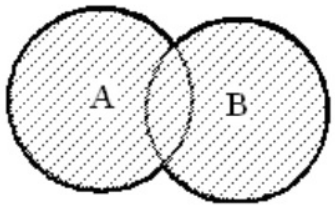
By Venn diagram:



2. Union $A \cup B = \{x | x \in A \text{ or } x \in B\}$

$$\begin{cases} x \in A \cup B \Leftrightarrow x \in A \text{ or } x \in B \\ x \notin A \cup B \Leftrightarrow x \notin A \text{ and } x \notin B \end{cases}$$

By Venn diagram:



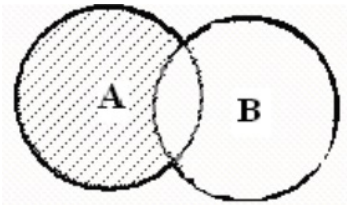
Operations on sets

Let A, B be two sets.

1) The difference of two sets $A \setminus B = \{x | x \in A \text{ and } x \notin B\}$

$$\begin{cases} x \in A \setminus B \Leftrightarrow x \in A \text{ and } x \notin B \\ x \notin A \setminus B \Leftrightarrow x \notin A \text{ or } x \in B \end{cases}$$

By Venn diagram:



2) The symmetric difference of two sets $A \triangle B = A \setminus B$ and $B \setminus A$

3) Complement $\overline{A}$
If $A \subset X$, then $\overline{A} = X \setminus A$ and is called absolute complement of A

Operations on sets

Exercise

Let

$$A = \{x \in R | x^2 - 4x + 3 \leq 0\}, B = \{x \in R | |x - 1| \leq 1\}$$

and

$$C = \{x \in R | x^2 - 5x + 6 \leq 0\}$$

Determine

a) $(A \cup B) \cap C$

b) $(A \cap B) \cup C$

c) $(A \setminus B) \setminus C$

d) $(A \cap B) \setminus C$

e) $(A \cup B) \setminus C$

Properties of sets

1) Commutative law:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

2) Associative law:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

3) Distributive law:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

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Properties of sets

4) Property of the complement
   If $A, B \subset X$, then $A \setminus B = A \cap \overline{B}$
5) De Moorgan's law

$$\overline{A \cap B} = \overline{A} \cup \overline{B}, \quad \overline{\cap A_i} = \cup \overline{A_i}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}, \quad \overline{\cup A_i} = \cap \overline{A_i}$$

The Cartesian product

Definition

1. Let A, B be two sets. The Cartesian product of A and B, denoted by $A \times B$, is given by

$$A \times B = \{(x, y) | (x \in A) \text{ and } (y \in B)\}$$

.

2. Let $A_1, A_2 \ldots A_n$ be the sets. The Cartesian product of $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$ is given by

$$A_1 \times A_2 \times \ldots \times A_n = \{(x_1, x_2, \ldots, x_n) | x_i \in A_i, \text{ where } i \in [n]\}$$

Note that if $A_1 = A_2 = \ldots = A_n = A$, then

$$A_1 \times A_2 \times \ldots \times A_n = A^n$$

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The Cartesian product

Definition

1. If $(a, b), (c, d)$ be two elements of $A \times B$, then
$$(a, b) = (c, d) \Leftrightarrow \begin{cases} a = c \\ b = d \end{cases}$$
2. If $(a_1, \ldots, a_n), (b_1, \ldots, b_n)$ be two elements of $A_1 \times A_2 \times \ldots \times A_n$, then
$$(a_1, \ldots, a_n) = (b_1, \ldots, b_n) \Leftrightarrow \begin{cases} a_1 = b_1 \\ \ldots \\ a_n = b_n \end{cases}$$

Three possible methods to prove set equality

1. Double inclusion
2. By using the properties
3. Membership table

The set equality

Four possible methods to prove two sets are equal

1. Double inclusion
2. By using the properties
3. Membership table
4. By using Venn diagram

Examples

Prove that $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$

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Overview

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4. Field of Complex Numbers

Definition

Let X, Y be two nonempty sets. A map f from domain X to codomain Y is an assignment of every element $x \in X$ to an unique element $y \in Y$. We denote by

$$f : X \to Y$$

or

$$x \mapsto y = f(x)$$

Examples

1. $f : \mathbb{R} \mapsto \mathbb{R}$; $f(x) = \sin x$, $forallx \in \mathbb{R}$
2. $f : X \mapsto X$; $f(x) = x \ \forall x \in X$. This is called the identity map on the set X, denoted by I_X

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Image, Preimage

Definition

Let $f : X \mapsto Y$ be a map and $A \subset X$, $B \subset Y$.

1) The image

$$f(A) = \{y \in Y | y = f(x) \text{ and } x \in A\} = \{f(x) | x \in A\}$$

2) The preimage

$$f^{-1}(B) = \{x \in X | f(x) \in B\}$$

$$\boxed{x \in f^{-1}(B) \Leftrightarrow f(x) \in B}$$

Image, Preimage

Examples

Ex1. Given a map $f : \mathbb{R} \setminus \{-1\} \mapsto \mathbb{R}, f(x) = \dfrac{2x+3}{x+1}$. Compute

a) $f((-1;0])$
b) $f^{-1}([4;7])$

Ex2. Let $f : X \mapsto Y$.

a) $f(A \cup B) = f(A) \cup f(B)$; $A, B \subset X$
b) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$; $A, B \subset Y$.

Ex3. Let $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$, $f(x,y) = (2x, 2y)$ and $A = \{(x,y) \in \mathbb{R}^2 | (x-4)^2 + y^2 = 4\}$. Find $f(A), f^{-1}(A)$.

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Maps

Injective, Surjective, Bijective maps

Let $f : X \mapsto Y$ be a map

1) The map f is called injective
   - i) $\forall x_1, x_2 \in X$ if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$, or
   - ii) $\forall x_1, x_2 \in X$ if $f(x_1) = f(x_2)$, then $x_1 = x_2$, or
   - iii) $\forall y \in Y$, the equation $y = f(x)$ has at most one solution
2) The map f is called surjective
   - i) $Y = f(X)$, or
   - ii) $\forall y \in Y$, $\exists x \in X$, $y = f(x)$, or
   - iii) $\forall y \in Y$, the equation $y = f(x)$ has at least one solution
3) The map f is called bijective
   - i) if it is both injective and surjective, or
   - ii) the equation $y = f(x)$ has a unique solution.

Injective, Surjective, Bijective maps

Examples

Which of the following maps are injective, surjective, bijective?

1. $f : \mathbb{R} \mapsto \mathbb{R}$, $f(x) = 3 - 2x$
2. $f : (1, +\infty) \mapsto (-1, +\infty)$, $f(x) = x^2 - 2x$
3. $f : \mathbb{R} \mapsto \mathbb{R}$, $f(x) = 3x - 2|x|$
4. $f : \mathbb{R} \setminus \{0\} \mapsto \mathbb{R}$, $f(x) = \dfrac{1}{x}$

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Composition of maps

Definition

Let $f : X \mapsto Y$, $g : Y \mapsto Z$ be two maps. The map $h : X \mapsto Z$ which is defined by $h(x) = g(f(x))$, $\forall x \in X$ is called the composition of g and f. We denote by $h = g \circ f$.

Examples

Given two following maps $f : \mathbb{R} \mapsto \mathbb{R}$, $x \mapsto f(x) = \dfrac{x^2 + 1}{3}$ and $g : \mathbb{R} \mapsto \mathbb{R}$, $x \mapsto g(x) = x^3 + 2$. Determine $g \circ f$ and $f \circ g$?

Inverse map

Definition

Let $f : X \mapsto Y$ be a bijective map. The map $f^{-1} : Y \mapsto X$ such that $f^{-1}(y) = x \Leftrightarrow f(x) = y$ is called the inverse map of f.

Properties

1. $h \circ (g \circ f) = (h \circ g) \circ f$
2. $f \circ Id = Id \circ f = f$
3. $(f^{-1})^{-1} = f$

1. $f \circ f^{-1} = f^{-1} \circ f = Id$
2. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Injectivity, Surjectivity, Bijectivity of the composition of maps

- If f and g are injective, then $g \circ f$ is injective;
- If f and g are surjective, then $g \circ f$ is surjective;
- If f and g are bijective, then $g \circ f$ is bijective.

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Composition of maps, inverse map

Exercises

Let $f : X \mapsto Y$ and $g : Y \mapsto Z$ be two maps. Prove that

a) if f is surjective and $g \circ f$ is injective, then g is injective.

b) give an example to show that $g \circ f$ is injective, but g is not.

c) if g is injective and $g \circ f$ is surjective, then f is surjective

d) give an example to show that $g \circ f$ is surjective, but f is not.

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Binary operations

Definition

Let G be a set. A binary operation is a map of sets:

$$* : G \times G \to G$$

For ease of notation we write $*(a, b) = a * b \ \forall a, b \in G$

When we are talking about a set G, together with a fixed binary operation $*$, we often write $(G, *)$

Example

Let $B = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$. Determine whether the operations of addition and multiplication are the binary operation on B.

Binary operations

Properties of binary operations

Let $*$ be a binary operation on G

1) $*$ is said to be commutative if $x * y = y * x \ \forall x, y \in G$.
2) $*$ is said to be associative if $(x * y) * z = x * (y * z) \ \forall x, y \in G$.

Examples

Determine whether the following binary operation $*$ is commutative, associative?

a) $x * y = xy - 2 \ \forall x, y \in \mathbb{R}$

b) $x * y = x^y \ \forall x, y \in \mathbb{R}^+$

c) $(x_1, x_2) * (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \ \forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$

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Groups

Definition

A group is a set G, together with a binary operation $*$, such that the following hold:

1) (Associativity): $(a * b) * c = a * (b * c)\ \forall a, b, c \in G$
2) (Existence of identity): $\exists e \in G$ such that $a * e = e * a = a\ \forall a \in G$
3) (Existence of inverses): Given $a \in G$, $\exists b \in G$ such that $a * b = b * a = e$. The inverse element is denoted by a'

If $(G, *)$ satisfies only (1)), then G is called a semigroup.

Definition

A group $(G, *)$ is called Abelian if it also satisfies $x * y = y * x\ \forall x, y \in G$

Properties of group

Properties

1) the identity element e is unique
2) the inverse element a' of a is unique

Examples

Which of the following are groups?

1) $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{N}, +)$
2) $(\mathbb{A} = \{\pm 1\}, .)$, $(\mathbb{Q}^*, .)$, $(\mathbb{R}^*, .)$

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Rings

Definition

A ring is a set R with two binary operations $*$ and $\circ$ such that:

1) $(R, *)$ is an Abelian group
2) The binary operation $\circ$ is associative $\forall x, y, z \in R$:

$$(x \circ y) \circ z = x \circ (y \circ z)$$

3) $*$ and $\circ$ are related by the distributive law $\forall x, y, z \in R$:

$$(x * y) \circ z = x \circ z * y \circ z$$

$$x \circ (y * z) = x \circ y * x \circ z$$

i) The ring is called commutative if the binary operation $\circ$ is commutative.

ii) If ring R has the identity for the binary operation $\circ$, then R is called a ring with identity.

Rings

Examples

a) $(\mathbb{Z}, +, .)$, $(\mathbb{Q}, +, .)$ are commutative rings with identity.

b) $(\mathbb{N}, +, .)$ is not a ring.

Exercises

Determine whether the following sets with addition and multiplication are rings ?

i) $X = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$

ii) $X = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$

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Fields

Definition

Let $(F, *, \circ)$ be a ring and e be the identity on $*$. We say that F is a field if

i) F is a commutative ring with identity 1 on $\circ$, where $1 \neq e$

ii) $\forall a \in F, a \neq e$, there always exist the inverse element x^{-1} such that $x \circ x^{-1} = 1$

Examples

a) $(\mathbb{Q}, +, .)$ is a field.

b) $(\mathbb{Z}, +, .)$ is not a field.

Fields

Exercises

Determine whether the following sets with addition and multiplication are fields?

i) $X = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$

ii) $X = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$

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Introduction

Let $\mathbb{C} = \mathbb{R} \times \mathbb{R} = \{(a; b) | a, b \in \mathbb{R}\}$ be a set such that:

i) $(a; b) = (c; d) \Leftrightarrow \begin{cases} a = c \\ b = d \end{cases}$

ii) Two binary operations $+$ and $.$ on $\mathbb{C}$ are defined as follows:

a) $(a; b) + (c; d) = (a + c; b + d)$

b) $(a; b).(c; d) = (ac - bd; ad + bc)$

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Field of the complex numbers

Proposition

The set $\mathbb{C}$ with two above binary operations $+$ and . is a field. We call it the field of complex numbers.

Remarks

i) The additive identity is $(0;0)$.

ii) The multiplicative identity is $(1;0)$.

iii) The inverse element of $(a;b) \neq (0;0)$ is $(a;b)^{-1} = (\frac{a}{a^2+b^2}; \frac{-b}{a^2+b^2})$.

The canonical form of complex numbers

Let $i = (0;1)$. Hence,

i) $z = (a;b) = (a;0) + (0;b) = (a;0) + (b;0)(0;1) = a + bi$

ii) $i^2 = (0;1)(0;1) = (-1;0) = -1$

Definition

For every complex number $z \in \mathbb{C}$, $z = a + bi$, where $a, b \in \mathbb{R}$ and $i^2 = -1$. Now $z = a + bi$ is called the canonical form of the complex number z.

i) The real number a is called the real part of z and denoted by $a = Re(z)$.

ii) The real number b is called the imaginary part of z and denoted by $b = Im(z)$.

iii) i is called the imaginary unit.

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The canonical form of complex numbers

Operations of complex numbers in the canonical form

i) Addition: $(a+bi)+(c+di)=(a+c)+(b+d)i$

ii) Subtraction: $(a+bi)-(c+di)=(a-c)+(b-d)i$

iii) Multiplication: $(a+bi)(c+di)=(ac-bd)+(ad+bc)i$

iv) Division: $\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{c^2+d^2}$, where $c^2+d^2\neq 0$

The conjugate and the modulus of a complex number

Let $z = a + bi$, where $a, b \in \mathbb{R}$, be a complex number

The conjugate of a complex number

The conjugate of z is defined by $\overline{z} = a - bi$

The modulus of a complex number

The modulus of z is defined by $|z| = \sqrt{a^2 + b^2}$

Remark

1) $|z| = |\overline{z}|$
2) $z + \overline{z} = 2Re(z) = 2a$
3) $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$
4) $\overline{\overline{z}} = z$
5) $z\overline{z} = |z|^2$

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Properties of a complex number

Properties

i) $z_1 + z_2 = z_2 + z_1$; $z_1 z_2 = z_2 z_1$

ii) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$; $z_1(z_2 z_3) = (z_1 z_2) z_3$

iii) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

iv) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$; $\overline{z_1 z_2} = \overline{z_1 z_2}$

v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

vi) $|z_1 z_2| = |z_1||z_2|$; $|z_1 + z_2| \leq |z_1| + |z_2|$

Examples

E1) Compute $A = \frac{1+2i}{4-3i} + \frac{1}{2i} - \frac{3}{4}$

E2) Given $|z_1| = 1$ and $\forall z_2 \neq z_1$. Prove that $\left|\frac{z_1 - z_2}{1 - \overline{z_1} z_2}\right| = 1$

The polar (trigonometric) form of complex numbers

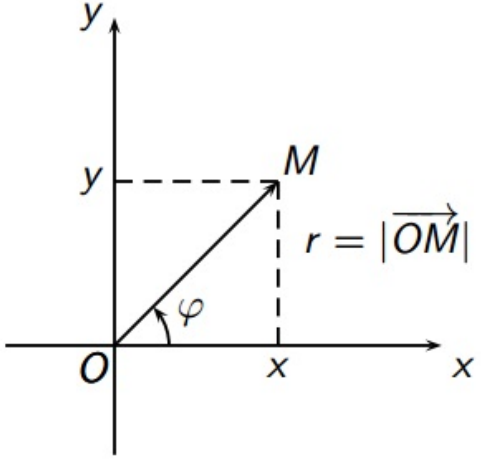
Each complex number $z = a + bi$ can be represented by a point $M(a, b)$ in the plane Oxy, where Ox is real axis and Oy is imaginary axis.

Let $\begin{cases} r = |\overrightarrow{OM}| \\ \varphi = (Ox, \overrightarrow{OM}) \end{cases}$. Now $z = a + bi = r(\cos\varphi + i\sin\varphi)$, where

i) $r = |z| = \sqrt{a^2 + b^2}$

ii) φ is called the Argument of z, denoted by $\mathrm{Arg}(z)$. Moreover,

$$\boxed{\cos\varphi = \frac{a}{\sqrt{a^2+b^2}};\ \sin\varphi = \frac{b}{\sqrt{a^2+b^2}}}$$



The polar (trigonometric) form of complex numbers

Examples

Determine the polar form of the following complex numbers

a) $z = \sqrt{3} + i$

b) $z = -\sqrt{3} - i$

c) $z = -2$

d) $z = 3i$

Operations of complex numbers in the polar form

Operations of complex numbers in the polar form

Let $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$, $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2$ be two complex numbers.

i) Multiplication: $z_1z_2 = r_1r_2(\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2))$
Hence $|z_1z_2| = |z_1||z_2|$, $\text{Arg}(z_1z_2)$=$\text{Arg}(z_1) + \text{Arg}(z_2)$.

ii) Division: $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2))$
Hence $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$, $\text{Arg}(\frac{z_1}{z_2})$=$\text{Arg}(z_1)-\text{Arg}(z_2)$.

Examples

Given $z_1 = 6(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$, $z_2 = 4(\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6})$. Compute $z_1.z_2$ and $\frac{z_1}{z_2}$

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Operations of complex numbers in the polar form

Remarks

If $z = r(\cos\varphi + i\sin\varphi)$, then

1. $\overline{z} = r(\cos(-\varphi) + i\sin(-\varphi))$
2. $z^{-1} = \frac{1}{r}(\cos(-\varphi) + i\sin(-\varphi))$

Exponentiation (Moivre's formula)

$$z = r(\cos\varphi + i\sin\varphi) \Rightarrow z^n = r^n(\cos n\varphi + i\sin n\varphi)$$

Clearly, $|z^n| = |z|^n$

Example

E1) Compute $z = (\sqrt{3} + i)^{2019}$

Operations of complex numbers in the polar form

Definition

n-roots of a complex number z is denoted by $\sqrt[n]{z}$. If $z = r(\cos\varphi + i\sin\varphi) \neq 0$, then

$$\sqrt[n]{z} = \sqrt[n]{r}(\cos\frac{\varphi + k2\pi}{n} + i\sin\frac{\varphi + k2\pi}{n}), k = \overline{0, n-1}$$

Each non-zero complex number has exactly n different n-roots.

Examples

Compute

E1) $\sqrt[3]{8(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})}$

E2) $\sqrt[3]{8}$

E3) $\sqrt{1+i}$

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Examples

E1) Compute the following complex numbers

a) $(1 + i\sqrt{3})^{2019}$

b) $(2 + i\sqrt{12})^5(\sqrt{3} - i)^{11}$

E2) Solve the following equations

a) $z^4 - 3iz^2 + 4 = 0$

b) $z^6 - 7z^3 - 8 = 0$

c) $\frac{(z+i)^4}{(z-i)^4} = 1$

d) $z^8(\sqrt{3} + i) = 1 - i$

The End