Linear Algebra

Part 2 - Matrices, Determinants, System of Linear equations

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Overview

1. Matrices
2. Determinant
3. Invertible matrix
4. Rank of a matrix
5. System of linear equations

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Matrices

Definition

A matrix is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1_n} \\ a_{21} & a_{22} & \dots & a_{2_n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1_n} \\ a_{21} & a_{22} & \dots & a_{2_n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

The numbers, symbols or expression in the matrix are called its entries or its elements.

The horizontal and vertical lines of entries in a matrix are called rows and columns, respectively.

The entry (element) in the i-th row and j-th column of a matrix A is denoted by a_{ij}

We denote by $A = [a_{ij}]_{mn}$

Matrices

Our lectures focuses on real and complex matrices, that is, matrices whose elements are real numbers or complex numbers, respectively.

Examples

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} i & 1+i \\ 2 & 4 \\ 1 & 2 \end{bmatrix}$$

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Matrices

Definition

i) The size of a matrix is defined by the number of rows and columns that it contains. A matrix with m rows and n columns is called an $m \times n$ matrix or m-by-n matrix.

ii) Matrices with a single row are called row vectors, and those with a single column are called column vectors.

iii) A matrix with the same number of rows and columns ($m = n$) is called a square matrix.

Matrices

Name	Size	Examples
Row vector	$1 \times n$	$\begin{bmatrix} 3 & -1 & 8 \end{bmatrix}$
Column vector	$m \times 1$	$\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$
Square matrix	$n \times n$	$\begin{bmatrix} 1 & -2 & 3 \\ 4 & 8 & 8 \\ 5 & 3 & 7 \end{bmatrix}$

Definition

Let $A = [a_{ij}]_{nn}$ be a square matrix

i) The elements $a_{11}, \ldots, a_{nn}$ form the main diagonal of a square matrix. They lie on the imaginary line that runs from the top left corner to the bottom right corner of the matrix.

ii) If all elements of A below (above) the main diagonal are zero, A is called an upper (lower) triangular matrix.

iii) If $a_{ij} = 0$, $\forall i \neq j$, then A is called a diagonal matrix.

Matrices

Name	Examples with $n = 3$
Diagonal	$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$
Lower triangular matrix	$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}$
Upper triangular matrix	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

Definition

The identity matrix I_n of size n is the n-by-n matrix in which all the elements on the main diagonal are equal to 1 ($a_{ii} = 1$) and all other elements are equal to 0 ($a_{ij} = 0, \forall i \neq j$).

$$I_1 = \begin{bmatrix} 1 \end{bmatrix}, I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ldots, I_n = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

Definition

A zero matrix or null matrix, denoted by 0, is a matrix all of whose entries are zero.

$$0_{1,1} = \begin{bmatrix} 0 \end{bmatrix}, 0_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 0_{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Equality of Matrices

Definition

We say that two matrices are equal if they have the same size and their entries are equal.

$$A = [a_{ij}]_{m\times n} = B[b_{ij}]_{m\times n} \Leftrightarrow a_{ij} = b_{ij}, \forall i, j$$

Consider three matrices shown below

$$A = \begin{bmatrix} 1 & x \\ y & -3 \end{bmatrix}, B = \begin{bmatrix} a & 7 \\ -8 & b \end{bmatrix}, C = \begin{bmatrix} 1 & 7 & z \\ m & n & p \end{bmatrix}$$

If $A = B$, then $\Leftrightarrow \begin{cases} a = 1 \\ b = -3 \\ x = 7 \\ y = -8 \end{cases}$ $C \neq A$ and $C \neq B$

Matrix addition

Definition

Let A, B be two matrices of the same size. The sum of A and B, denoted by $A + B$, is computed by adding the corresponding elements of A and B.

$$[a_{ij}]_{mn} + [b_{ij}]_{mn} = [a_{ij} + b_{ij}]_{mn}$$

Examples

$$\begin{bmatrix} 1 & 2 \\ 3 & -3 \\ 4 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 0 \\ 7 & 0 \end{bmatrix}$$

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Scalar Multiples

Definition

Let r be a scalar and A be a matrix. The scalar multiple rA is the matrix whose entries are r times the corresponding entries in A

$$\boxed{(-1)A = -A \Leftrightarrow A - B = A + (-1)B}$$

Examples

$$2\begin{bmatrix} 1 & 2 \\ 3 & -3 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & -6 \\ 8 & -8 \end{bmatrix};$$

$$\begin{bmatrix} 2 & 3 & -5 \\ -1 & 0 & -6 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & -4 \\ -3 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 3 \\ 5 & 4 & -6 \end{bmatrix}$$

Properties

Let A, B and C be three matrices of the same size, and r, s be scalars.

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

Matrix multiplication

Matrix multiplication

Let $A = [a_{ij}]_{m\times p}$ be an $m \times p$ matrix and $B = [b_{jk}]_{p\times n}$ be an $m \times n$ matrix. The matrix product $C = AB$ is an $m \times n$ matrix defined by $C = [c_{ik}]_{m\times n}$ such that

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{ip}b_{pk}, \forall i = \overline{1, m}; k = \overline{1, n}$$

$$\begin{bmatrix} \ldots & \ldots & \ldots \\ a_{i1} & \ldots & a_{ip} \\ \ldots & \ldots & \ldots \end{bmatrix} \begin{bmatrix} \ldots & b_{1k} & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & b_{pk} & \ldots \end{bmatrix} = \begin{bmatrix} \ldots & \ldots & \ldots \\ \ldots & c_{ik} & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}$$

Examples

$$\begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1.1 + (-2).3 & 1.2 + (-2).4 \\ 3.1 + (-1).3 & 3.2 + (-1).4 \\ 2.1 + (-3).3 & 2.2 + (-3).4 \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ 0 & 2 \\ -7 & -8 \end{bmatrix}$$

Properties of matrix multiplication

Properties

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined

a) $A(BC) = (AB)C$ (associative law of multiplication)

b) $A(B + C) = AB + AC$ (left distributive law)

c) $(B + C)A = BA + CA$ (right distributive law)

d) $r(AB) = (rA)B = A(rB)$ (for any scalar)

e) $I_m A = A = AI_n$ (identity for matrix multiplication)

Note that $\boxed{AB \neq BA}$

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Power and polynomial of a matrix

Power of a matrix

Let A be a squared matrix and k be a positive integer. Now, A^k denotes the product of k copies of A

$$A^k = A \dots A$$

The polynomial of a matrix

Let A be a squared matrix and I be an identity matrix of the same size. If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, then

$$f(A) = a_n A^n + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I$$

Example

Find $f(A)$, where $f(x) = x^2 + 3x - 2$ and $A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$

Definition

Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Examples: Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 4 & 1 & 2 \\ 5 & 0 & -1 & -3 \end{bmatrix}$$

Then

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & -3 \end{bmatrix}, B^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}, C^T = \begin{bmatrix} 2 & 5 \\ 4 & 0 \\ 1 & -1 \\ 2 & -3 \end{bmatrix}$$

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Properties of the transpose of a matrix

Let A and B denote matrices whose sizes are appropriate for the following sums and products

a) $(A^T)^T = A$

b) $(A+B)^T = A^T + B^T$

c) For any scalar r, $(rA)^T = rA^T$

d) $(AB)^T = B^T A^T$

Row echelon form

A matrix is in row echelon form if:

1. all non-zero rows are above any rows of all zeros
2. Each leading element of a row is in a column to the right of the leading element of the row above it
3. all elements in a column below a leading element are zeros

Elementary row (column) operations

Elementary row (column) operations

There are three kinds of elementary matrix operations.

i) Interchange two rows (or columns).

ii) Multiply each element in a row (or column) by a non-zero number.

iii) Multiply a row (or column) by a non-zero number and add the result to another row (or column).

Example

Determine elementary row operations on the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & -3 \\ -1 & 1 & 0 & 3 \\ 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

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Symmetric and antisymmetric matrices

Definition

Let A be a square matrix.

i) A is called symmetric if $A^T = A$

ii) A is called anti-symmetric or skew-symmetric if $A^T = -A$

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Submatrix

Submatrix

For any square matrix A of size $n \geq 2$, let M_{ij} denote the (i, j)-submatrix of size $n - 1$ formed by deleting the i-th row and j-th column of A.

Examples

Given matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$. Find all the submatrices M_{ij} of A

Cofactor

Cofactor

Given $A = [a_{ij}]_n$ be a square. The (i, j)-cofactor of element a_{ij} is the number A_{ij} given by

$$A_{ij} = (-1)^{i+j} detM_{ij}$$

Examples

Given matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$. Find all the cofactor A_{ij} of A.

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Determinant

Definition

The determinant of a matrix A, denoted by $|A|$ or $\det A$, is defined as follows

i) 1-square matrix $|A| = |a| = a$,

ii) 2-square matrix $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

iii) 3-square matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

iii) n-square matrix, $n \geq 3$

$$detA = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$$

, where A_{ij} the cofactor of a_{ij}.

Determinant

Examples

Compute the determinant of the following matrices

$$A = \begin{vmatrix} 4 & -3 \\ -2 & -4 \end{vmatrix}, B = \begin{vmatrix} 1 & -1 & -2 \\ 2 & -3 & 1 \\ 3 & -4 & 2 \end{vmatrix}$$

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Properties of the determinant

Properties

Let A, B be a square matrix

1) $detA = detA^T$, i.e. Column operations have the same effects on determinant as row operations.
2) The Laplace formula:

$$detA = \sum_{j=1}^{n} a_{ij}A_{ij} = \sum_{i=1}^{n} a_{ij}A_{ij}$$

3) $det(AB) = detA.detB$
4) If A is a triangular matrix, then $detA = a_{11}a_{22}\dots a_{nn}$.
5) If all elements of any row (column) in A equal to zero, then its determinant is zero.

Properties of the determinant

Properties

6) If any two rows (columns) in A are identical (or proportional), then its determinant is zero.
7) If a multiple of one row (column) of A is added to another row (column) to produce a matrix B, then $detB = detA$.
8) If two rows (columns) of A are interchanged to produce B, then $detB = -detA$
9) If one row of A is multiplied by k to produce B, then $detB = k.detA$.
10) $det(cA) = c^n detA$, where n is the size of matrix A.
11) If elements of a row (column) in a determinant can be expressed as the sum of two or more elements, then the given determinant can be expressed as the sum of two or more determinants.

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Properties of the determinant

Remarks

The properties 4); 7); 8) allow to use the elementary row operations to deduce the matrix to the upper triangular matrix.

Examples

Find the determinant of the matrix $A = \begin{bmatrix} -1 & 1 & 2 & 0 \\ 3 & -1 & 0 & 4 \\ -2 & 0 & 5 & -2 \\ 0 & 3 & 6 & -1 \end{bmatrix}$

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Invertible matrix

Definition

Let A be a square matrix of size n. Matrix A is said to be invertible (or non-singular) if there exists a square matrix B of the same size such that

$$AB = BA = I_n$$

,where I_n is the identity matrix of size n. Now, B is called the inverse of A and denoted by A^{-1}.

Remarks:

i) $(I_n)^{-1} = I_n$

ii) Null matrix is not invertible.

Invertible matrix

Uniqueness

If A is an invertible matrix, then the inverse of matrix A, say A^{-1}, is unique.

Inverse matrix

Let A be a square matrix. If $detA \neq 0$, then

$$A^{-1} = \frac{1}{detA} C^T$$

where $C = [c_{ij}]_n$ and $c_{ij} = A_{ij}$

Examples

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 2 \end{bmatrix}$

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Inverse of a matrix

Properties

1) $(A^{-1})^{-1} = A$
2) $(kA)^{-1} = \frac{1}{k}A^{-1}$
3) $(AB)^{-1} = B^{-1}A^{-1}$

Gauss-Jordan elimination

1) Adjoin the identity matrix to the right side of A.
2) Apply row operations to this matrix until the left side is reduced to the identity matrix.
3) If A is reduced to I, then I becomes A^{-1}.

Applying Gauss-Jordan elimination $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 2 \end{bmatrix}$

Inverse of a matrix

Let A be a square matrix with $detA \neq 0$. Find matrix X such that

1) $AX = B \Rightarrow X = A^{-1}B$
2) $XA = B \Rightarrow X = BA^{-1}$
3) $AXB = C \Rightarrow X = A^{-1}CB^{-1}$
4) $AX + kB = C \Rightarrow X = A^{-1}(C - kB)$

Examples

Find X such that

$$X\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + 2\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix}$$

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Submatrix

Definition

Let A be a matrix of size $m \times n$ and $k \leq min\{m, n\}$. A submatrix of size k of matrix A, that is a square matrix of size k is obtained from A by deleting $m - k$ rows and $n - k$ columns.

Rank of matrix

The rank of matrix A, denoted by $r(A)$ or $rank(A)$, is the largest size of submatrix with non-zero determinant.

Remarks

If A is a matrix with $r(A) = k$, then

i) A has at least one submatrix of size k with non-zero determinant.

ii) The determinant of all submatrices whose size is larger than k is zero.

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Properties of the rank

Properties

i) $r(0_{m\times n}) = 0$

ii) $0 \leq r(A) \leq min\{m, n\}$

iii) Let A be a square matrix of size n. Hence,

   a) If $detA \neq 0$, then $r(A) = n$.

   b) If $detA = 0$, then $r(A) < n$.

The rank of a matrix

Theorem

The rank of a matrix, in a row echelon form, is the number of non-zero rows.

Remark

1. Any matrix can be transformed to row echelon form by using elementary row operations.
2. The elementary row operations on a matrix will not change its rank.

Example

Find the rank of matrix $A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 1 & -1 & 3 \\ -4 & 5 & 2 & -1 \\ -1 & 7 & 3 & 2 \end{bmatrix}$

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The rank of a matrix

Examples

Determine the rank of the following matrix which depends on m

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & m & 1 \\ -1 & 4 & 5 \end{bmatrix}$$

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System of linear equations

Definition

A general system of m linear equations with n unknowns can be written as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n = b_m \end{cases}$$

where $x_1, x_2, \ldots, x_n$ are unknowns, a_{ij}, $\forall i = \overline{1, m}, j = \overline{1, n}$ are the coefficients of the system, and $b_1, b_2, \ldots, b_m$ are the constant terms.

The matrix equation

The system of linear equations can be written in the following form

$$Ax = b$$

where $A = [a_{ij}]_{m\times n}, x = [x_1, x_2, \ldots, x_n]^T$, is coefficient matrix and $b = [b_1, b_2, \ldots, b_m]^T$.

Example

$$\begin{cases} x_1 + x_2 - 2x_3 = 6 \\ 2x_1 + 3x_2 - 7x_3 = 16 \\ 5x_1 + 2x_2 + x_3 = 16 \\ x_1 + x_2 - 2x_3 = -4 \end{cases} \Rightarrow A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & -7 \\ 5 & 2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

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Cramer's rule

Cramer's rule

Consider a system of n linear equations for n unknowns, represented in matrix multiplication form as follows:

$$Ax = b$$

where the $n \times n$ matrix A has a non-zero determinant, and the vector $x = (x_1, \ldots, x_n)^T$ is the column vector of the variables.
In this case the system has a unique solution, whose individual values for the unknowns are given by

$$x_i = \frac{detA_i}{detA} \quad i = 1, \ldots, n$$

where A_i is the matrix formed by replacing the i-th column of A by the column vector b.

Cramer's rule

Example

Solve the following equation

$$\begin{cases} x_1 + x_2 - 2x_3 = 6 \\ 2x_1 + 3x_2 - 7x_3 = 16 \\ 5x_1 + 2x_2 + x_3 = 16 \end{cases}$$

Gauss-Jordan elimination

Gauss-Jordan elimination is an algorithm for solving systems of linear equations by applying row operations.
Let $Ax = b$ be a matrix equation. The augmented matrix, denoted by $\overline{A}$, is obtained by adding matrix b to the right side of A. Now, $\overline{A} = [A|b]$.

Given the following system of linear equations $\begin{cases} 2x + y - z = 8 \\ -3x - y + 2z = -11 \\ -2x + y + 2z = -3 \end{cases}$

Determine the augmented matrix? Applying Gauss-Jordan elimination on this system.

Kronecker-Capelli's Theorem

Theorem

Let $Ax = b$ be a matrix equation of n unknowns (system of linear equations). The system has solution if and only if $r(\overline{A}) = r(A)$

Corollary

1. If $r(\overline{A}) \neq r(A)$, then the system has no solution.
2. If $r(\overline{A}) = r(A) = n$, then the system has a unique solution.
3. If $r(\overline{A}) = r(A) = r < n$, then the system has infinitely many solutions.

Homogeneous systems

Definition

A system of linear equations is homogeneous if all of the constant terms are zero:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{cases}$$

or in matrix form $Ax = 0$

Homogeneous systems

Properties

1. Every homogeneous system has at least one solution, known as the zero solution (or trivial solution), which is $x = 0$, i.e., $x_1 = x_2 = \ldots = x_n = 0$.
2. If $det(A) \neq 0$, then it is also the only solution.
3. If $det(A) = 0$, then there is a solution set with infinite number of solutions. Moreover:
   i) if u, v are solutions, then $u + v$ is also a solution.
   ii) if v is a solution and k is a scalar, then kv is also a solution.

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