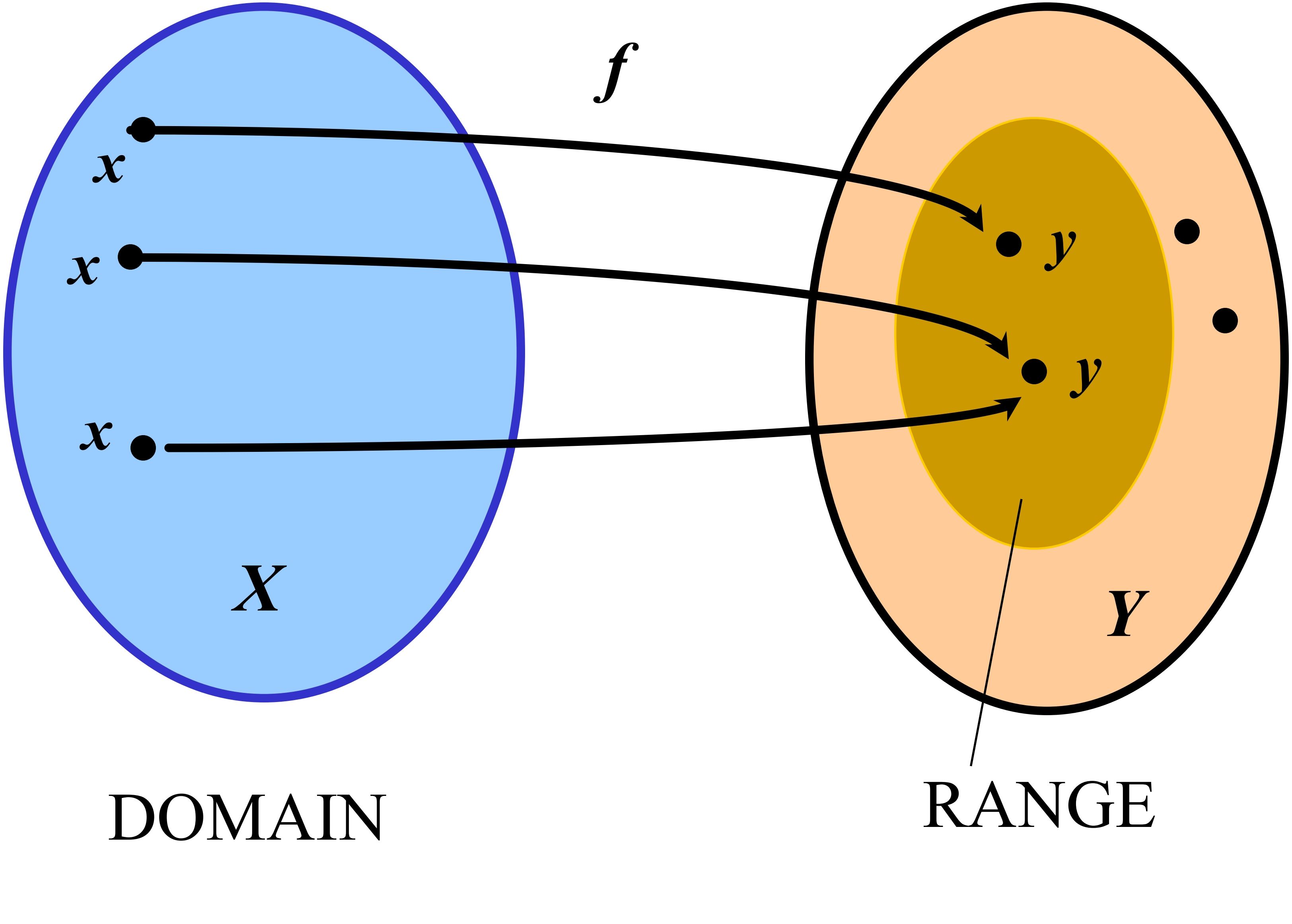
Functions

Let X and Y be two nonempty sets of real numbers. A **function** from X into Y is a rule or a correspondence that associates with each element of X a **unique** element of Y.

The set X is called the **domain** of the function.

For each element x in X, the corresponding element y in Y is called the **image** of x. The set of all **images** of the elements of the domain is called the **range** of the function.



f
x
x
x
y
y
X
Y
DOMAIN
RANGE

Determine which of the following relations represent functions.

$$\{(-2,3),(1,3),(-2,5),(10,5)\}$$

Not a function.

$$\{(1,2),(2,2),(3,1),(-4,2)\}$$

Function.

$$\{(0,0),(1,1)\}$$

Function.

$$(x-2)^2+(y+4)^2=25$$

Not a function.
(2,1) and (2,-9)both work.

Find the domain of the following functions:

A) $f(x) = 2x - 1$

Domain is all real numbers.

B) $g(x) = \dfrac{x}{x-1}$

Domain is all real numbers but $x \neq 1$.

C) $h(x) = \sqrt{4 - x}$

Square root is real only for nonnegative numbers.

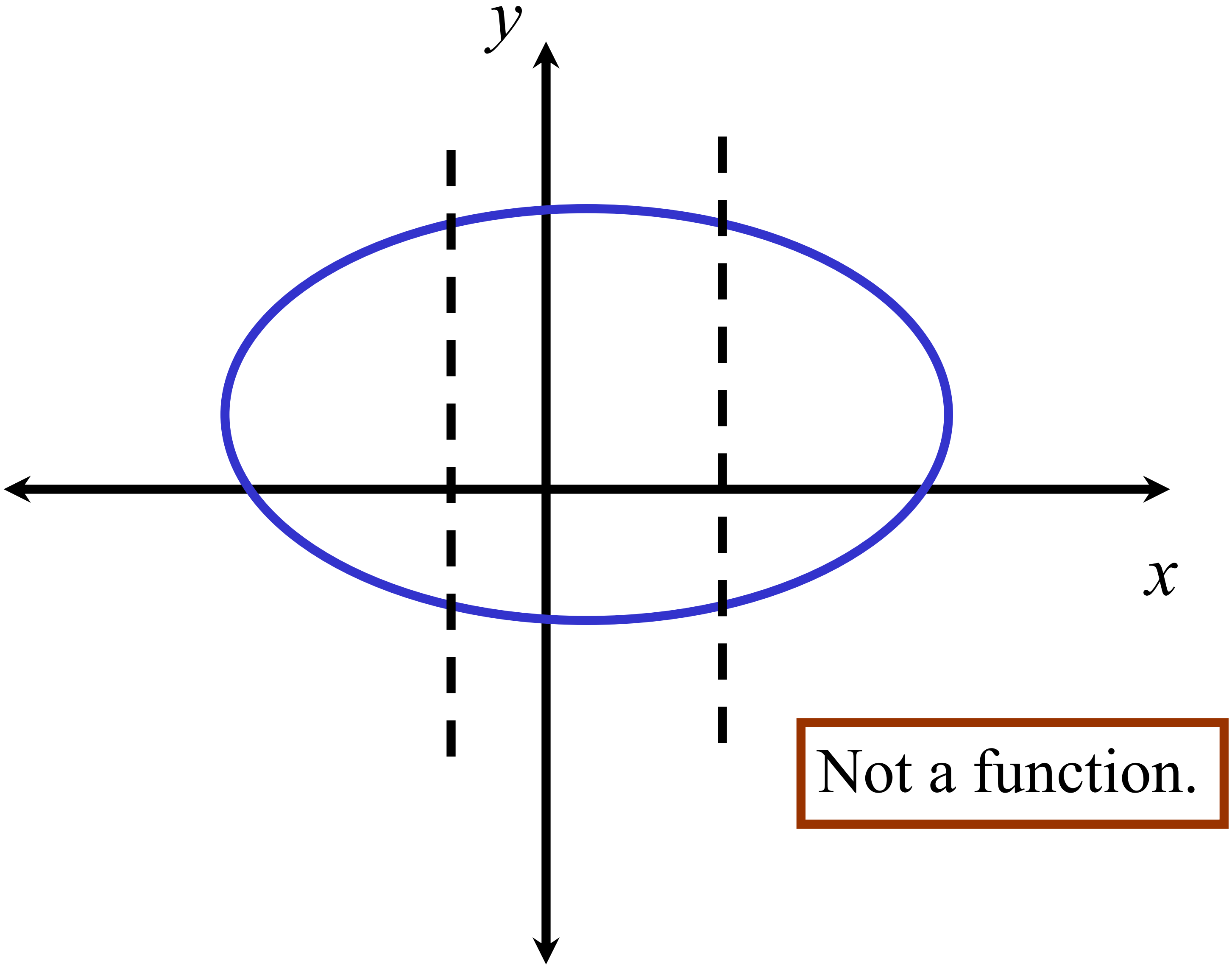
$$4 - x \geq 0$$

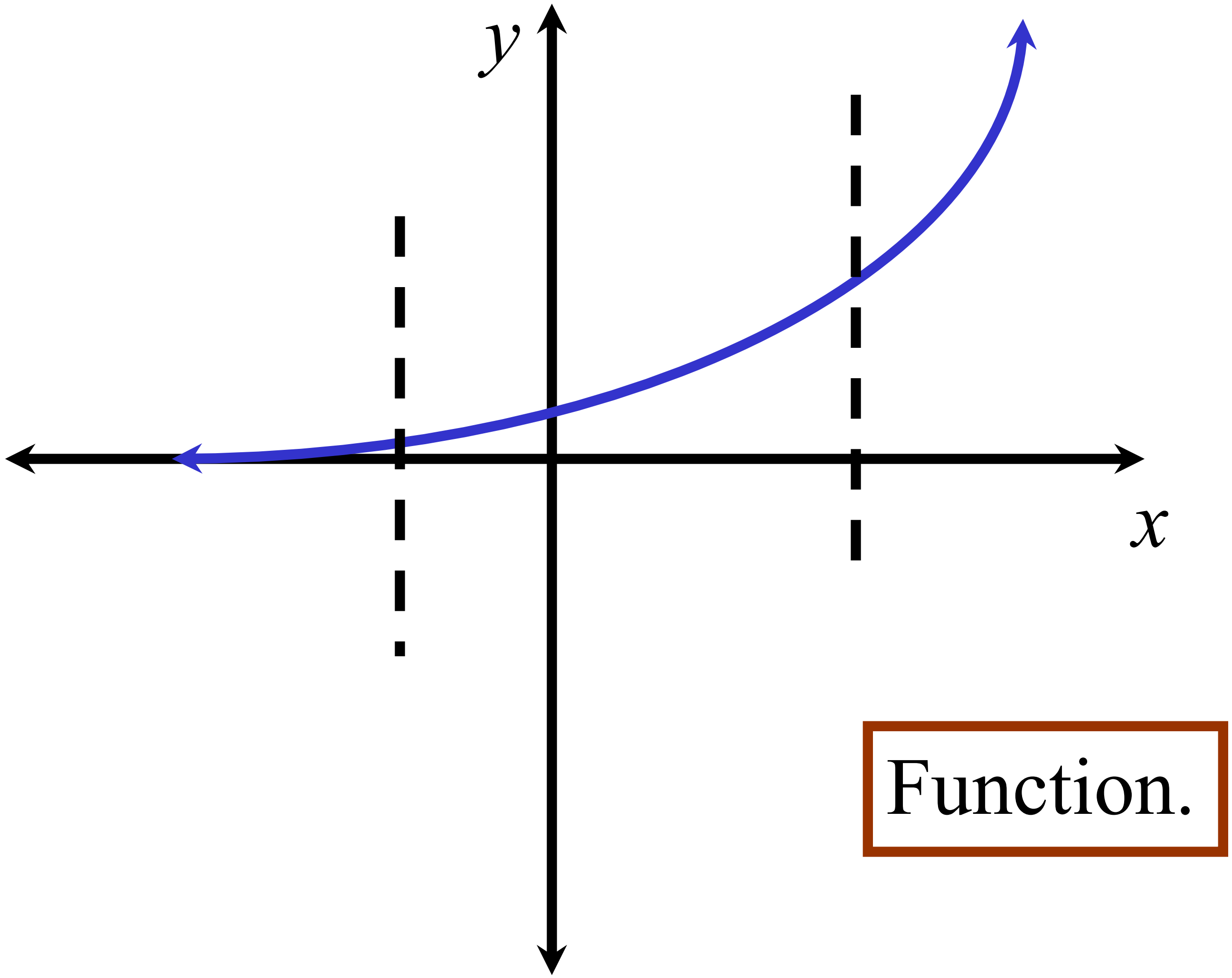
$$4 \geq x$$

$$\boxed{\text{Thus the domain is } (-\infty, 4].}$$

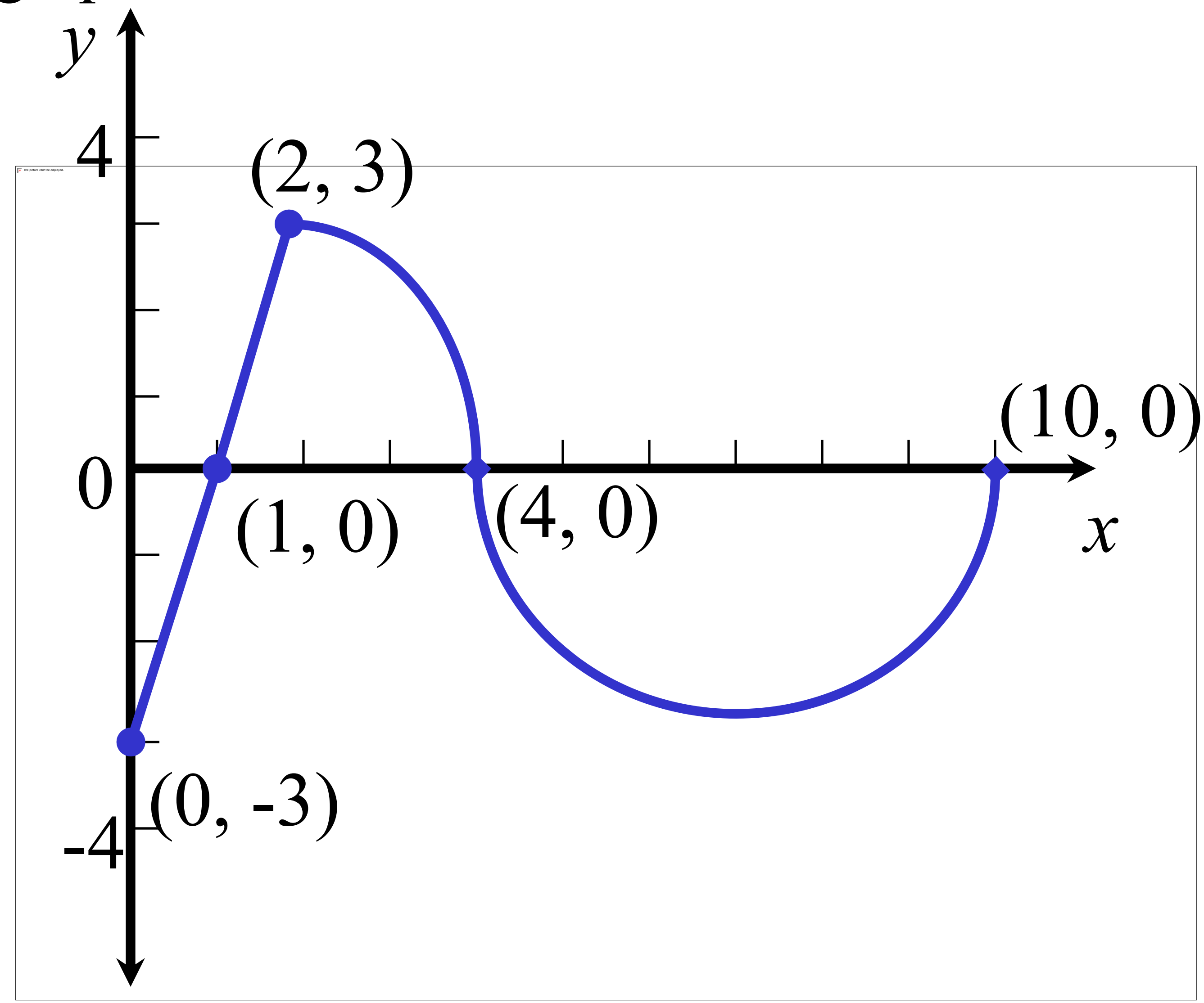
Theorem Vertical Line Test

A set of points in the *xy* - plane is the graph of a function if and only if a vertical line intersects the graph in at most one point.


y
x
Not a function.


y
x
Function.

Determine the domain, range, and intercepts of the following graph.



Linear Functions and Models

A **linear function** is a function of the form

$$\boxed{f(x) = mx + b}$$

The graph of a linear function is a line with a slope m and y-intercept b.

Scatter Diagrams

A **relation** is a correspondence between two sets. If x and y are two elements and a relation exists between x and y, then we say that x **corresponds to** y or that y **depends on** x and write $x \rightarrow y$ or we write it as an ordered pair (x,y).

y - **dependent** variable
x - **independent** variable

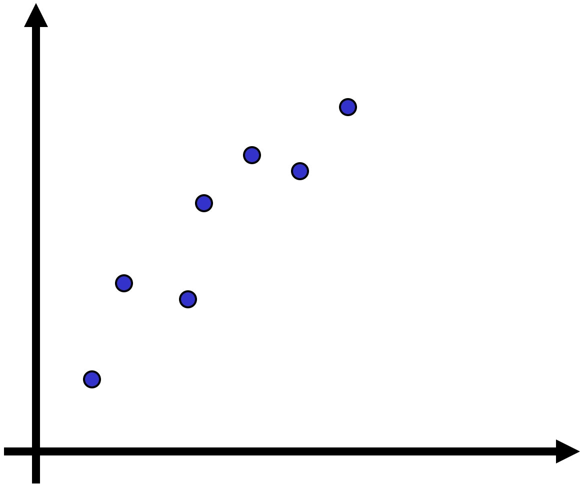
The first step in finding whether a relation might exist between two variables is to plot the ordered pairs using rectangular coordinates.

The resulting graph is called a **scatter diagram**.

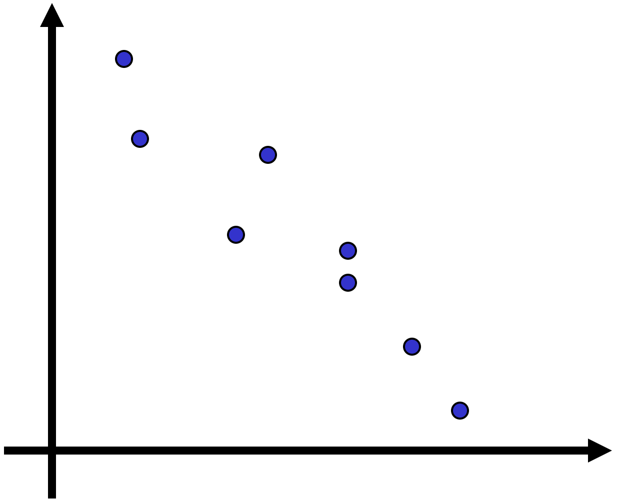
Curve Fitting

Scatter diagrams help us to see the type of relation that exists between two variables.

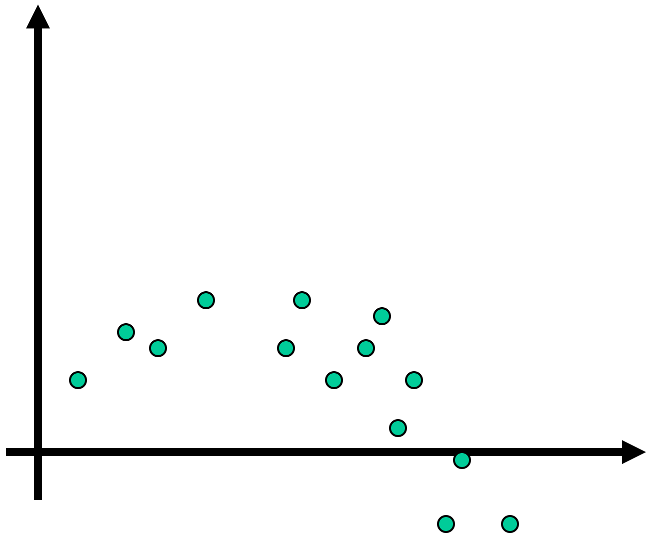
Distinguishing between Linear and Nonlinear Relations



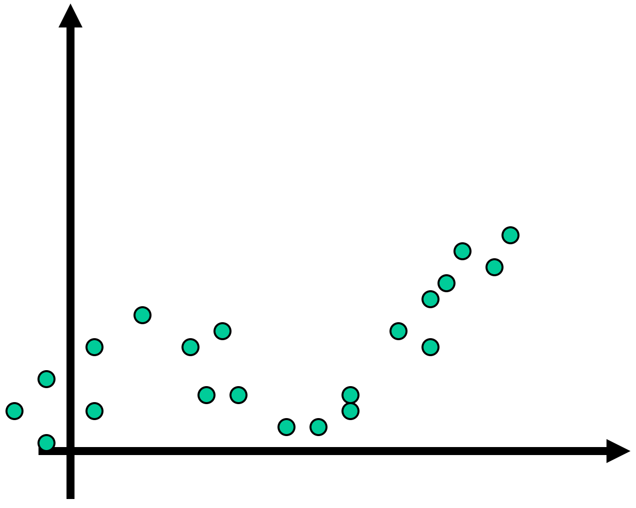
Linear
$y=mx+b$
$m>0$



Linear
$y=mx+b$
$m<0$



Nonlinear



Nonlinear

Direct Variation

Let x and y denote two quantities. Then y **varies directly** with x, or y is **directly proportional** to x, if there is a nonzero number k such that

$$\boxed{y=kx}$$

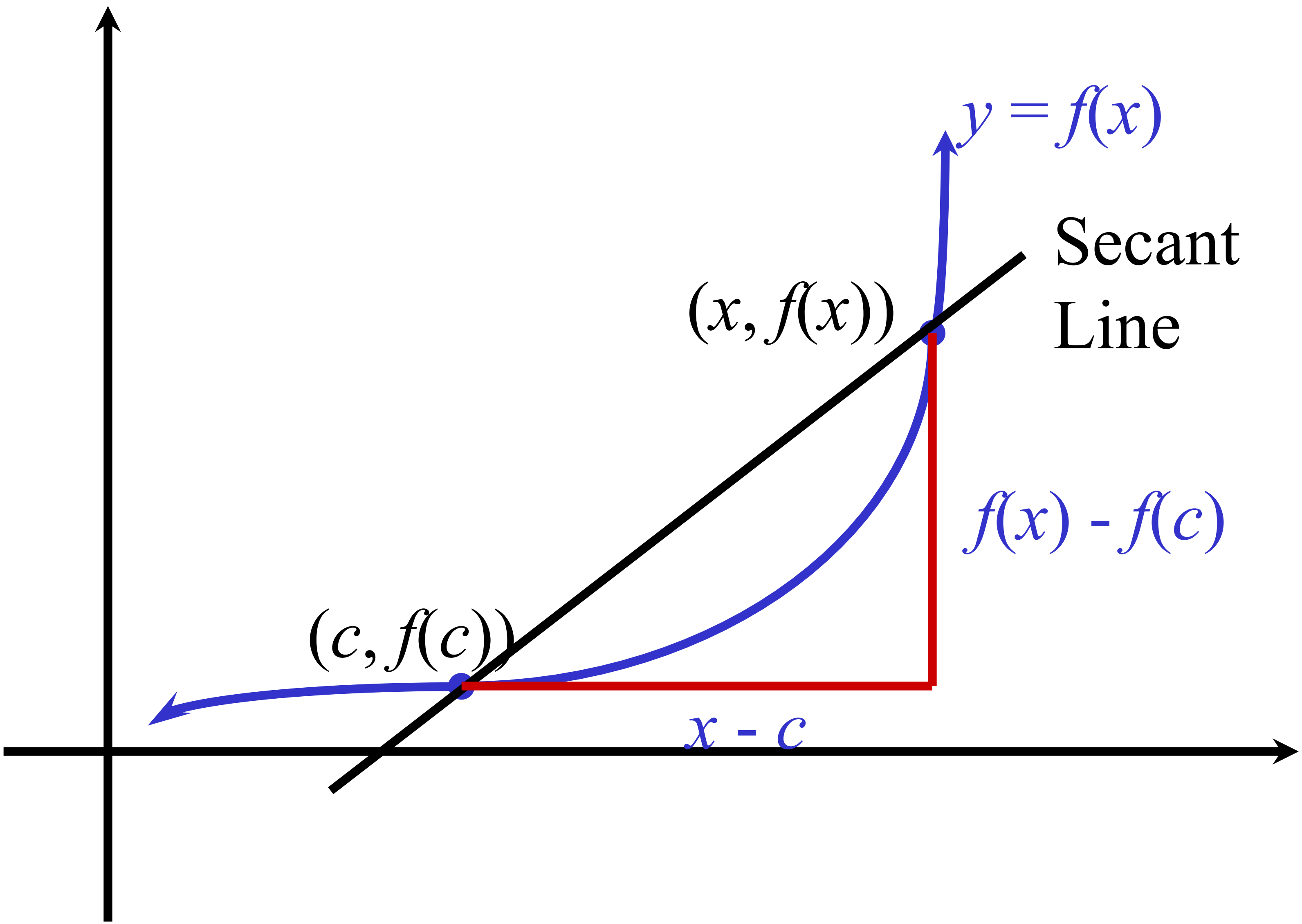
The number k is called the constant of proportionality.

Properties of Functions

If c is in the domain of a function $y=f(x)$, the **average rate of change** of f from c to x is defined as

$$\boxed{\text{average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(c)}{x - c}}$$

This expression is also called the **difference quotient** of f at c.



$y = f(x)$
Secant
Line
$(x, f(x))$
$f(x) - f(c)$
$(c, f(c))$
$x - c$

The function $s(t) = -16t^2 + 100t + 6$ represents the height (in feet) of a ball thrown up as a function of time t in seconds.

(a) Find the average rate of change of the height of the ball between 1 and t seconds.

(b) Using the result found in (a), find the average rate of change of the height of the ball between 1 and 2 seconds.

$$s(t) = -16t^2 + 100t + 6$$

$$\frac{\Delta s}{\Delta t} = \frac{s(t) - s(1)}{t - 1}, \quad t \neq 1$$

$$s(1) = -16(1)^2 + 100(1) + 6 = 90$$

$$\frac{s(t) - s(1)}{t - 1} = \frac{-16t^2 + 100t + 6 - 90}{t - 1}$$

$$= \frac{-16t^2 + 100t - 84}{t - 1} = \frac{-4\left(4t^2 - 25t + 21\right)}{t - 1}$$

$$= \frac{-4\left(4t^2 - 25t + 21\right)}{t-1} = \frac{-4(4t-21)(t-1)}{t-1}$$

$$= \boxed{-4(4t-21)}$$

The average rate of change between 1

and 2 seconds is

$$-4(4(2) - 21) = -4(-13)$$

$$= \boxed{52 \text{ feet per second.}}$$

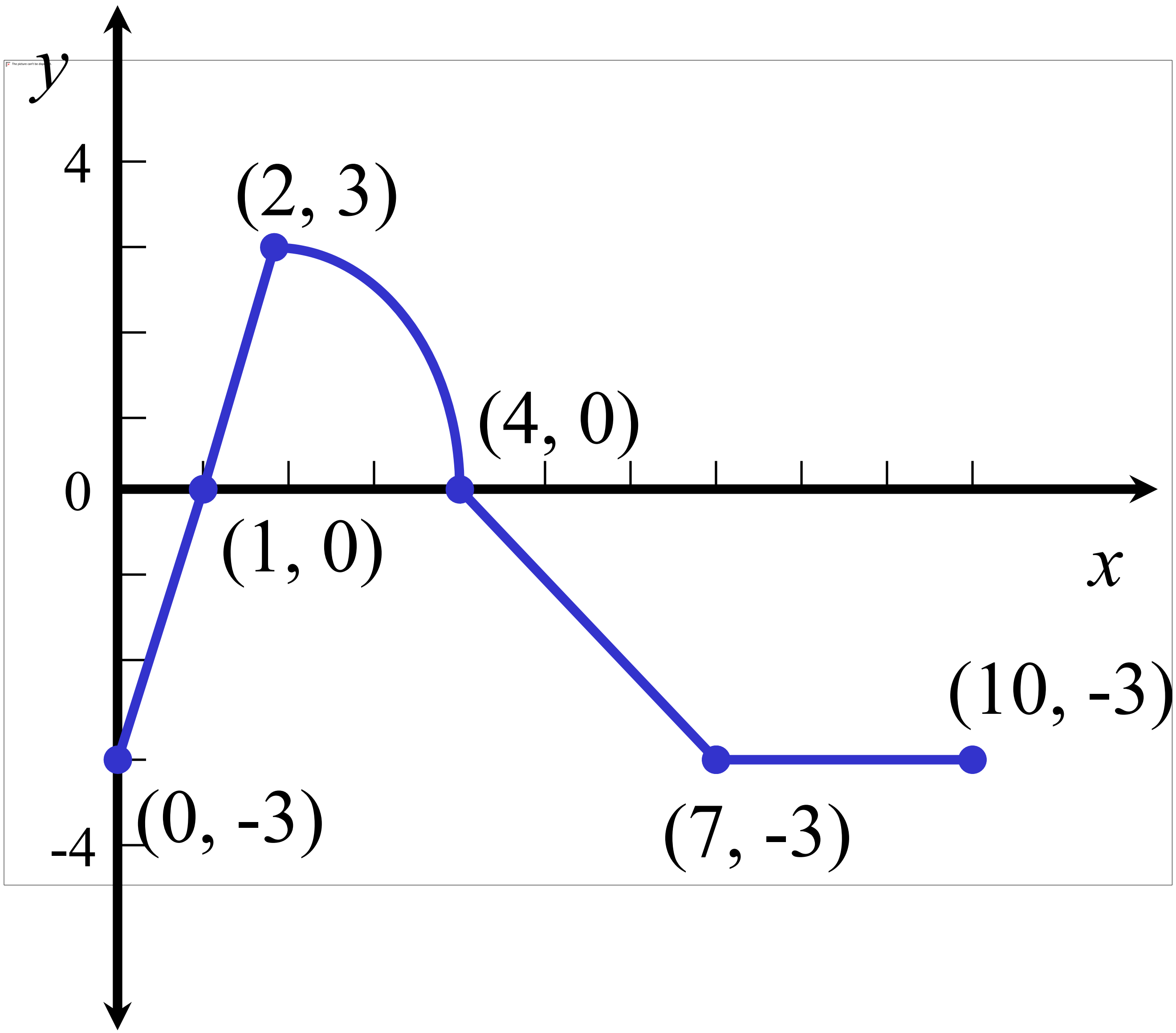
Increasing and Decreasing Functions

- A function f is **increasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$ we have $f(x_1)<f(x_2)$.
- A function f is **decreasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$ we have $f(x_1)>f(x_2)$.
- A function f is **constant** on an open interval I if, for all choices of x, the values $f(x)$ are equal.

Local Maximum, Local Minimum

- A function f has a **local maximum at *c*** if there is an open interval I containing c so that, for all $x \neq c$ in I, $f(x) < f(c)$. We call $f(c)$ a **local maximum** of f.
- A function f has a **local minimum at *c*** if there is an open interval I containing c so that, for all $x \neq c$ in I, $f(x) > f(c)$. We call $f(c)$ a **local minimum** of f.

Determine where the following graph is increasing, decreasing and constant. Find local maxima and minima.



y
4
0
-4
x
(2, 3)
(4, 0)
(1, 0)
(0, -3)
(7, -3)
(10, -3)

The graph is increasing on an interval (0,2).

The graph is decreasing on an interval (2,7).

The graph is constant on an interval (7,10).

The graph has a local maximum at $x=2$ with value $y=3$. And no local minima.

A function f is **even** if for every number x in its domain the number $-x$ is also in its domain and

$$f(-x) = f(x)$$

A function f is **odd** if for every number x in its domain the number $-x$ is also in its domain and

$$f(-x) = -f(x)$$

Theorem

A function is even if and only if its graph is symmetric with respect to the y-axis. A function is odd if and only if its graph is symmetric with respect to the origin.

Determine whether each of the following functions is even, odd, or neither. Then decide whether the graph is symmetric with respect to the y-axis, or with respect to the origin.

(a) $g(z) = -z^2 + 2$

$$g(-z) = (-z)^2 + 2 = z^2 + 2$$

$$g(z) = g(-z)$$

Even function, graph symmetric with respect to the *y*-axis.

(b) $f(x) = -4x^5 + 3x$

$$f(-x) = -4(-x)^5 + 3(-x) = 4x^5 - 3x$$

$$f(x) \neq f(-x)$$

Not an even function.

$$-f(x) = -(-4x^5 + 3x) = 4x^5 - 3x$$

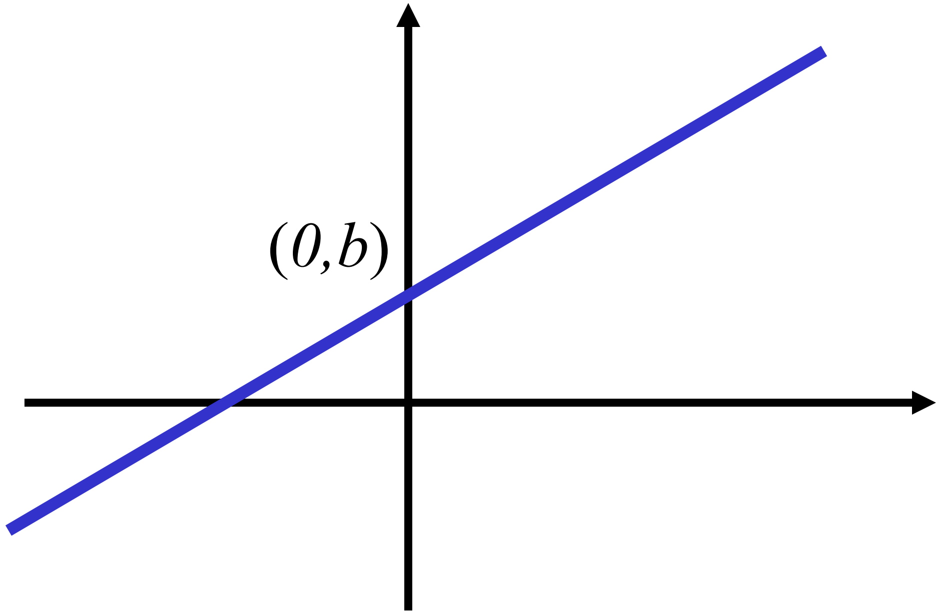
Odd function. The graph will be symmetric with respect to the origin.

Library of Functions, Piecewise-Defined Functions

A **linear function** is a function of the form

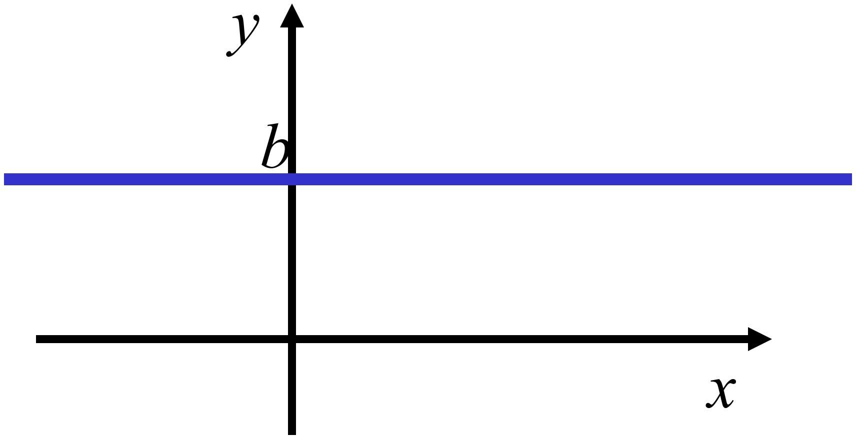
$$\boxed{f(x) = mx + b}$$

The graph of a linear function is a line with a slope m and y-intercept b.



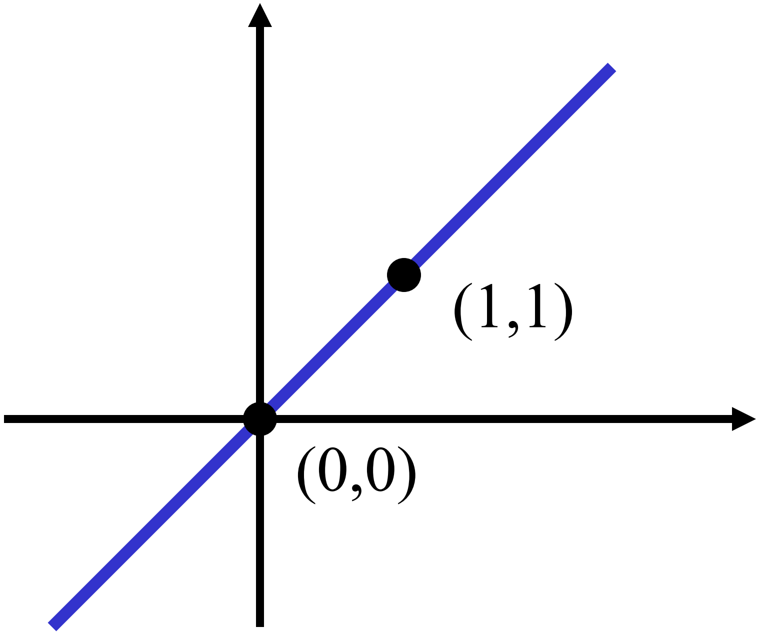
A constant function is a function of the form

$$\boxed{f(x) = b}$$



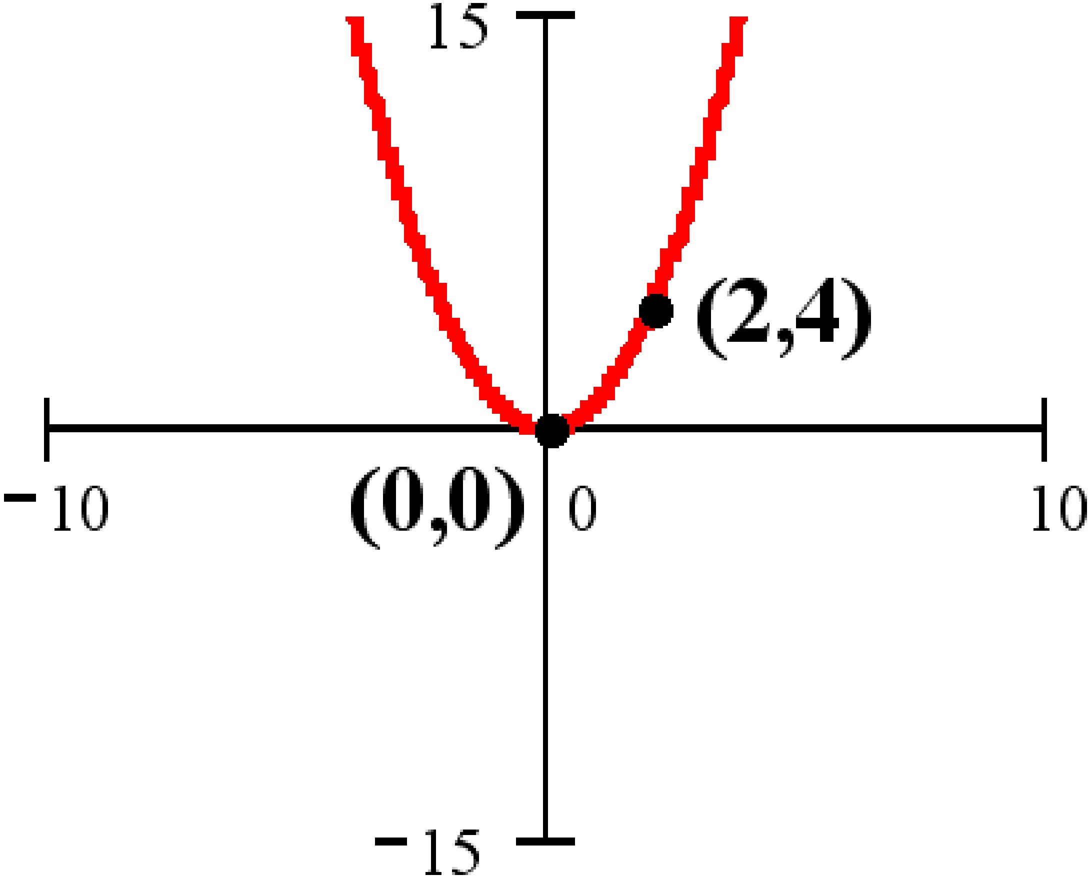
Identity function is a function of a form:

$$\boxed{f(x)=x}$$



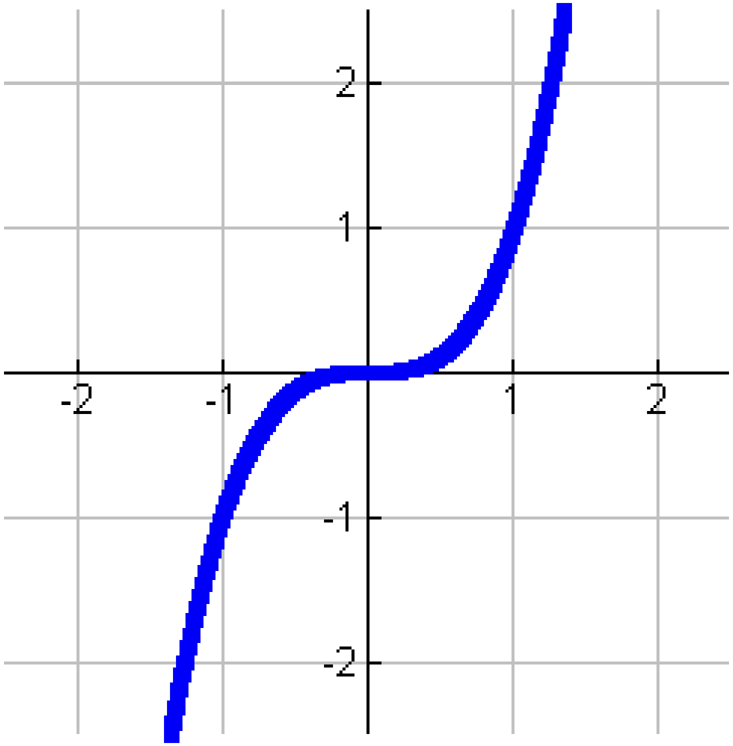
The picture can't be displayed.

The square function



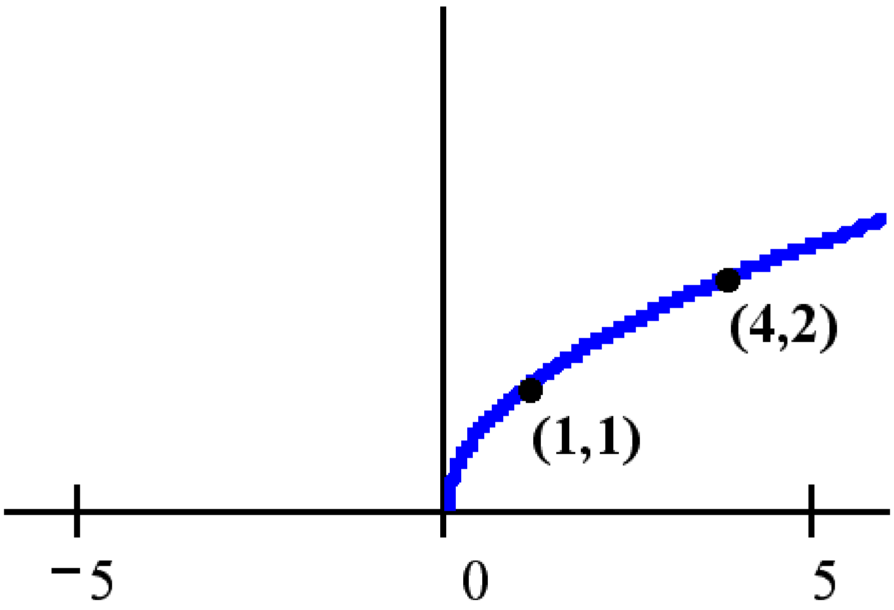
Cube Function

$$\boxed{f(x) = x^3}$$



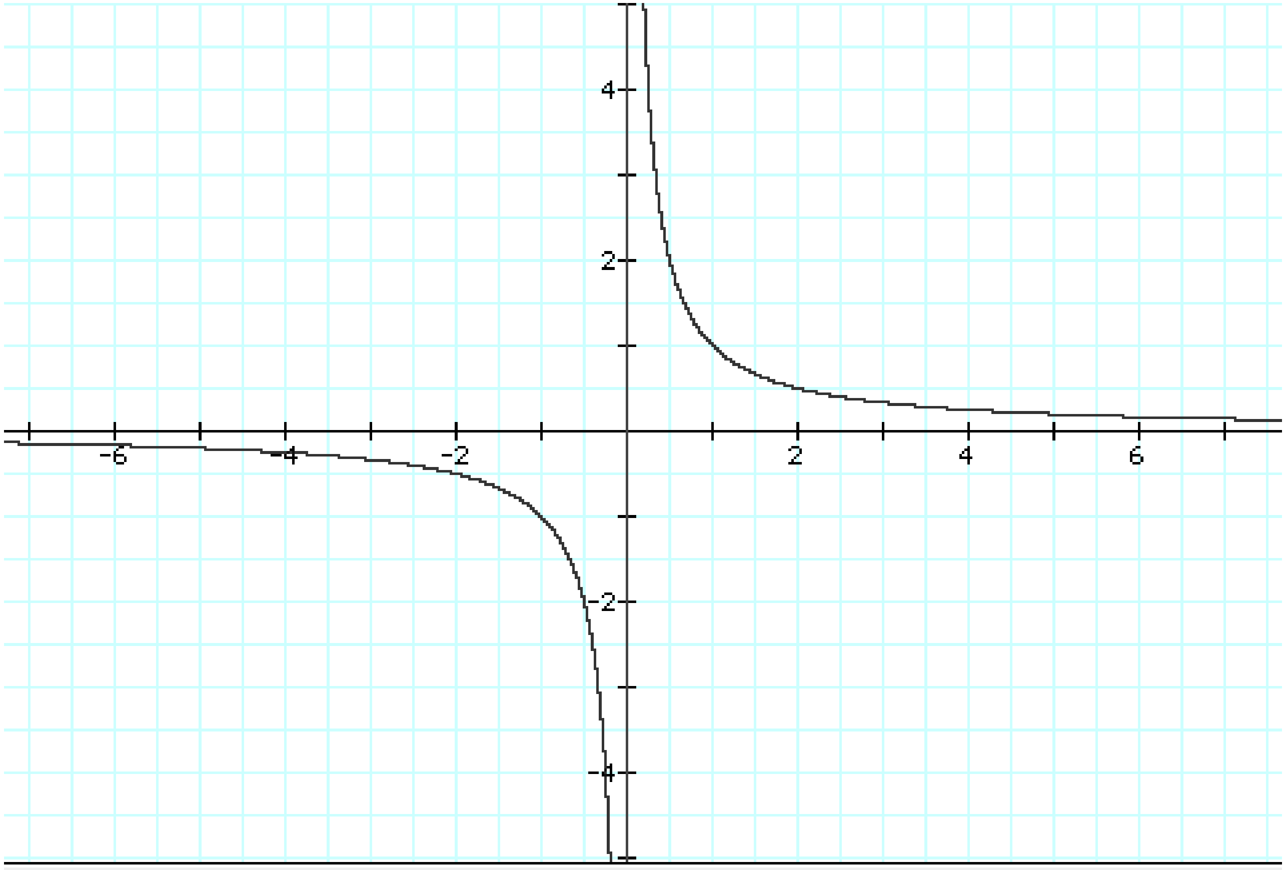
Square Root Function

$$\boxed{f(x) = \sqrt{x}}$$



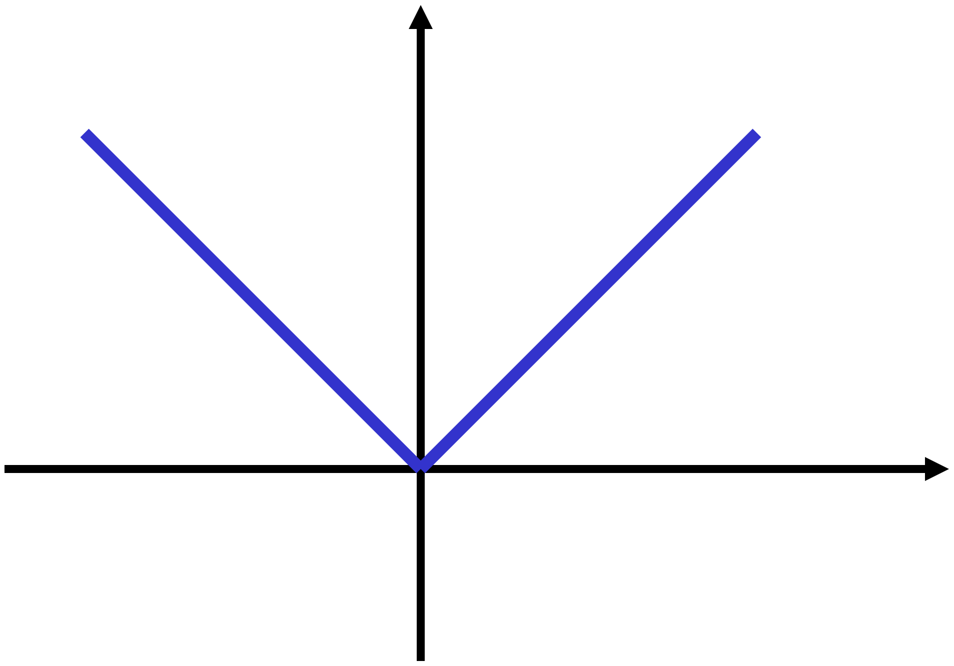
Reciprocal Function

$$\boxed{f(x) = \frac{1}{x}}$$



Absolute Value Function

$$\boxed{f(x) = |x|}$$



When functions are defined by more than one equation, they are called **piece-wise defined** functions.

For the following function

$$f(x)=\begin{cases} x+3 & -2\le x<1 \\ 3 & x=1 \\ -x+3 & x>1 \end{cases}$$

a) Find $f(-1)$, $f(1)$, $f(3)$.

b) Find the domain.

c) Sketch the graph.

a)

$f(1) = 3$

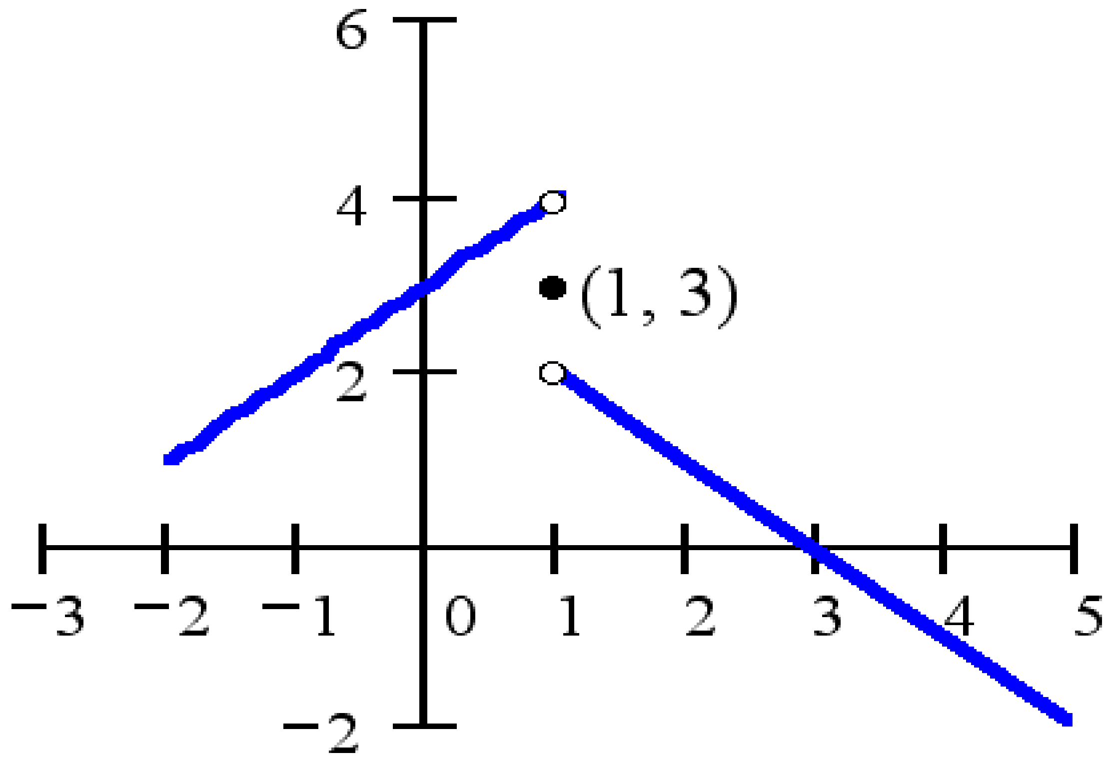
$f(-1) = -1 + 3 = 2$

$f(3) = -3 + 3 = 0$

b)

Domain is $[-2, \infty)$.

c)

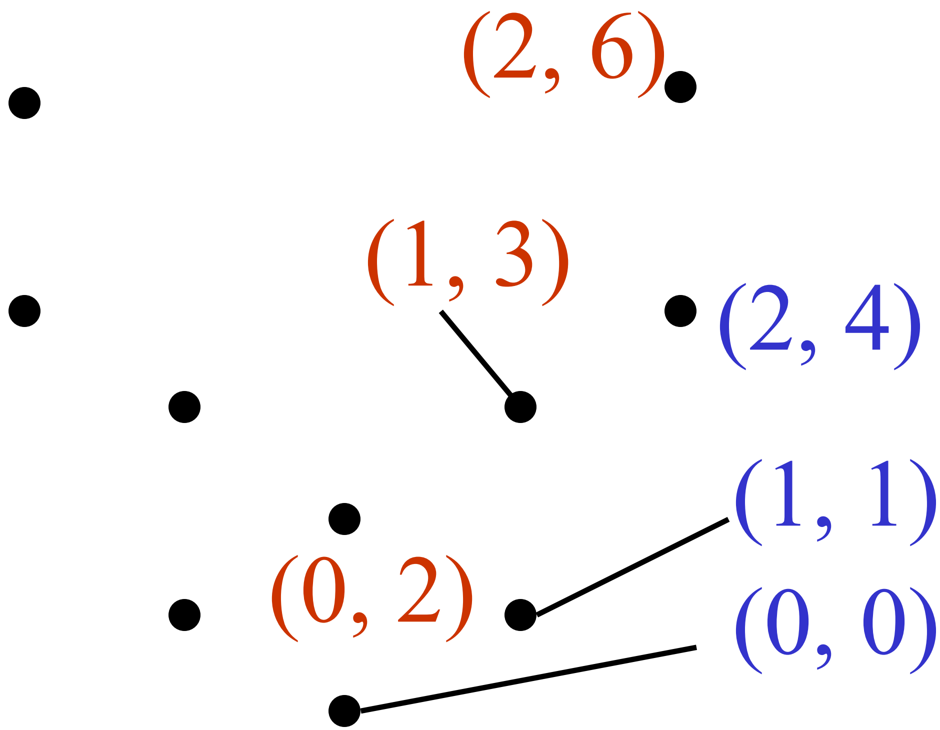


Graphing Techniques; Transformations

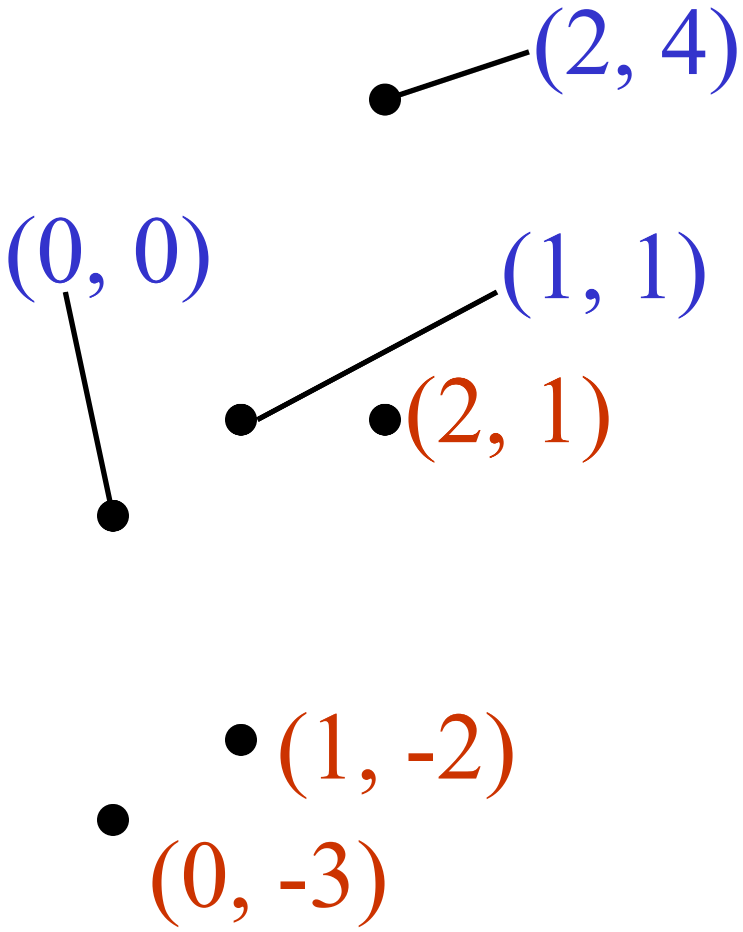
Use the graph of $f(x) = x^2$ to obtain the graph of $g(x) = x^2 + 2$

The picture can't be displayed.

The picture can't be displayed.



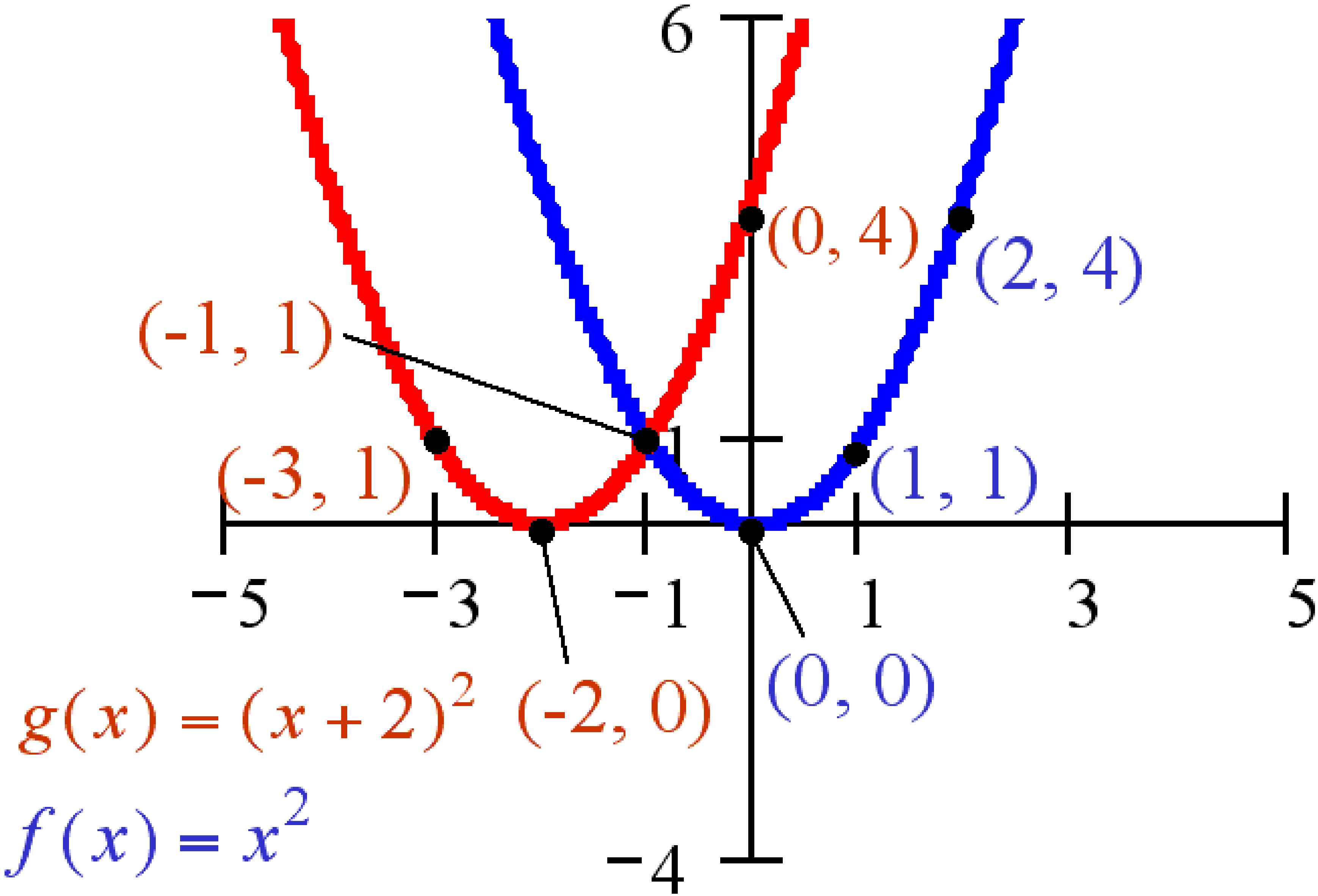
Use the graph of $f(x) = x^2$ to obtain the graph of $g(x) = x^2 - 3$


(2, 4)
(0, 0)
(1, 1)
(2, 1)
(1, -2)
(0, -3)

Vertical Shifts

- $c>0$
  The graph of $f(x)+c$ is the same as the graph of $f(x)$ but shifted UP by c.
- For example: $c = 2$ then $f(x)+2$ shifts $f(x)$ up by 2.
- $c<0$
  The graph of $f(x)+c$ is the same as the graph of $f(x)$ but shifted DOWN by c.
- For example: $c = -3$ then $f(x)+(-3) = f(x)-3$ shifts $f(x)$ down by 3.

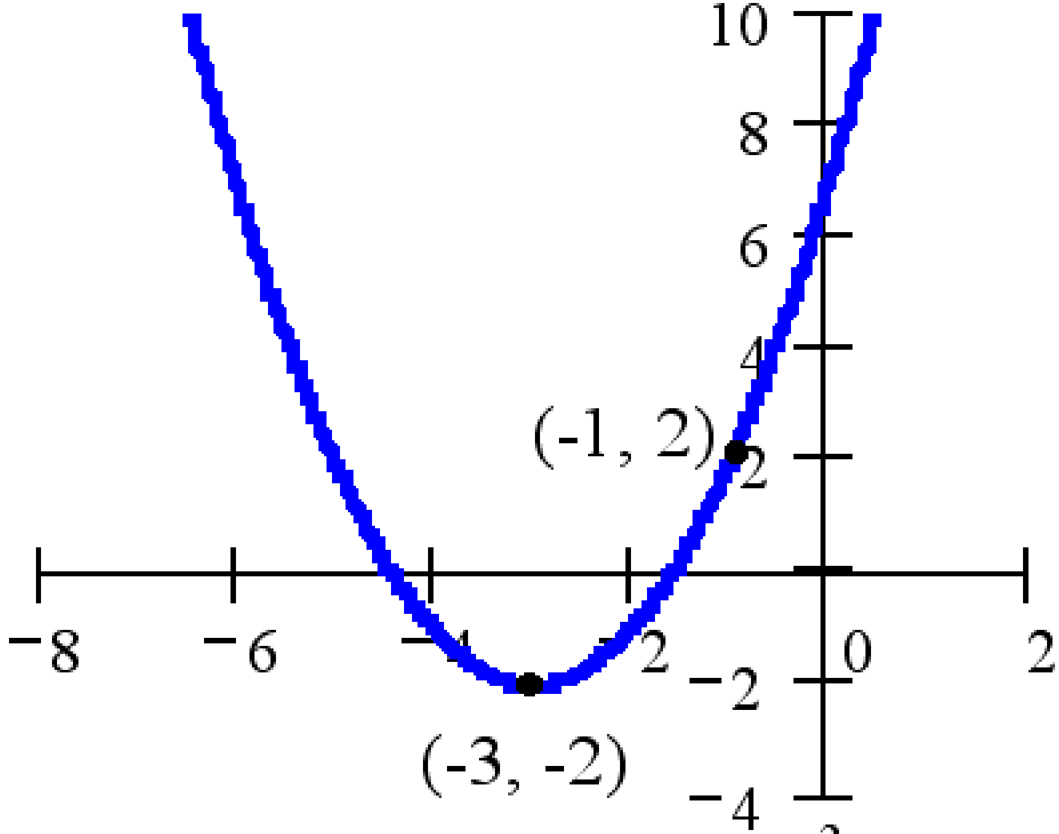
Use the graph of $f(x) = x^2$ to obtain the graph of $g(x) = (x+2)^2$.


6
1
−4
−5
−3
−1
1
3
5
(0, 4)
(2, 4)
(-1, 1)
(-3, 1)
(1, 1)
(-2, 0)
(0, 0)
$g(x) = (x+2)^2$
$f(x) = x^2$

Horizontal Shifts

If the argument x of a function f is replaced by $\boldsymbol{x-h}$, h a real number, the graph of the new function $y = f(x-h)$ is the graph of f shifted horizontally left (if $h < 0$) or right (if $h > 0$).

Use the graph of $f(x) = x^2$ to obtain the graph of $g(x) = (x+3)^2 - 2$.

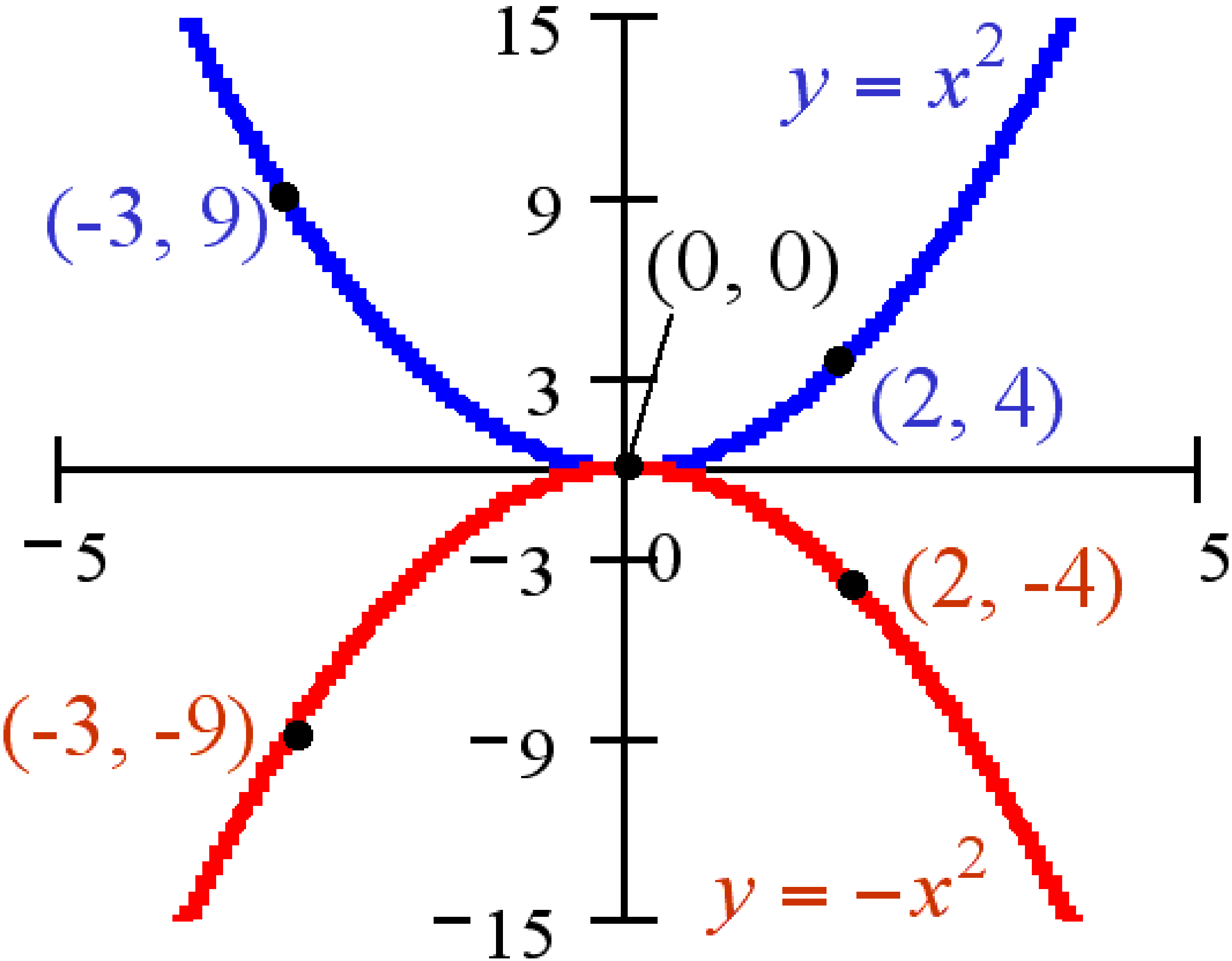


$$y = f(x+3) - 2 = (x+3)^2 - 2$$

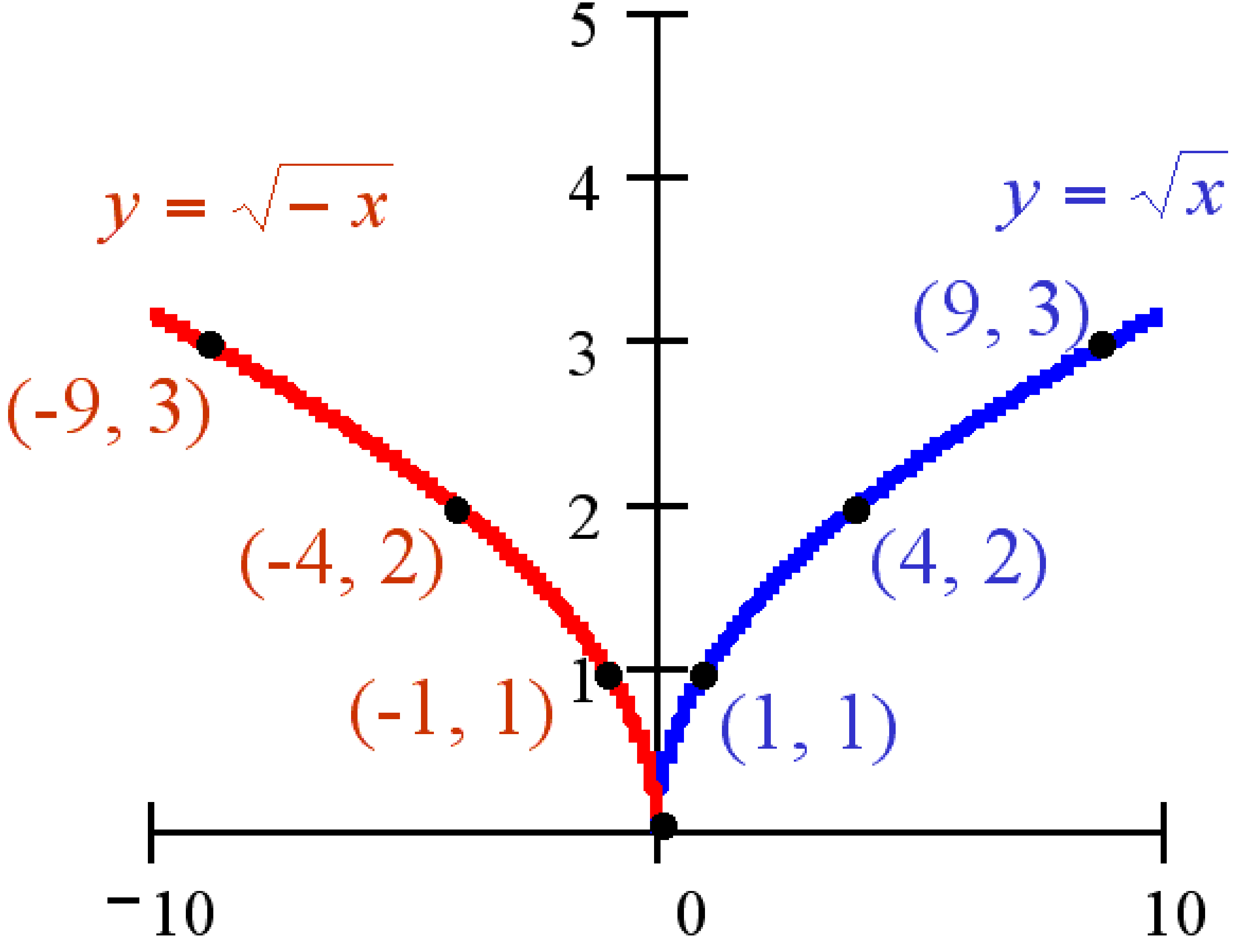
Reflections about the x-Axis and the y-Axis

- The graph of $\boldsymbol{g(x) = -f(x)}$ is the same as graph of $f(x)$ but reflected about the x-axis.

- The graph of $\boldsymbol{g(x) = f(-x)}$ is the same as graph of $f(x)$ but reflected about the y-axis.

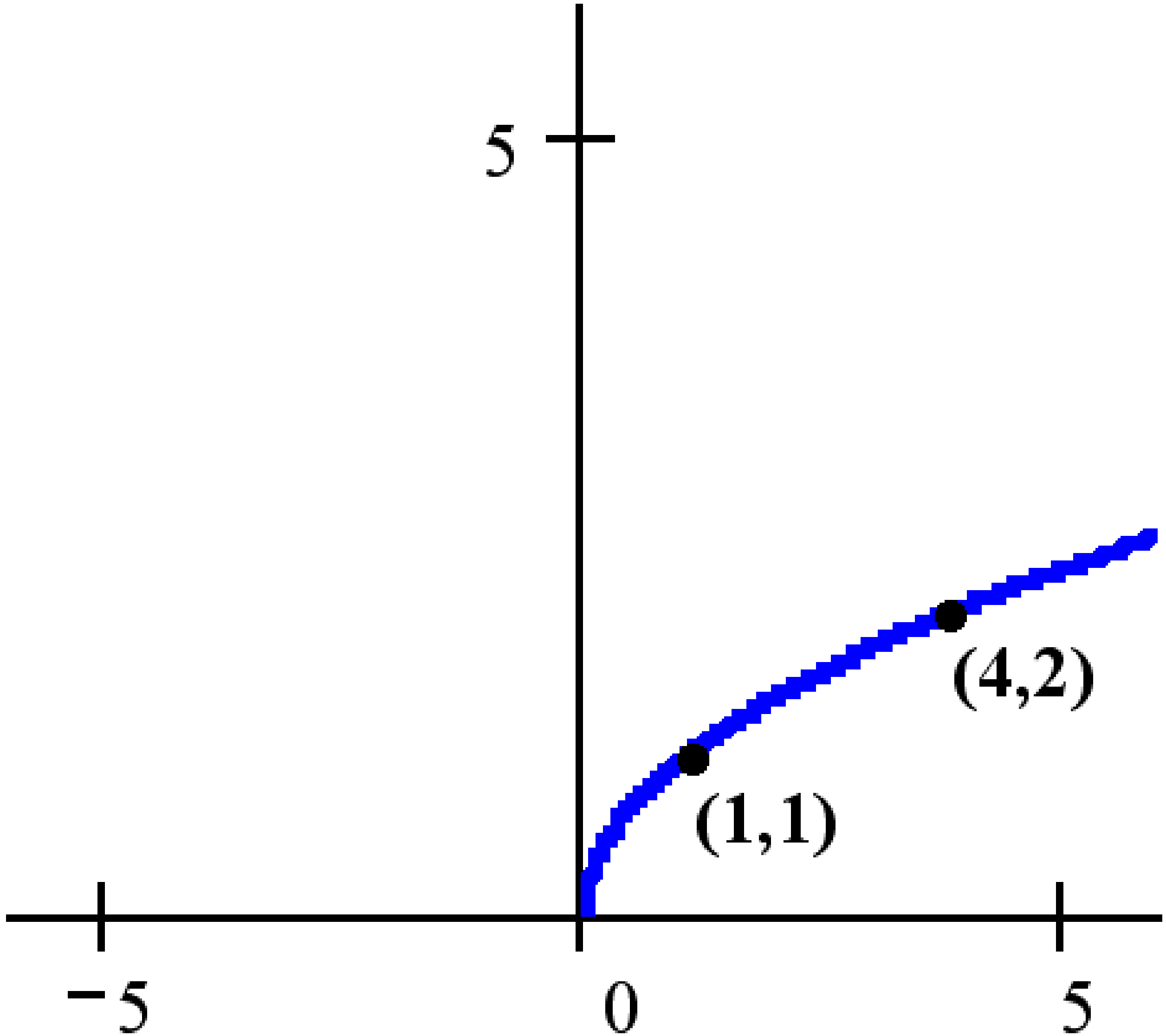


15
9
3
−3
−9
−15
−5
0
5
$y = x^2$
$y = -x^2$
(-3, 9)
(0, 0)
(2, 4)
(2, -4)
(-3, -9)

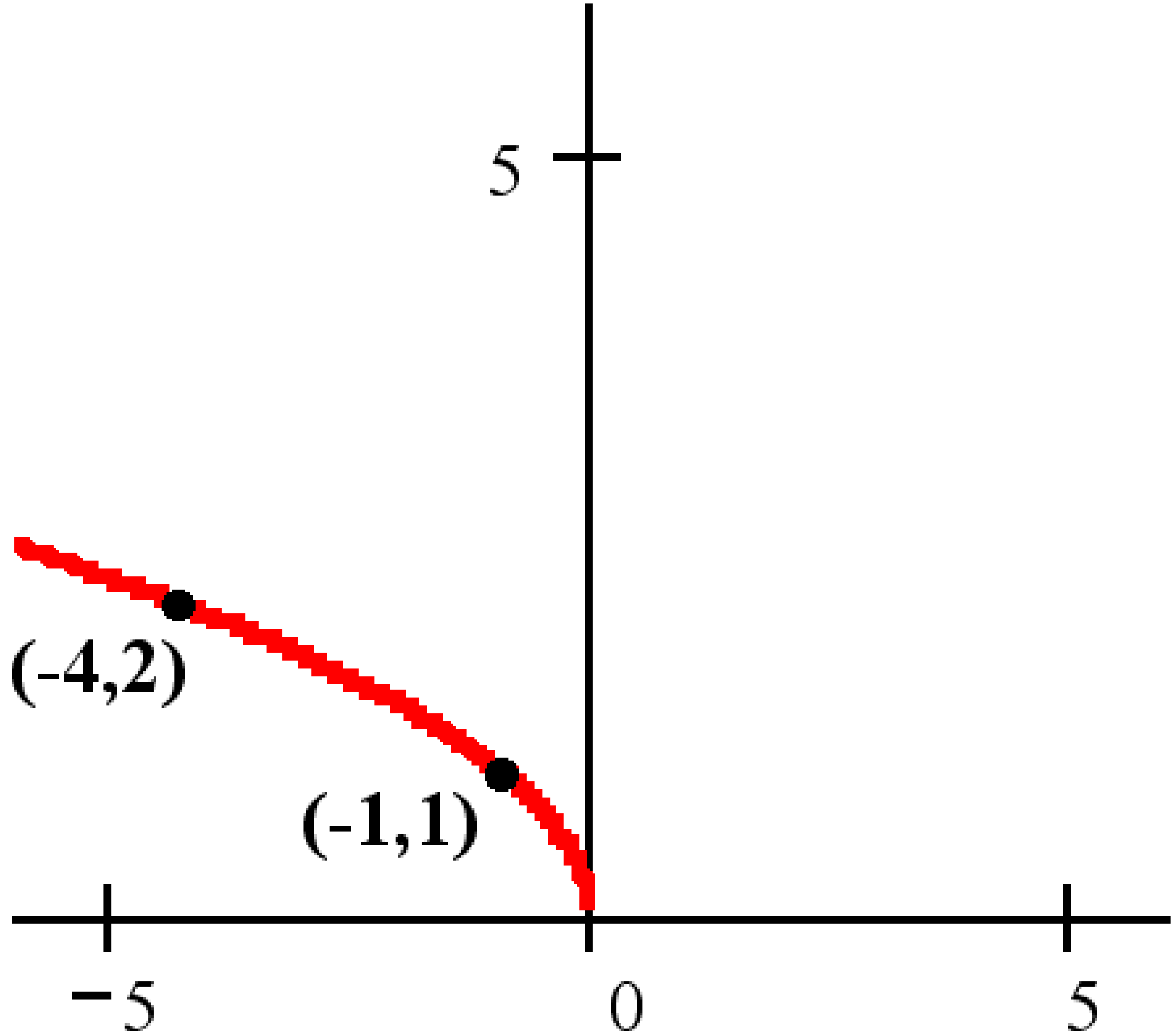


$y = \sqrt{-x}$
$y = \sqrt{x}$
5
4
3
2
1
(-9, 3)
(-4, 2)
(-1, 1)
(9, 3)
(4, 2)
(1, 1)
−10
0
10

Use the graph of $f(x) = \sqrt{x}$ to obtain the graph of $g(x) = -\sqrt{-x}$.

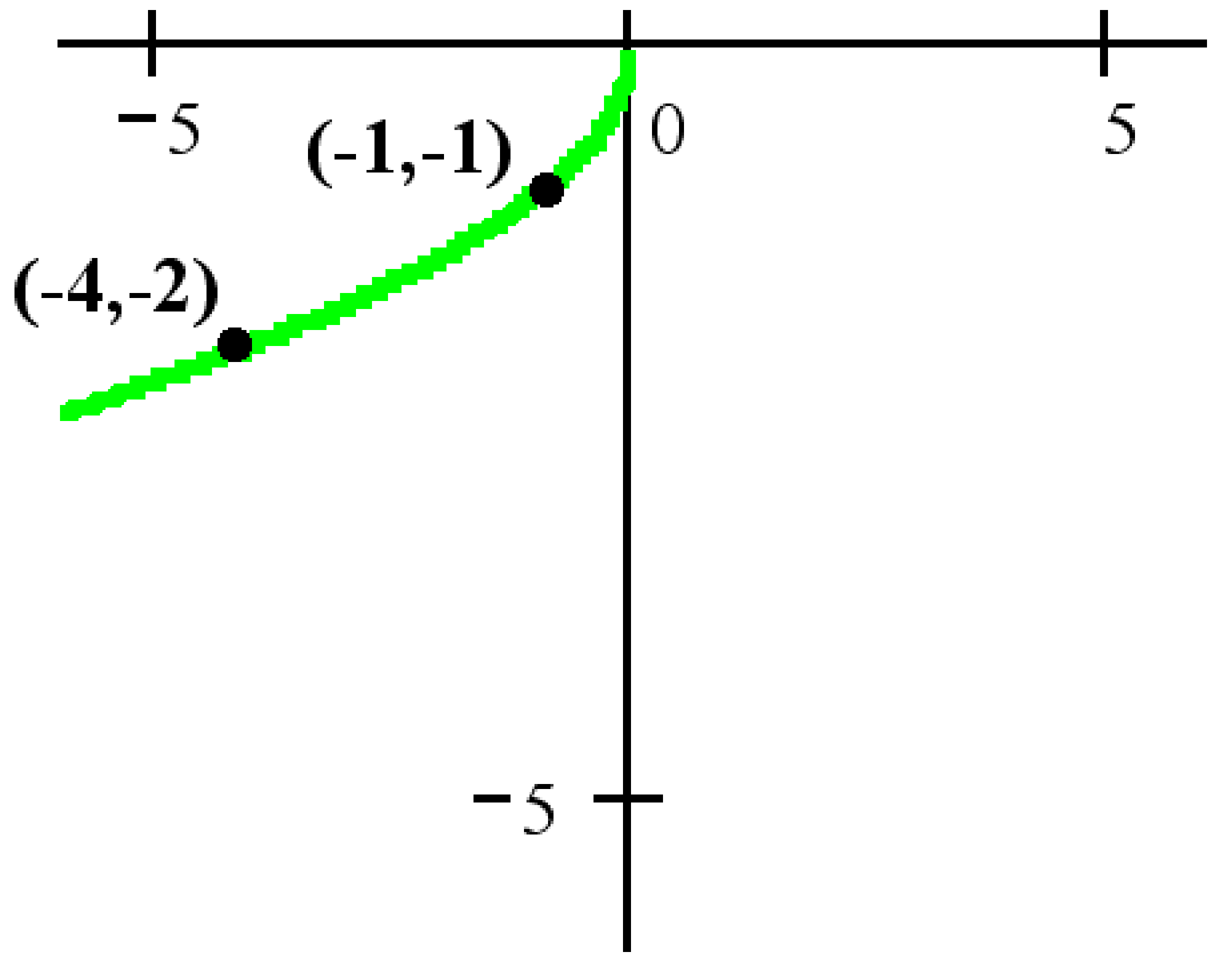


$$y = \sqrt{x}$$



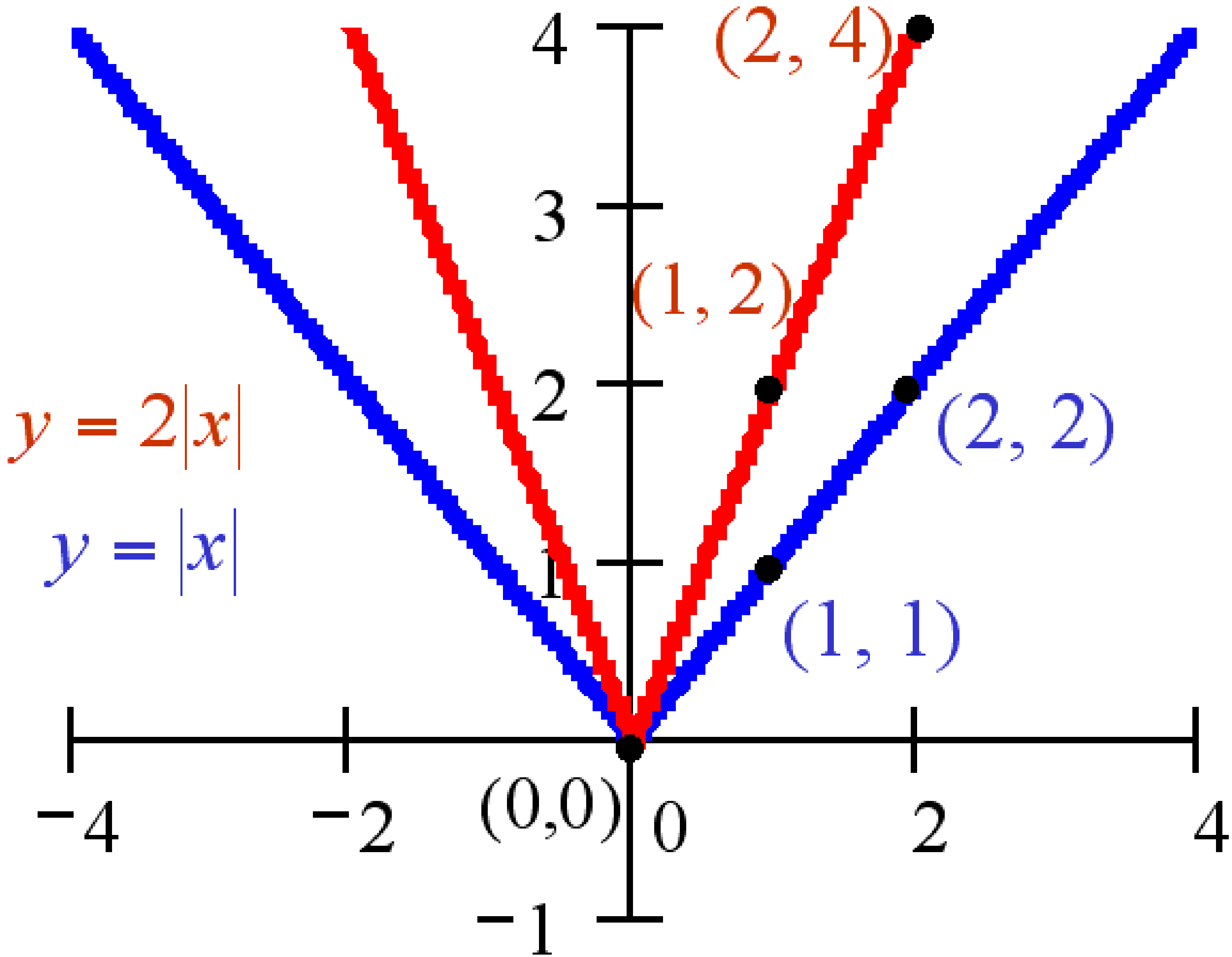
$$y = \sqrt{-x}$$

$$g(x) = -\sqrt{-x}$$



Compression and Stretches

- The graph of $y=af(x)$ is obtained from the graph of $y=f(x)$ by vertically stretching the graph if $a > 1$ or vertically compressing the graph if $0 < a < 1$.
- The graph of $y=f(ax)$ is obtained from the graph of $y=f(x)$ by horizontally compressing the graph if $a > 1$ or horizontally stretching the graph if $0 < a < 1$.



(2, 4)
(1, 2)
(2, 2)
(1, 1)
(0,0)
y = 2|x|
y = |x|
4
3
2
1
−1
−4
−2
0
2
4

Operations on Functions

The **sum** $\boldsymbol{f+g}$ is the function defined by

$$\boxed{(f+g)(x) = f(x) + g(x)}$$

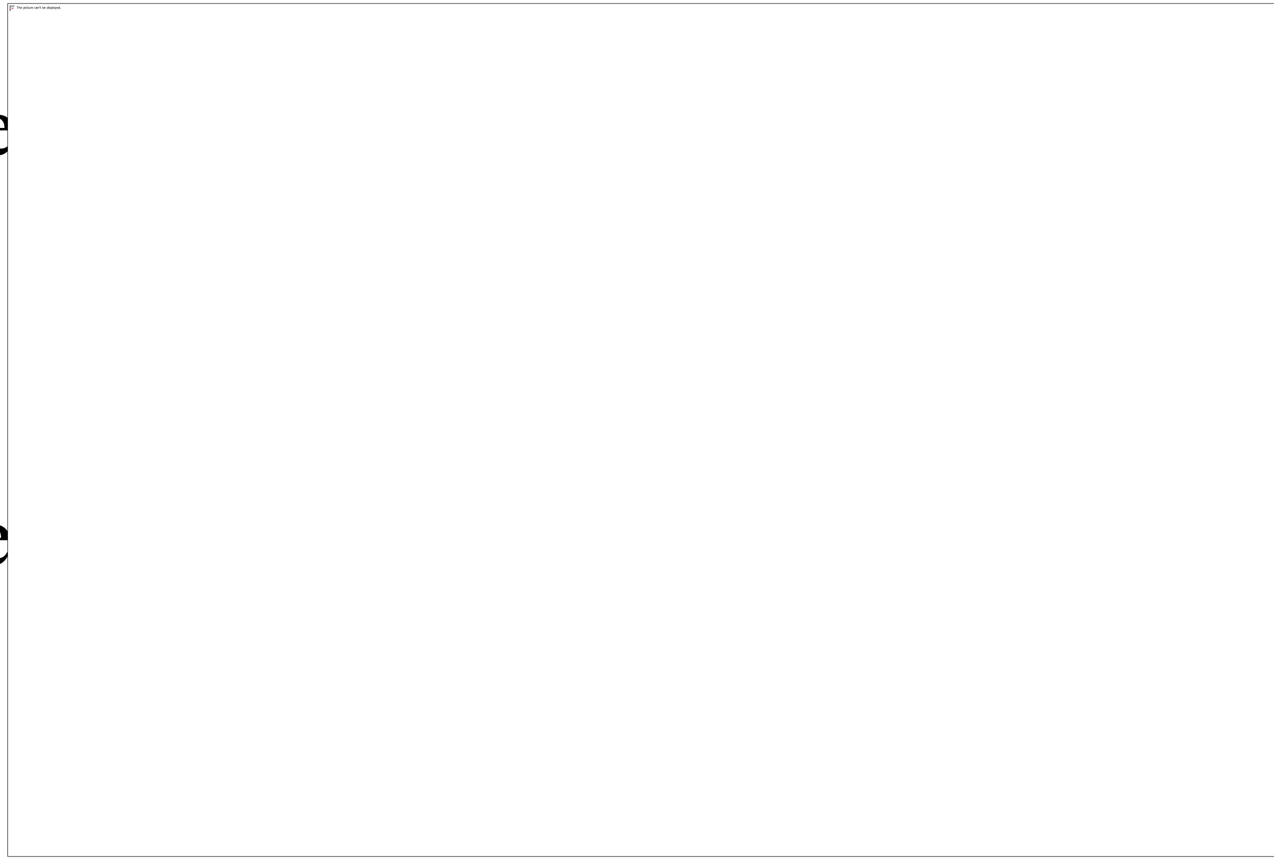
The domain of $f+g$ consists of numbers x that are in the domain of both f and g.

The **difference** $\boldsymbol{f\text{-}g}$ is the function defined by

$$\boxed{(f - g)(x) = f(x) - g(x)}$$

The domain of f-g consists of numbers x that are in the domain of both f and g.

The by



The that
are

The quotient $\dfrac{f}{g}$ is the function

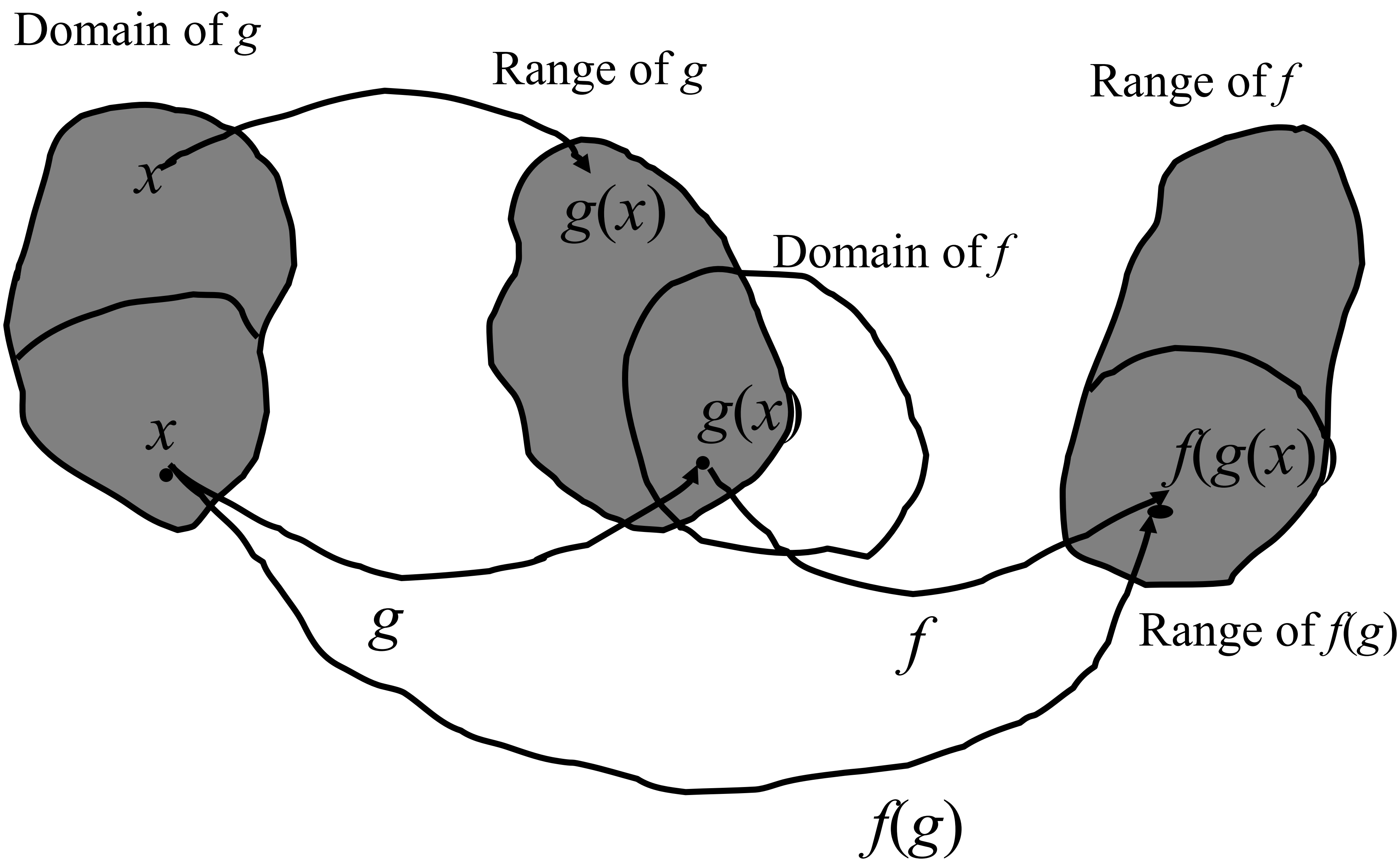
$$\boxed{\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0}$$

The domain of $\dfrac{f}{g}$ consists of numbers x for which $g(x) \neq 0$ and are in the domains of both f and g.

Given two functions *f* and *g*, the co

$$\boxed{(f \circ g)(x) = f(g(x))}$$

The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.



Domain of g
Range of g
Range of f
x
$g(x)$
Domain of f
x
$g(x)$
$f(g(x))$
g
f
Range of $f(g)$
$f(g)$

In general

$$\boxed{(f \circ g)(x) \neq (g \circ f)(x)}$$

Suppose $f(x) = \dfrac{1+x}{\sqrt{x-1}}$ and $g(x) = x^2 - 2$.

Find $f \circ g$ and $g \circ f$ and $g \circ g$, and their domains.

$$(f \circ g)(x) = f(g(x)) = f(x^2 - 2) =$$

$$= \frac{1+(x^2-2)}{\sqrt{(x^2-2)-1}} = \frac{x^2-1}{\sqrt{x^2-3}}$$

Domain of $f(x)$ is $x-1>0$, meaning $x>1$.

Domain of $g(x)$ is all real numbers.

Domain of $(f \circ g)(x)$ consists of those real numbers from the domain of g that have $g(x)>1$. That is $\left(x^2-2\right)>1$

$$x^2>3$$

$$|x|>\sqrt{3}$$

$$\boxed{\text{Domain } (f \circ g)(x) = \left\{x \middle| x>\sqrt{3} \text{ or } x<-\sqrt{3}\right\}.}$$

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{1+x}{\sqrt{x-1}}\right) =$$

$$= \left(\frac{1+x}{\sqrt{x-1}}\right)^2 - 2 = \frac{(1+x)^2}{(x-1)} - 2 =$$

$$\boxed{= \frac{1+2x+x^2-2(x-1)}{x-1} = \frac{x^2+3}{x-1}}$$

Domain of $(g \circ f)(x)$ consists of those real numbers from the domain of f that have $g(x)$ defined, but that is defined for all real numbers. The domain of $g \circ f$ is then equal to the domain of f.

$$\boxed{(g \circ g)(x)} = g(g(x)) = g(x^2 - 2) =$$

$$(x^2 - 2)^2 - 2 = x^4 + 4x^2 + 4 - 2 =$$

$$= \boxed{x^4 + 4x^2 + 2}$$

Domain of $g \circ g$ is all real numbers.

One-to-one and Inverse Functions

What is an Inverse?

An inverse relation is a relation that performs the opposite operation on *x* (the domain).

Examples:

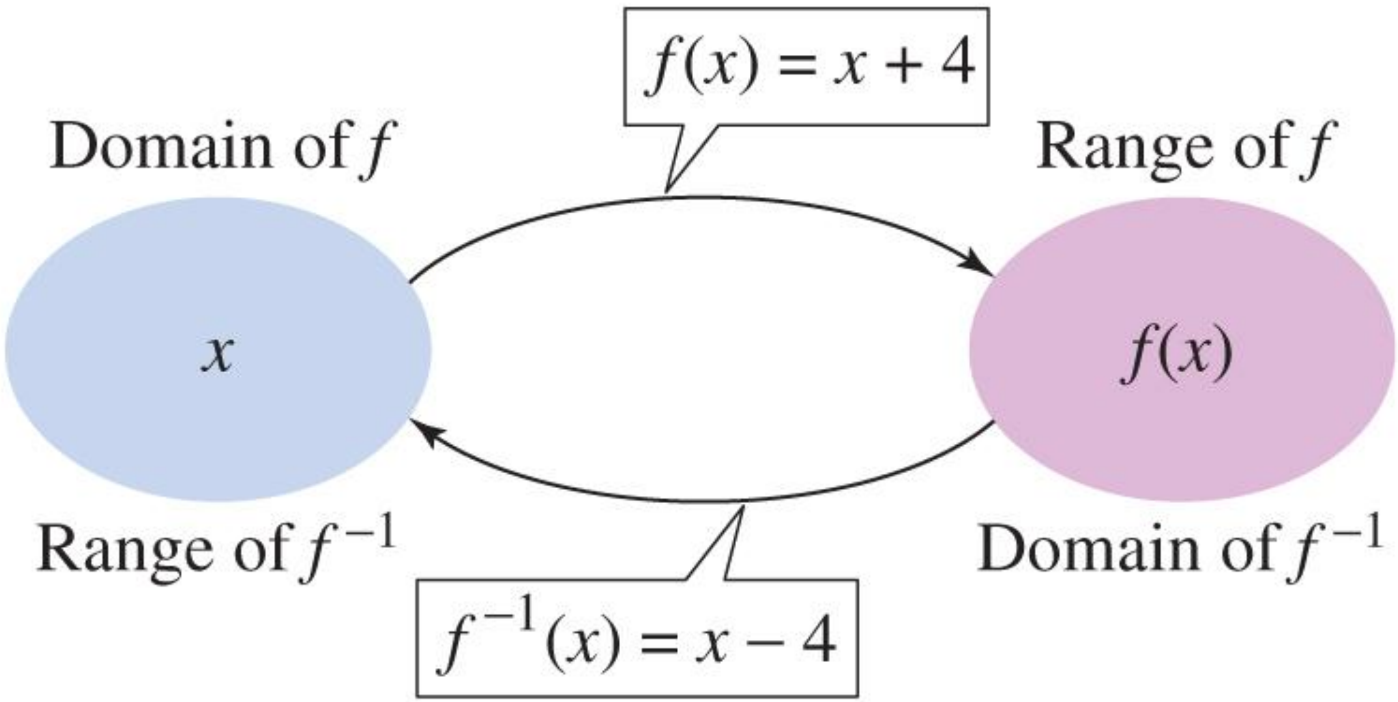
$f(x) = x - 3$ $\quad$ $f^{-1}(x) = x + 3$

$g(x) = \sqrt{x}$, $x \geq 0$ $\quad$ $g^{-1}(x) = x^2$, $x \geq 0$

$h(x) = 2x$ $\quad$ $h^{-1}(x) = \frac{1}{2}x$

$k(x) = -x + 3$ $\quad$ $k^{-1}(x) = -(x - 3)$

Section 1.9 : Illustration of the Definition of Inverse Functions



The ordered pairs of the function f are *reversed* to produce the ordered pairs of the inverse relation.

Example: Given the function
$f = \{(1, 1), (2, 3), (3, 1), (4, 2)\}$, its domain is $\{1, 2, 3, 4\}$ and its range is $\{1, 2, 3\}$.

The inverse ________ of f is $\{(1, 1), (3, 2), (1, 3), (2, 4)\}$.

The *domain* of the inverse relation is the *range* of the original function.

The *range* of the inverse relation is the *domain* of the original function.

How do we know if an inverse function exists?

- Inverse functions only exist if the original function is one to one. Otherwise it is an inverse relation and cannot be written as $f^{-1}(x)$.

- What does it mean to be one to one?

That there are no repeated y values.

Horizontal Line Test

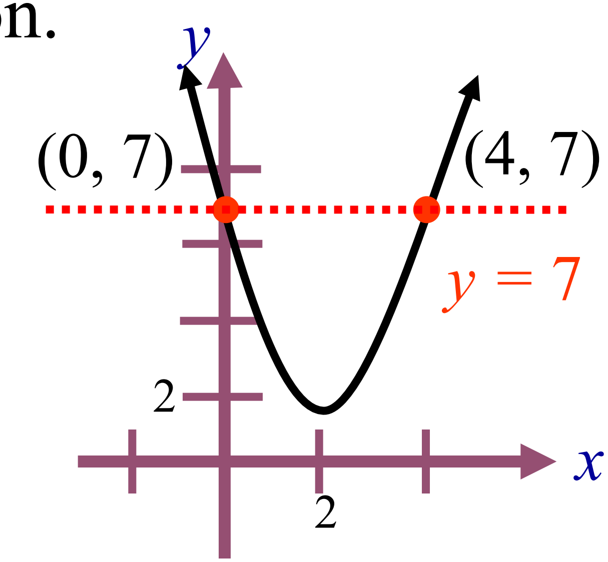
Used to test if a function is one-to one

If the line intersection more than once then it is not one to one.

Therefore there is not inverse function.

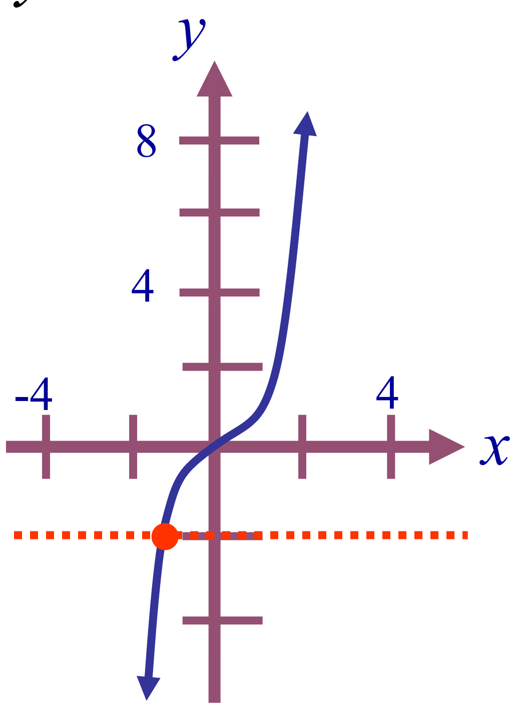
Example: The function $y = x^2 - 4x + 7$ is **not one-to-one** because a horizontal line can intersect the graph twice.

Examples points: (0, 7) & (4, 7).



Example: Apply the *horizontal line test* to the graphs below to determine if the functions are one-to-one.

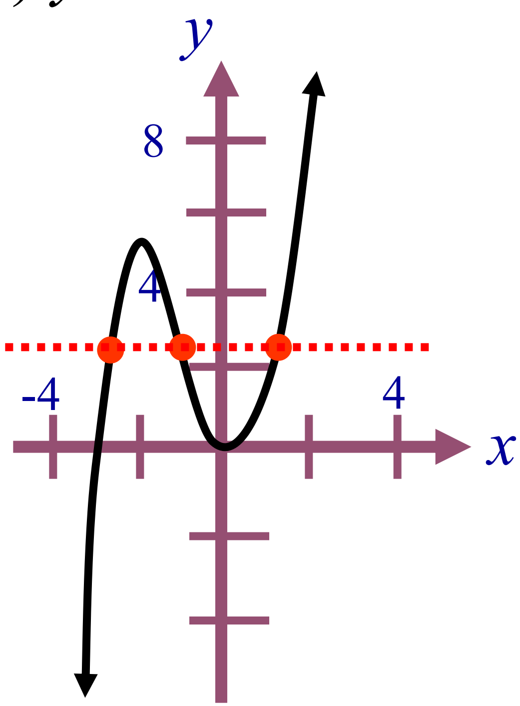
a) $y = x^3$



one-to-one

The Inverse is a Function

b) $y = x^3 + 3x^2 - x - 1$



not one-to-one

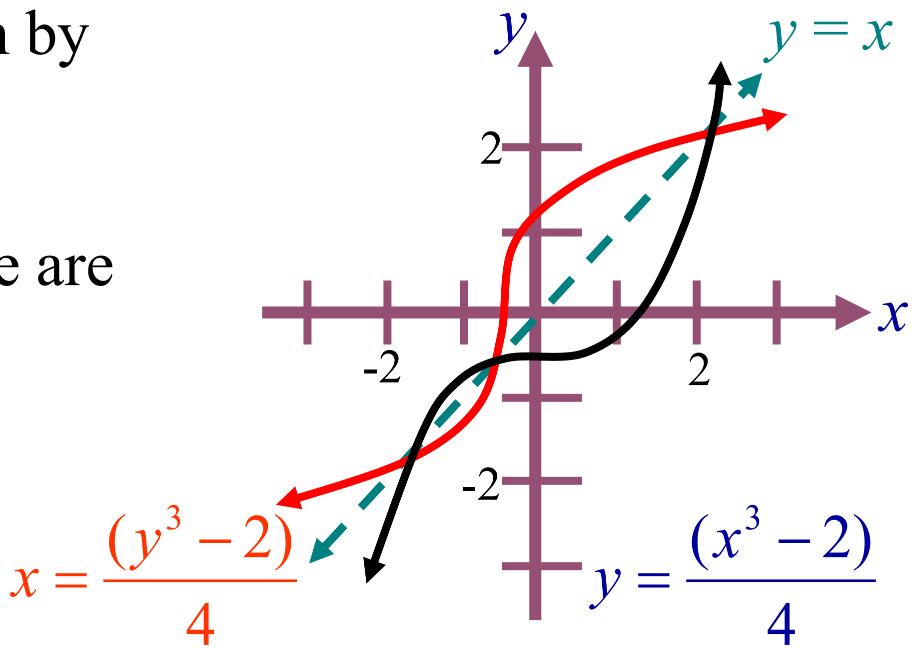
The Inverse is a Relation

The graphs of a relation and its inverse are reflections in the line $y = x$.

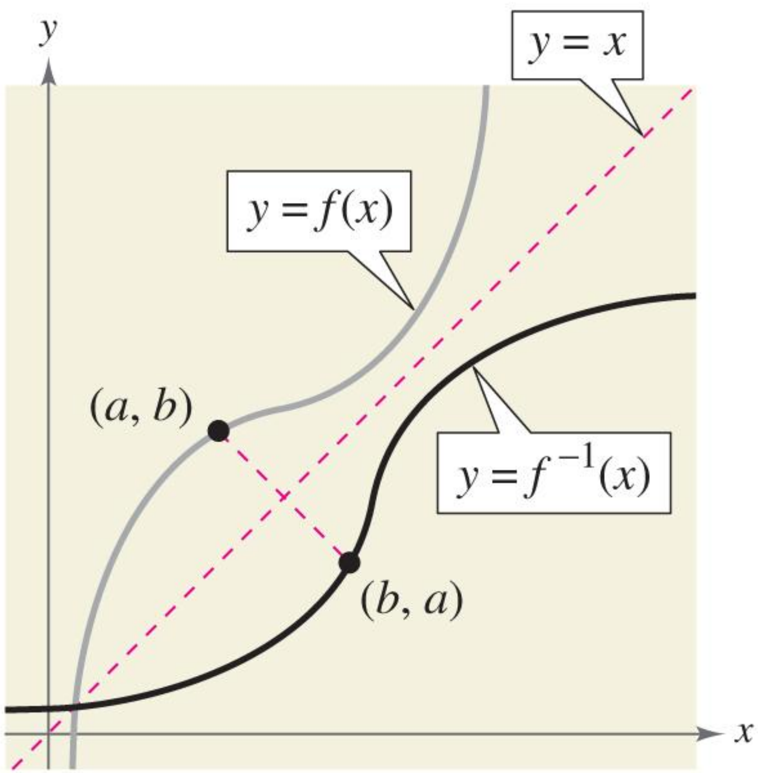
Example: Find the graph of the inverse relation *geometrically* from the graph of $f(x) = \frac{1}{4}(x^3 - 2)$

The ordered pairs of f are given by the equation $y = \frac{1}{4}(x^3 - 2)$.

The ordered pairs of the inverse are given by $x = \frac{1}{4}(y^3 - 2)$.



Section 1.9 : Figure 1.93, Graph of an Inverse Function



Functions and their inverses are symmetric over the line y =x

To find the inverse of a relation *algebraically*, interchange x and y and solve for y.

Example: Find the inverse relation *algebraically* for the function $f(x) = 3x + 2$.

DETERMINING IF 2 FUNCTIONS ARE INVERSES:

The inverse function "undoes" the original function, that is, $f^{-1}(f(x)) = x$.

The function is the *inverse* of its inverse function, that is, $f(f^{-1}(x)) = x$.

Example: The inverse of $f(x) = x^3$ is $f^{-1}(x) = \sqrt[3]{x}$.

$f^{-1}(f(x)) = \sqrt[3]{x^3} = x$ and $f(f^{-1}(x)) = (\sqrt[3]{x})^3 = x$.

Example: Verify that the function $g(x) = \dfrac{x+1}{2}$ is the *inverse* of $f(x) = 2x - 1$.

$$g(f(x)) = \frac{(f(x)+1)}{2} = \frac{((2x-1)+1)}{2} = \frac{2x}{2} = x$$

$$f(g(x)) = 2g(x) - 1 = 2\left(\frac{x+1}{2}\right) - 1 = (x + 1) - 1 = x$$

It follows that $g = f^{-1}$.

Review of Today's Material

- A function must be 1-1 (pass the horizontal line test) to have an inverse function (written $f^{-1}(x)$) otherwise the inverse is a relation (y =)

- To find an inverse: 1) Switch x and y
  2) Solve for y
- Original and Inverses are symmetric over y =x
- " " " have reverse domain & ranges

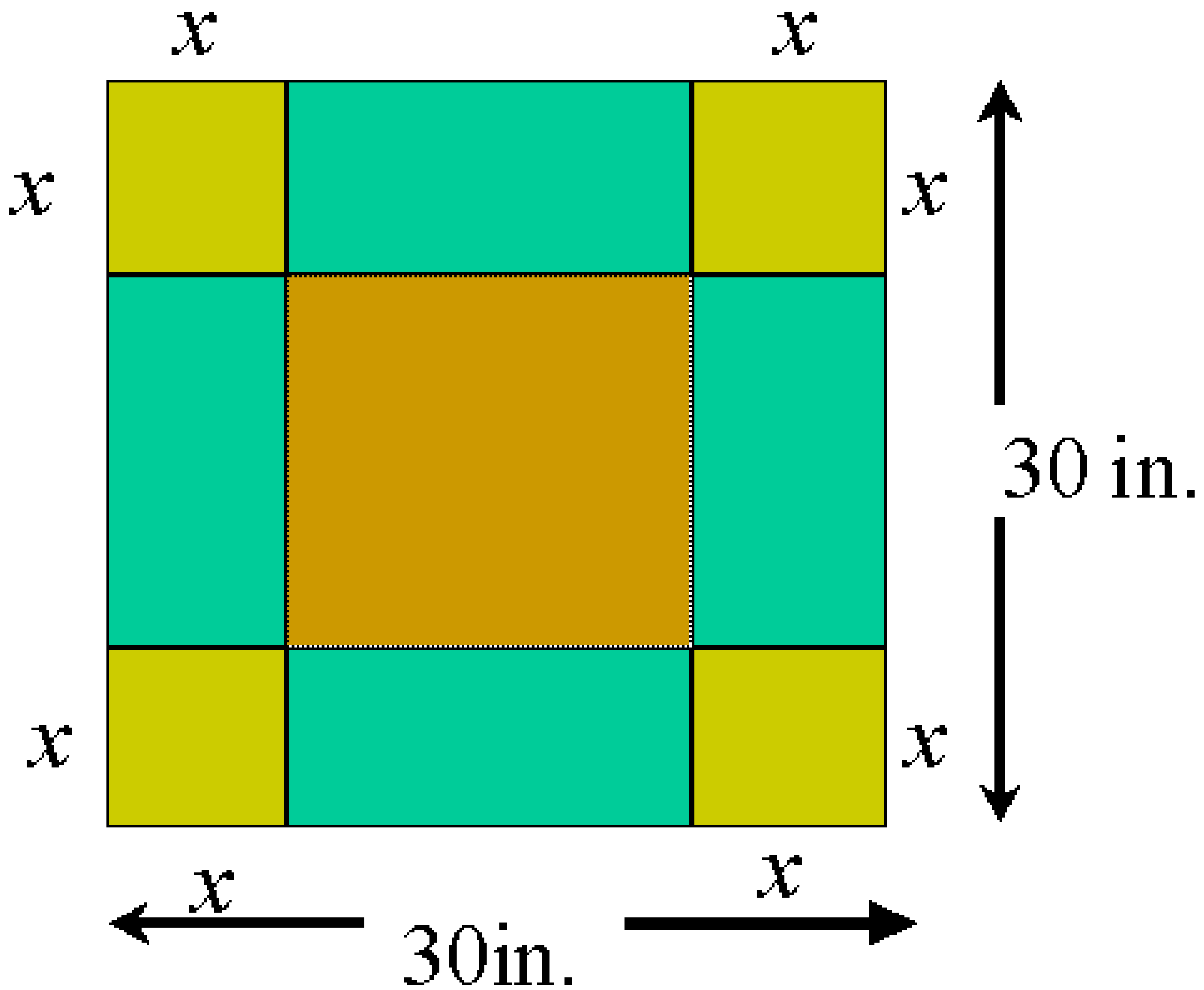
- Given two relations to test for inverses.

$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ **both must be true**

Mathematical Models

An open box with a square base is to be made from a square piece of cardboard 30 inches wide on a side by cutting out a square from each corner and turning up the sides.

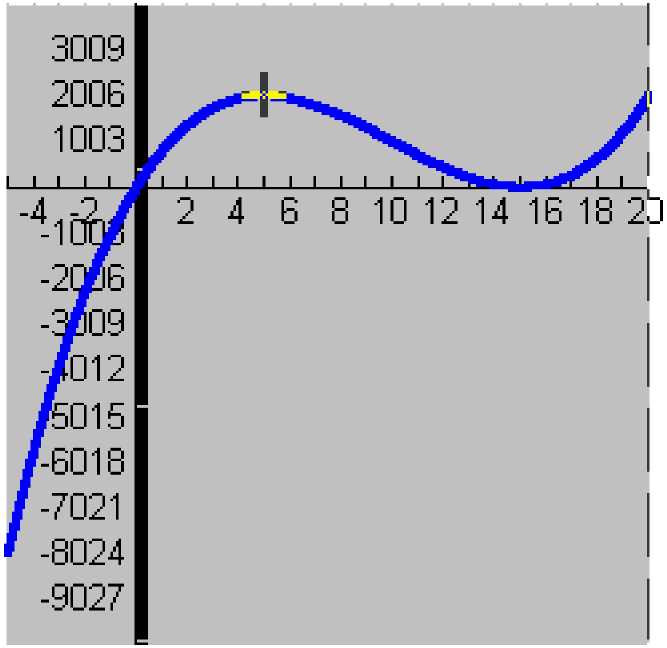
(A) Express the volume V of the box as a function of the length x of the side of the square cut from each corner.


x
x
x
x
x
x
x
x
30 in.
30in.

Volume = (length)(width)(height)

$$\boxed{V(x) = (30 - 2x)(30 - 2x)x = (30 - 2x)^2 x}$$

(B) Graph $V = V(x)$.



(C) For what value of x is V largest?

V is largest (2000 cubic inches), when $x = 5$ inches.