SECTION 2.5: FINDING ZEROS OF POLYNOMIAL FUNCTIONS

Assume $f(x)$ is a nonconstant polynomial with real coefficients written in standard form.

PART A: TECHNIQUES WE HAVE ALREADY SEEN

Refer to:
- Notes 1.31 to 1.35
- Section A.5 in the book
- Notes 2.45

	Refer to
1) Factoring	(Notes 1.33)
2) Methods for Dealing with Quadratic Functions	(Book Section A.5: pp.A49-51)
a) Square Root Method	(Notes 1.31, 2.45)
b) Factoring	(Notes 1.33)
c) QF	(Notes 1.34, 2.45)
d) CTS (Completing the Square)	(Book Section A.5: p.A49)
3) Bisection Method (for Approximating Zeros)	(Notes 2.20 to 2.21)
4) Synthetic Division and the Remainder Theorem (for Verifying Zeros)	(Notes 2.33)

PART B: RATIONAL ZERO TEST

Rational Zero Test (or Rational Roots Theorem)

Let $f(x)$ be a polynomial with integer (i.e., only integer) coefficients written in standard form:

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$
$$\left(\text{each constant } a_i \in \mathbf{Z};\ a_n \neq 0;\ a_0 \neq 0;\ n \in \mathbf{Z}^+\right)$$

If $f(x)$ has rational zeros, they must be in the list of $\pm\frac{p}{q}$ candidates, where:

p is a factor of a_0, the constant term, and
q is a factor of a_n, the leading coefficient.

Note: We require $a_0 \neq 0$. If $a_0 = 0$, try factoring out the GCF first.

Example

Factor $f(x) = 4x^3 - 5x^2 - 7x + 2$ completely, and find **all** of its real zeros.

Solution

Since the $\text{GCF} = 1$, and Factoring by Grouping does not seem to help, we resort to using the Rational Zero Test. We will now list the candidates for possible rational zeros of $f(x)$.

p (factors of the constant term, 2): $\pm 1,\ \pm 2$
q (factors of the leading coefficient, 4): $\pm 1,\ \pm 2,\ \pm 4$

Note: You may omit the $\pm$ symbols above if you use them below.

List of $\pm\frac{p}{q}$ candidates:

$$\pm\frac{1}{1},\ \pm\frac{1}{2},\ \pm\frac{1}{4},\ \pm\frac{2}{1},\ \pm\frac{2}{2},\ \pm\frac{2}{4}$$

Simplified:

$$\pm 1,\ \pm\frac{1}{2},\ \pm\frac{1}{4},\ \pm 2,\ \underbrace{\pm 1,\ \pm\frac{1}{2}}_{\text{Redundant}}$$

Use Synthetic Division to divide $f(x)$ by $(x-k)$, where k is one of our rational candidates. Remember that the following are equivalent for a nonzero polynomial $f(x)$ and a real number k:

$(x-k)$ is a factor of $f(x)$ $\Leftrightarrow$
k is a zero of $f(x)$ $\left(\text{i.e., } f(k)=0\right)$ $\Leftrightarrow$
We get a 0 remainder in the Synthetic Division process.

The first $\Leftrightarrow$ is the Factor Theorem, and the second $\Leftrightarrow$ comes from the Remainder Theorem. See Notes on Section 2.3: 2.32-2.34.

We use trial-and-error and proceed through our list of candidates (k) for rational zeros: $\pm 1,\ \pm\frac{1}{2},\ \pm\frac{1}{4},\ \pm 2$

Let's try $k=1$.

<u>Method 1</u>

We may directly evaluate $f(1)$ and see if it is 0.

$$f(x)=4x^3-5x^2-7x+2$$
$$\begin{aligned} f(1) &= 4(1)^3-5(1)^2-7(1)+2 \\ &= -6 \quad (\neq 0) \end{aligned}$$

Therefore, 1 is **not** a zero of $f(x)$.

Method 2

We may also use the Synthetic Division process and see if we get a 0 remainder.

$$\begin{array}{r|rrrr} 1 & 4 & -5 & -7 & 2 \\ & & 4 & -1 & -8 \\ \hline & 4 & -1 & -8 & -6 \end{array}$$

We do **not** get a 0 remainder, so 1 is **not** a zero of $f(x)$.

Method 3

Observe from both previous methods that we can compute $f(1)$ by simply adding up the coefficients of $f(x)$ in standard form. This does not work in general for other values of *k*, though.

Let's try $k = 2$.

Let's use the Synthetic Division / Remainder Theorem method:

$$\begin{array}{r|rrrr} 2 & 4 & -5 & -7 & 2 \\ & & 8 & 6 & -2 \\ \hline & 4 & 3 & -1 & 0 \end{array}$$

We **do** get a 0 remainder, so 2 **is** a zero of $f(x)$.

This turns out to be the key that cracks the whole problem. Incidentally, this is the same $f(x)$ that we saw in Notes 2.33-2.35. Now we know how our "little bird" got its info!

By the Factor Theorem, $(x-2)$ must be a factor of $f(x)$.

We can find $q(x)$, the other (quadratic) factor, by using the last row of the table.

$$f(x)=(x-2)\cdot\left(4x^2+3x-1\right)$$

Factor $q(x)$ completely over the reals:

$$\boldsymbol{f(x)=(x-2)(4x-1)(x+1)}$$

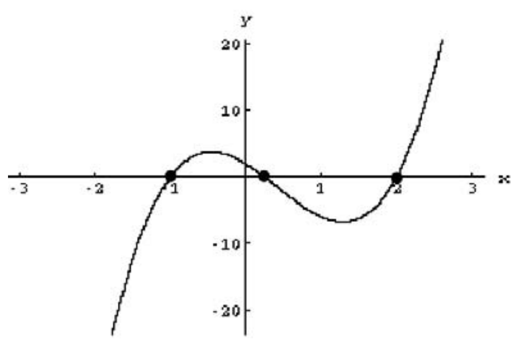
Remember that we always know how to break down a quadratic (use the QF if you have to), although its zeros may or may not be real. We no longer have to rely on rational zeros at this point.

The zeros of $f(x)$ are the zeros of these factors:

$$\mathbf{2,\ \frac{1}{4},\ -1}$$

Observe that all three are rational and appeared in our list of candidates for rational zeros. Any one of these three could have been used to start cracking the problem.

Below is a graph of $f(x)=4x^3-5x^2-7x+2$. Where are the x-intercepts?



Note: If we can get a graph of $f(x)$ beforehand, then we may be able to choose our guesses for rational zeros more wisely.

Warning: Remember that the template for our list of candidates is:

$$\pm\frac{\text{factor of constant term}}{\text{factor of leading coefficient}},$$

not the reciprocal. One way to remember which way the template goes is to use a simple example such as $x-2$. The list must be $\pm 1, \pm 2$ and not $\pm 1, \pm\frac{1}{2}$.

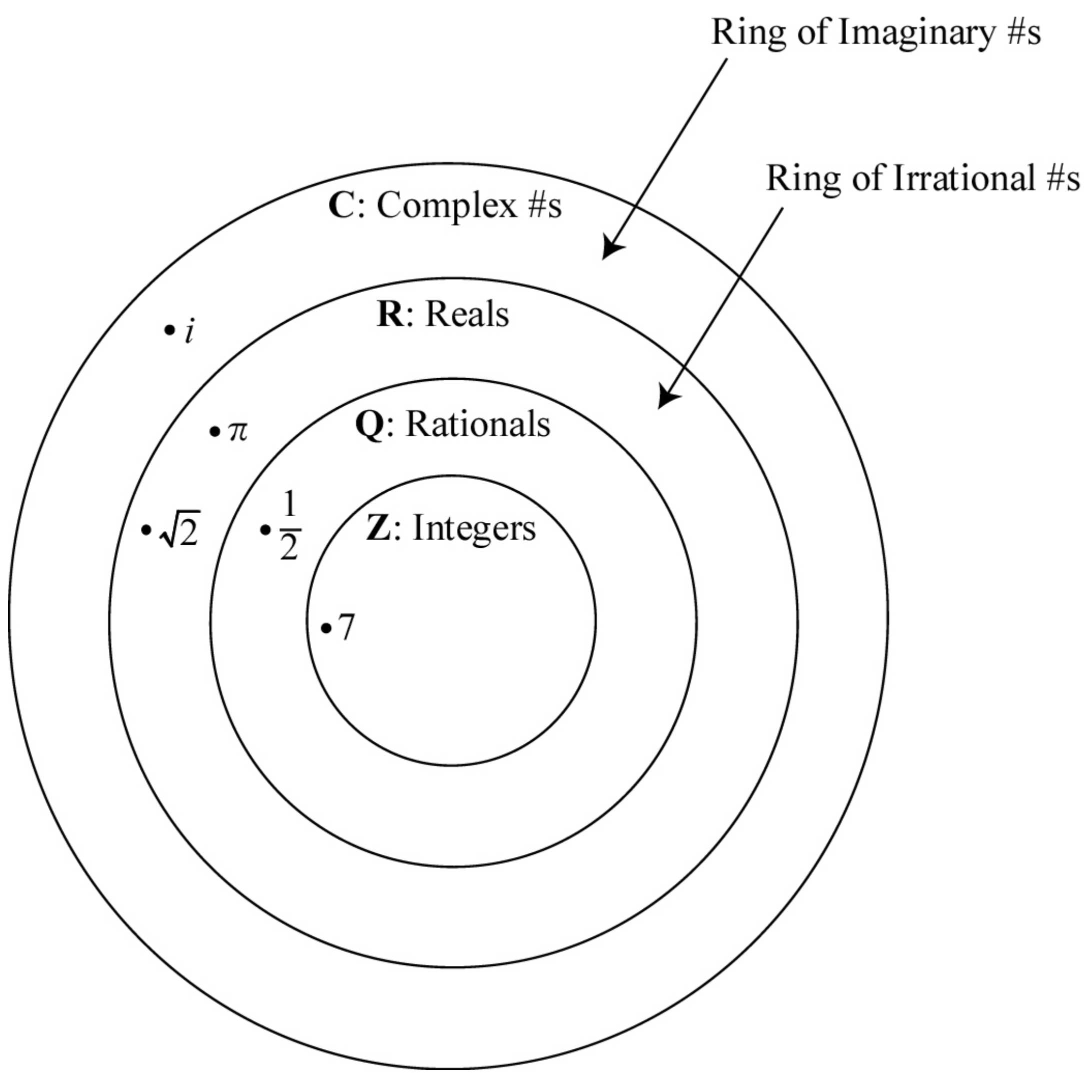
Note: If none of our rational candidates work, then $f\left(x\right)$ has no rational zeros. For example, x^2-3 has no rational zeros.

Note: Synthetic Division may be applied repeatedly. It may be a good idea to revise the list of rational candidates before each new application (the textbook does not do this). A candidate that "worked" as a zero before may work again (in which case it is a repeated zero), but a candidate that "failed" before will never work. You may want to stop using Synthetic Division when you get down to a quadratic, or when you get down to something you think you can factor, such as x^4-1.

Note: Synthetic Division may be used when we are dealing with imaginary zeros and imaginary coefficients. This is reflected in the students' Study and Solutions Guide.

PART C: FACTORING OVER VARIOUS SETS

Recall our Venn diagram from Notes P.03:



Factoring over **Z** (the Integers)

Example

In our Example in Part B, we factored:

$$4x^3 - 5x^2 - 7x + 2 = (x - 2)(4x - 1)(x + 1)$$

This is an example of factoring over **Z**, because we only use integers as coefficients (including constant terms within factors).

<u>Factoring over **Q** (the Rationals)</u>

<u>Example</u>

Let's factor a 4 out of the second factor in the previous Example.

$$4x^3 - 5x^2 - 7x + 2 = (x-2)(4x-1)(x+1) \quad \leftarrow \text{Factored over } \mathbf{Z}$$
$$= 4(x-2)\left(x-\frac{1}{4}\right)(x+1) \quad \leftarrow \text{Factored over } \mathbf{Q}$$

The "Factored over **Z**" expression is also an example of factoring over **Q**, but this new factorization over **Q** immediately identifies $\frac{1}{4}$ as a zero.

<u>Factoring over **R** (the Reals)</u>

<u>Example</u>

$x^2 - 3$ is <u>prime</u> (or <u>irreducible</u>) over **Z** and **Q**; it cannot be factored further (nontrivially; breaking out a 1 or a -1 doesn't count) using only integer or rational coefficients. However, it can be factored over **R**.

$$x^2 - 3 = \left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$$

Note that $-\sqrt{3}$ and $\sqrt{3}$ are immediately identified as zeros.

Factoring over **C** (the Complex Numbers)

Recall our work from Notes 2.40. We found that:
$a^2 + b^2 = (a+bi)(a-bi)$, where a and b were real numbers.
However, this form is also appropriate if a and/or b represent variable expressions.

Example

$x^2 + 9$ is prime (or irreducible) over **R**. However, it can be factored over **C**.

$$x^2 + 9 = (x+3i)(x-3i)$$

Note that $-3i$ and $3i$ are immediately identified as zeros here.

Example

Factor (i.e., factor completely) $x^5 + x^3 - 6x$ over **R** and find all of its real zeros.

Solution

First factor out the GCF, x.

$$x^5 + x^3 - 6x = x\left(x^4 + x^2 - 6\right)$$

The second factor is in Quadratic Form, because it is of the form $u^2 + u - 6$, where $u = x^2$. How do we know $x^4 + x^2 - 6$ is in Quadratic Form? Observe that the exponent on x in the first term is twice the exponent on x in the second term, and the third term is a constant.

Substitute $u = x^2$ (optional, but it may help):

$$\begin{aligned} x^5 + x^3 - 6x &= x\left(x^4 + x^2 - 6\right) \\ &= x\left(u^2 + u - 6\right) \\ &= x(u+3)(u-2) \\ &= x\left(x^2+3\right)\left(x^2-2\right) \qquad \leftarrow \text{Substitute back.} \end{aligned}$$

We have factored completely over **Z** (and **Q**).
Let us now factor completely over **R**.

$$x^5 + x^3 - 6x = x\left(x^2+3\right)\left(x^2-2\right) \qquad \leftarrow \text{Reminder}$$
$$= \boldsymbol{x\left(x^2+3\right)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)} \qquad \leftarrow \text{Factored over } \mathbf{R}$$

$\left(x^2+3\right)$ is a quadratic that is irreducible over the reals. It therefore yields no real zeros. (We need more theorems to show this.)

From the other three factors, we obtain $\mathbf{0, -\sqrt{2}, \text{and } \sqrt{2}}$ as real zeros.

<u>Example</u>

Factor $f(x) = x^5 + x^3 - 6x$ over **C** and find all of its real zeros.

<u>Solution</u>

We continue with our work from the previous Example.

We can factor $\left(x^2+3\right)$ further over **C**.

$$x^5 + x^3 - 6x = x\left(x^2+3\right)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right) \qquad \leftarrow \text{Factored over } \mathbf{R}$$
$$= \boldsymbol{x\left(x+i\sqrt{3}\right)\left(x-i\sqrt{3}\right)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)} \qquad \leftarrow \text{Factored over } \mathbf{C}$$

We immediately obtain the five complex zeros of $f(x)$:

$\mathbf{0,\ -i\sqrt{3},\ i\sqrt{3},\ -\sqrt{2},\ \text{and } \sqrt{2}}$

PART D: COMPLEX CONJUGATE PAIRS OF ZEROS

Complex Conjugate Pairs Theorem

Let $a, b \in \mathbf{R}$.

If $f\left(x\right)$ is a polynomial with (only) real coefficients, then:
$a + bi$ is a zero of $f\left(x\right) \Leftrightarrow a - bi$ is a zero of $f\left(x\right)$.

In our last Example in Part C, if we know that $i\sqrt{3}$ is a zero of $f\left(x\right)$, then we can conclude that $-i\sqrt{3}$ must also be a zero.

Technical Note: The theorem requires real coefficients. Observe that $x - i$ has i as a zero but not $-i$.

Note: There is a Conjugate Pairs Theorem for a quadratic polynomial $f\left(x\right)$ with (only) rational coefficients. Consider $f\left(x\right) = ax^2 + bx + c$, where $a, b, c \in \mathbf{Q}$ and $a \neq 0$. As an example, $2 + \sqrt{3}$ is a zero of such an $f\left(x\right) \Leftrightarrow 2 - \sqrt{3}$ is.
The structure of the QF implies this.

PART E: THE FUNDAMENTAL THEOREM OF ALGEBRA (FTA)

The Fundamental Theorem of Algebra (FTA)

If $f(x)$ is a nonconstant n^{th}-degree polynomial in standard form with real coefficients, then it must have at least one complex (possibly real) zero.

Put Another Way: It must have exactly n complex zeros, where the zeros may be repeated based on their multiplicities.

Technical Note: The Fundamental Theorem of Arithmetic states that any integer greater than or equal to 2 is either prime or can be decomposed uniquely as a product of (possibly repeated) primes (or "prime powers"), up to a reordering of the factors. For example, 6 can only be decomposed in one way: $6 = 2 \cdot 3$. The decomposition $6 = 3 \cdot 2$ does not count as a different one.

Technical Note: The Fundamental Theorem of Calculus will allow you to evaluate definite integrals, which are used in finding areas, volumes, arc lengths, surface areas, and much more.

Historical Note: The FTA was first proven by the great Gauss. For more history, see p.193 of the textbook.

PART F: THE LINEAR FACTORIZATION THEOREM (LFT)

The Linear Factorization Theorem (LFT)

If $f(x)$ is a nonconstant polynomial in standard form with real coefficients, then it must have a factorization into linear factors of the form:

$$f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n)$$
$$\left(a_n \in \mathbf{R};\ a_n \neq 0;\ \text{each } c_i \in \mathbf{C}\right)$$

Note: The zeros of $f(x)$ are then $c_1, c_2, \ldots, c_n$.

Note: There may be repetitions of a zero c_i, based on the multiplicity of c_i.

Note: a_n is the leading coefficient of $f(x)$.

Technical Note: The LFT is proven using the FTA and the Factor Theorem. See p.193 of the textbook.

Technical Note: This helps explain the Complex Conjugate Pairs Theorem in Notes 2.56.

Example

Let $f(x) = x^5 - 8x^4 + 16x^3$.

$$\begin{aligned} x^5 - 8x^4 + 16x^3 &= x^3\left(x^2 - 8x + 16\right) \\ &= x^3(x-4)^2 \end{aligned}$$

It may be said that $f(x)$ has 5 zeros: 0, 0, 0, 4, and 4.

$f(x)$ has only 2 distinct zeros: 0 and 4.
They are both repeated zeros:
The multiplicity of 0 is 3, and the multiplicity of 4 is 2.

You can think of x as $(x - 0)$.

Recall our Examples in Notes 2.52 and 2.53.

$$4x^3 - 5x^2 - 7x + 2 = (x-2)(4x-1)(x+1) \qquad \leftarrow \text{Factored over } \mathbf{Z}$$

$$= 4(x-2)\left(x - \frac{1}{4}\right)(x+1) \qquad \leftarrow \text{Factored over } \mathbf{Q}$$

The second factorization is in "LFT Form." The zeros can be immediately read off (watch out for signs, though).

PART G: FACTORING OVER R

"Factoring Over **R**" Theorem

Let $f(x)$ be a nonconstant polynomial in standard form with real coefficients. Its complete factorization over **R** (the reals) consists of:

1) Linear factors,
2) Quadratic factors that are irreducible over **R** (i.e., have no real zeros), or
3) Some product of the above, possibly including repeated factors, and
4) Maybe a nonzero constant factor.

One consequence: A 3rd- or higher-degree polynomial $f(x)$ with real coefficients **must** be factorable (reducible) over the reals. Knowing **how** to factor such an $f(x)$ may pose a problem, however!

Note: We will need this theorem when we do Partial Fraction Decompositions in Section 2.7.

Technical Note: See the Proof on p.193 of the textbook. Consider the "LFT Form" of $f(x)$. If all the zeros of $f(x)$ are real, then it can be factored accordingly. If there exists an imaginary zero c_i, then its conjugate $\overline{c_i}$ must also be a zero by the Complex Conjugate Pairs Theorem, and the product $(x-c_i)(x-\overline{c_i})$ of their corresponding factors must have real coefficients (see p.193); this product would be a quadratic factor of $f(x)$ with real coefficients that is irreducible over **R** (because its zeros are not real). Keep pairing off complex conjugate pairs of imaginary zeros until the remaining factors have only real coefficients.

PART H: DESCARTES'S RULE OF SIGNS

Historical Note: In addition to the Cartesian plane, this is also named after René Descartes. See Notes P.27.

Assumptions and Preliminaries

Let $f(x)$ be a polynomial with real coefficients written in standard form:

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$
$$\left(\text{each constant } a_i \in \mathbf{R};\ a_n \neq 0;\ a_0 \neq 0;\ n \in \mathbf{Z}^+\right)$$

Note: As with the Rational Zero Test, we require $a_0 \neq 0$. If $a_0 = 0$, factor out the GCF first. For example, $x^3 + x^2$ factors as $x^2(x+1)$, and we know that 0 is a real zero of multiplicity 2.

Variations in Sign

The number of variations in sign in $f(x)$ is given by the number of "sign flips" as the nonzero coefficients of $f(x)$ are read from left to right in the standard form.

Example

Find the number of variations in sign in $f(x) = 7x^6 - 2x^3 + 4x + 5$.

Solution

Because the leading coefficient is positive, we may want to clearly place a + sign in front of it:

$$f(x) = +7x^6 - 2x^3 + 4x + 5$$

There are **2** variations in sign. (Don't worry about "missing terms"; they have 0 coefficients.)

Parity

Two integers have the same parity $\Leftrightarrow$ They are both even or both odd.

Descartes's Rule of Signs

We want information about z^+ and z^-, where:

z^+ is the number of **positive** real zeros of $f(x)$, and
z^- is the number of **negative** real zeros of $f(x)$.

Let v^+ be the number of variations in sign in $f(x)$ (written in standard form).
Let v^- be the number of variations in sign in $f(-x)$ (written in standard form).

Then,

$0 \le z^+ \le v^+$, where z^+ has the same parity as v^+, and
$0 \le z^- \le v^-$, where z^- has the same parity as v^-.

Warning: A zero of multiplicity k is counted k times here.
For example, $f(x) = x^3 + 3x^2 + 3x + 1$, which factors as $(x+1)^3$, is said to have 3 real zeros: -1, -1, and -1.

Example

Based on Descartes's Rule of Signs, give the possible values of z^+ and z^- for $f(x) = 4x^3 - 5x^2 - 7x + 2$. (We've used this $f(x)$ in previous sections.)

Solution

Find possible values for z^+:

$$f(x) = +4x^3 - 5x^2 - 7x + 2$$

We see that $v^+ = 2$, an even integer.
Therefore, $0 \le z^+ \le 2$, where z^+ is also even.
The possible values for z^+ are then **0 and 2**.

Tip: Observe that we start with 2 and count down by twos; we stop before reaching negative numbers. This is similar to listing possible numbers of turning points for polynomial graphs in Section 2.2.

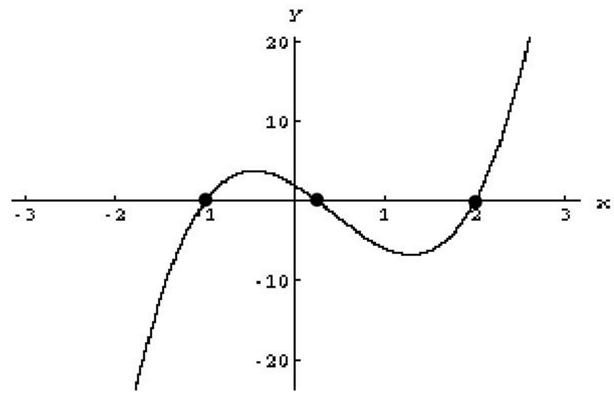
Find possible values for z^-:

$$\begin{aligned} f(-x) &= 4(-x)^3 - 5(-x)^2 - 7(-x) + 2 \\ &= -4x^3 - 5x^2 + 7x + 2 \end{aligned}$$

We see that $v^- = 1$, an odd integer.
Therefore, $0 \le z^- \le 1$, where z^- is also odd.
The **only** possible value for z^- is **1**.

(In other words, $f(x)$ must have exactly one negative real zero.)

Note: We earlier found that $f(x)$ had **2** positive real zeros, namely 2 and $\frac{1}{4}$, and **1** negative real zero, namely -1. See the graph below.



Note: Consider the form $f(x) = x^n \pm 1$ as a source of basic examples.

PART I: UPPER AND LOWER BOUND RULES FOR ZEROS

a is a lower bound for the real zeros of f, and b is an upper bound for them $\Leftrightarrow$ **All** the real zeros of f lie in the interval $\left[a, b\right]$.

It is easier to demonstrate the Upper and Lower Bound Rules rather than to state them in general.

We require that the leading coefficient of $f\left(x\right)$, a_n, be positive, and that all the coefficients be real.

Example and Demonstration

Show that all the real zeros of $f\left(x\right) = 4x^3 - 5x^2 - 7x + 2$ must lie in the interval $\left[-1, 3\right]$.

Solution

Use Synthetic Division to divide $f\left(x\right)$ by $x - 3$:

$$\begin{array}{r|rrrr} 3 & 4 & -5 & -7 & 2 \\ & & 12 & 21 & 42 \\ \hline & 4 & 7 & 14 & 44 \end{array}$$

Because $\mathbf{3 > 0}$, and all the entries in the last row are **nonnegative**, 3 is an **upper bound** for the real zeros of f.

Use Synthetic Division to divide $f(x)$ by $x-(-1)$:

$$
\begin{array}{r|rrrr}
-1 & 4 & -5 & -7 & 2 \\
 & & -4 & 9 & -2 \\
\hline
 & 4 & -9 & 2 & 0
\end{array}
$$

Because $\mathbf{-1 < 0}$, and the entries in the last row **alternate between nonnegative and nonpositive entries**, -1 is a **lower bound** for the real zeros of f.

In fact, because we get a 0 remainder, -1 must be a zero of f.

Therefore, all the real zeros of $f(x)$ must lie in the interval $[-1, 3]$.

Note: These rules can be used (possibly in conjunction with Descartes's Rule of Signs and/or a graph) to shrink the list of candidates for zeros resulting from the Rational Zero Test. The information obtained from these rules can also help us use the Intermediate Value Theorem (see Notes 2.20-2.21 on Section 2.2) more effectively in attempting to locate where zeros may be.

THE "QF" METHOD FOR FACTORING QUADRATICS

Remember our old friend $f(x) = 4x^3 - 5x^2 - 7x + 2$.
Let's say we want to factor this completely over **C**.

From Part B on the Rational Zero Test, we found a list of candidates for rational zeros. It turned out that 2 was, in fact, a zero. Therefore, by the Factor Theorem, $(x-2)$ was a factor of $f(x)$. After performing Synthetic Division, we found that:

$$4x^3 - 5x^2 - 7x + 2 = (x-2)\cdot(4x^2 + 3x - 1)$$

Trial-and-error can be used to factor the quadratic factor, but this method makes some people nervous. There is a more systematic alternative offered to us by the Quadratic Formula (QF). Ordinarily, we factor before finding zeros, but we will reverse that here.

Use the QF to find the zeros of $4x^2 + 3x - 1$; in other words, solve $4x^2 + 3x - 1 = 0$.
Observe: $a = 4$, $b = 3$, and $c = -1$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(3) \pm \sqrt{(3)^2 - 4(4)(-1)}}{2(4)} \\ &= \frac{-3 \pm \sqrt{25}}{8} \\ &= \frac{-3 \pm 5}{8} \end{aligned}$$

$$\begin{aligned} x &= \frac{-3+5}{8} \\ &= \frac{2}{8} \\ &= \frac{1}{4} \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= \frac{-3-5}{8} \\ &= \frac{-8}{8} \\ &= -1 \end{aligned}$$

The zeros of $4x^2 + 3x - 1$ are $\frac{1}{4}$ and -1.

Remember that the leading coefficient of $4x^2 + 3x - 1$ was $a_n = 4$.
An LFT Form of $4x^2 + 3x - 1$ is, therefore:

$$4x^2 + 3x - 1 = 4\left(x - \frac{1}{4}\right)(x+1)$$

However, factorizations over **Z** tend to be more useful in simplifications, so we will distribute the "4" through the $\left(x - \frac{1}{4}\right)$ factor.

Warning: You may distribute the "4" through one of the other factors, but not both!!

We obtain:

$$4x^2 + 3x - 1 = (4x-1)(x+1)$$

This is the kind of factorization we are typically used to, and it often helps us in simplification problems.

Example

Simplify $\dfrac{x+1}{4x^2+3x-1}$.

Solution

$$\begin{aligned} \frac{x+1}{4x^2+3x-1} &= \frac{\cancelto{1}{x+1}}{(4x-1)\cancelto{1}{(x+1)}}, \quad x \neq -1 \\ &= \frac{1}{4x-1}, \quad x \neq -1 \end{aligned}$$

By the way, we must complete the original problem! We had to factor $f(x) = 4x^3 - 5x^2 - 7x + 2$ over **C**.

Don't forget the $(x-2)$ factor that we obtained earlier:

$$\begin{aligned} 4x^3 - 5x^2 - 7x + 2 &= (x-2)(4x^2 + 3x - 1) \\ &= (x-2)(4x-1)(x+1) \end{aligned}$$

SECTION 2.6: RATIONAL FUNCTIONS

PART A: ASSUMPTIONS

Assume $f(x)$ is rational and written in the form $f(x) = \dfrac{N(x)}{D(x)}$,

where $N(x)$ and $D(x)$ are polynomials, and $D(x) \neq 0$ (i.e., the zero polynomial).

Assume **for now** that $N(x)$ and $D(x)$ have no real zeros in common.

Note: The textbook essentially makes this last assumption when it assumes that $N(x)$ and $D(x)$ have no common factors (over **R**) aside from ± 1, though, in Part E, we will consider what happens when we relax this assumption.

Warning: Even though $\dfrac{x^2 + x}{x} = x + 1 \quad (\forall x \neq 0)$, we do **not** consider the rational function [rule] $f(x) = \dfrac{x^2 + x}{x}$ to be a polynomial function [rule].

PART B: VERTICAL ASYMPTOTES (VAs)

An asymptote for a graph is a line that the graph approaches.

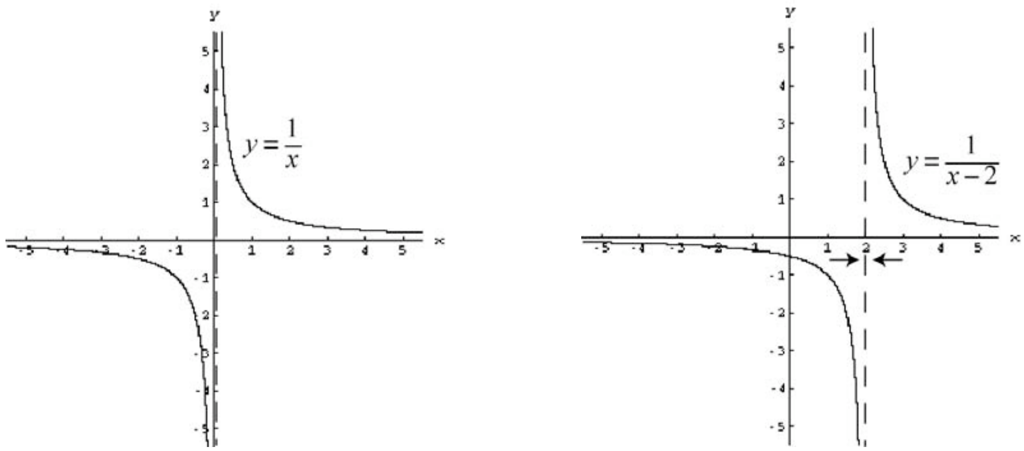
Example

Let $f(x) = \dfrac{1}{x - 2}$. Find any VAs for the graph of f.

Solution

Observe that 1 and $x - 2$ have no real zeros in common.
$x - 2 = 0 \iff x = 2$, so the only VA has equation $\boldsymbol{x = 2}$.

The graph of $y = \dfrac{1}{x}$ (on the left) is translated 2 units to the right to obtain the graph of $y = \dfrac{1}{x-2}$ (on the right). We typically use dashed lines to indicate asymptotes.



$x = 2$ is a VA for the graph on the right, because:
(Actually, just one of the two statements below would be sufficient.)

$f(x) \to \infty$ as $x \to 2^+$
(i.e., as x approaches 2 from the right, or from higher numbers), and

$f(x) \to -\infty$ as $x \to 2^-$
(i.e., as x approaches 2 from the left, or from lesser numbers).

Under our Assumptions in Part A,
the graph of $f(x) = \dfrac{N(x)}{D(x)}$ has a VA at $x = c$ $(c \in \mathbf{R})$

$\Leftrightarrow$ c is a real zero of $D(x)$

$\Leftrightarrow$ $f(x) \to \infty$ or $-\infty$ as $x \to c^+$, and
$f(x) \to \infty$ or $-\infty$ as $x \to c^-$.

Note: When we study logarithmic functions in Chapter 3, we will see graphs that have "one-sided" VAs. Graphs of rational functions "shoot off" on both sides of any VAs.

PART C: HORIZONTAL ASYMPTOTES (HAs)

The graph of $f(x)=\dfrac{N(x)}{D(x)}$ has a HA at $y=L$ $(L\in\mathbf{R})$

$\Leftrightarrow$ $f(x)\to L$ as $x\to\infty$ **and** as $x\to-\infty$.

Note: The graph of a **rational** function can have **at most one** HA.
The graph of a function that is **not rational** can have **at most two** HAs; $f(x)$ may approach different real values when $x\to\infty$ as opposed to when $x\to-\infty$.

Case 1

If $\deg(N)<\deg(D)$, then f is a proper rational function, and the x-axis $(y=0)$ is the only HA of its graph.

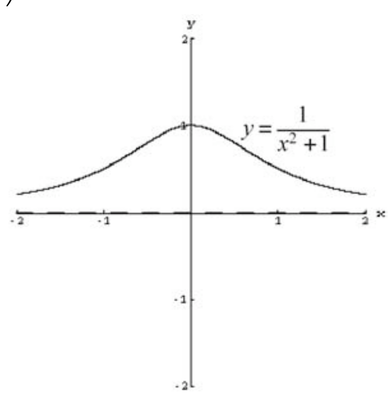
Example: $f(x)=\dfrac{1}{x}$. See the graph on Notes 2.67.

Example

Consider $f(x)=\dfrac{1}{x^2+1}$.

$\deg(N)<\deg(D)$, because $0<2$.
The x-axis is the only HA of the graph of f.

Observe that $f(x)>0$ $(\forall x\in\mathbf{R})$, f is even, and its graph (below) has no VAs, because $D(x)=x^2+1$ has no real zeros.



Case 2

If $\deg(N) = \deg(D)$, then $y = L$ is the only HA of the graph of f,

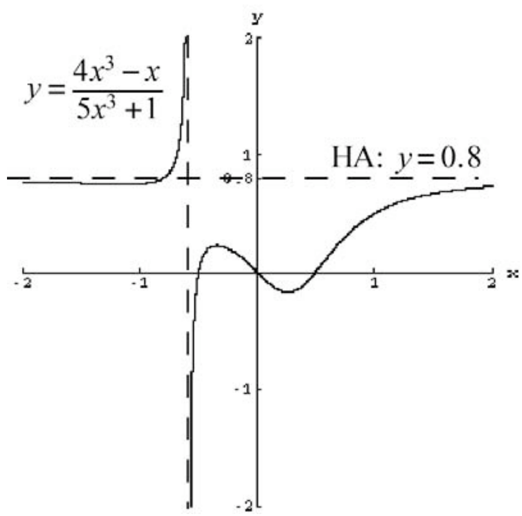
where $L = \dfrac{\text{the leading coefficient of } N(x)}{\text{the leading coefficient of } D(x)}$.

Example

Consider $f(x) = \dfrac{4x^3 - x}{5x^3 + 1}$.

$\deg(N) = \deg(D)$, because $3 = 3$.

$y = \dfrac{4}{5}$ (or 0.8) is the only HA of the graph of f (below).



Warning: Observe from the graph above that it is possible for the graph of $y = f(x)$ to cross a HA. However, its graph can never cross a VA.

Note: The idea is that, in the "long run," the "Zoom Out" Dominance Property for polynomials applies to the numerator and the denominator:

$$f(x) = \frac{4x^3 - x}{5x^3 + 1} \approx \frac{4x^3}{5x^3} = \frac{4}{5} \text{ if } x \text{ is "extreme"}$$

Case 3

If $\deg(N) > \deg(D)$, then the graph of f has no HAs.

The "Zoom Out" Property described in Part D will tell us about the "long-run" behavior of such graphs.

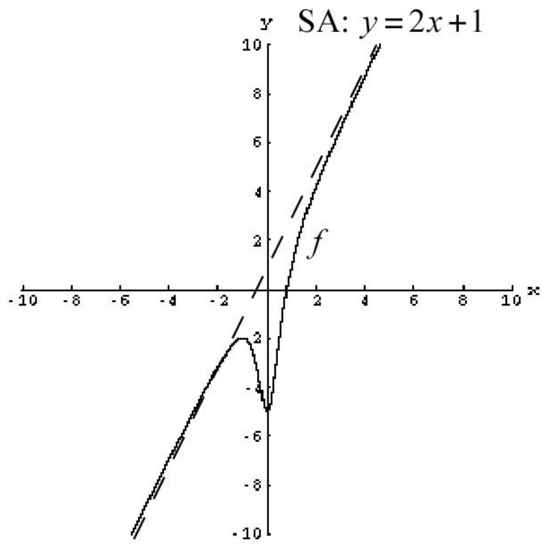
PART D: THE "ZOOM OUT" PROPERTY FOR RATIONAL FUNCTIONS; SLANT ASYMPTOTES

Example (from Section 2.3, Part A)

We have used long division to express $f(x) = \dfrac{-5 + 3x^2 + 6x^3}{1 + 3x^2}$ as:

$$f(x) = \underbrace{2x + 1}_{\substack{\text{polynomial} \\ \text{part, } p(x)}} \underbrace{- \frac{2x + 6}{3x^2 + 1}}_{\substack{\text{proper rational} \\ \text{part, } r(x)}}$$

This makes the corresponding graph (below) much easier to analyze.



Case 1 from Part C tells us that $r(x) \to 0$ as $x \to \infty$ **and** as $x \to -\infty$.
(The proper rational part "decays" in the long run.)
Therefore, the graph of $f(x)$ approaches the graph of $p(x)$ as $x \to \infty$ **and** as $x \to -\infty$.

Here, the graph of $f(x)$ approaches the graph of $p(x) = 2x + 1$.
Because $p(x)$ is linear, we call $y = 2x + 1$ a <u>slant asymptote (SA)</u> or an <u>oblique asymptote</u> for the graph of f.

<u>Note</u>: The graph of f has a SA $\Leftrightarrow$ $\deg(N) = \deg(D) + 1$.

<u>Note</u>: The graph of f has no VAs, because $D(x) = 3x^2 + 1$ has no real zeros.

PART E : WHAT IF $N(x)$ AND $D(x)$ HAVE REAL ZEROS IN COMMON?

The graph of f may have a VA or a hole at such a common real zero.

In Calculus: The "limit form" $\frac{0}{0}$ is important in Calculus. The limit definitions of the derivative described in Notes 1.24-1.25 and 1.57-1.58 involve this form.

Example

Graph $f(x) = \dfrac{x^2 - 9}{x - 3}$.

Solution

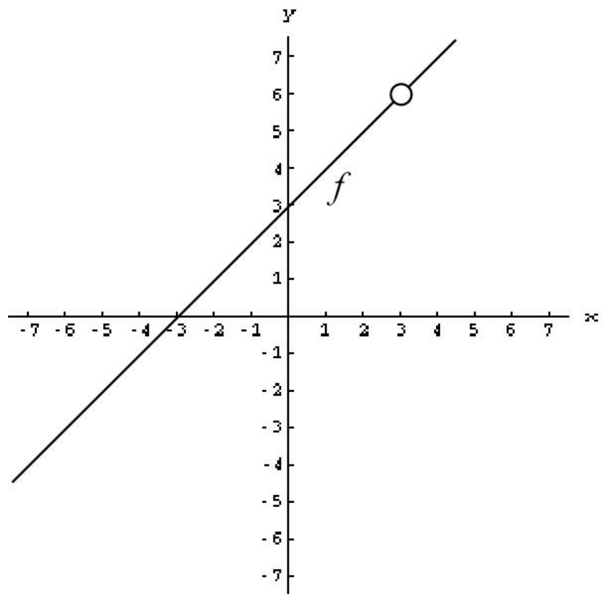
Observe that 3 is a real zero of $D(x) = x - 3$, so 3 is excluded from the domain of f, and 3 is also a real zero of $N(x) = x^2 - 9$. By the Factor Theorem, $(x - 3)$ must be a factor of $N(x)$.

$$\begin{aligned} f(x) &= \frac{x^2 - 9}{x - 3} \\ &= \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} \\ &= x + 3 \quad (x \neq 3) \end{aligned}$$

We include $(x \neq 3)$ as part of the final expression, because it is not apparent from the expression $x + 3$ that 3 is excluded from the domain of f.

The graph of f is essentially the line $y = x + 3$, except that the point $(3, 6)$ is deleted from the graph.

<u>In Calculus</u>: We say that the resulting "hole" reflects the fact that f has a <u>removable discontinuity</u> at $x = 3$.



Example

Graph $f(x) = \dfrac{x-2}{x^2 - 4x + 4}$.

Solution

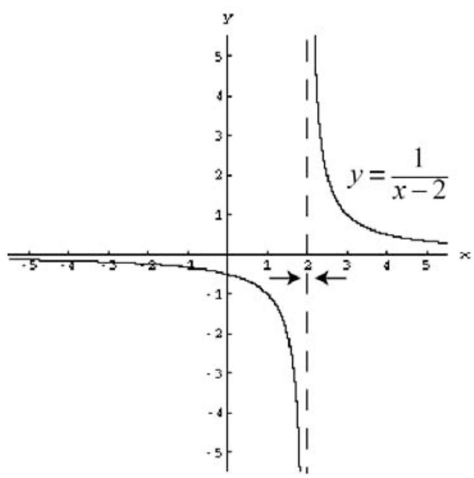
$$\begin{aligned} f(x) &= \frac{x-2}{x^2-4x+4} \\ &= \frac{x-2}{(x-2)^2} \end{aligned}$$

Observe that 2 is the only real number excluded from $\mathrm{Dom}(f)$.

$$= \frac{1}{x-2}$$

It is apparent from the final expression, $\dfrac{1}{x-2}$, that 2 is excluded from the domain of f, so we need not write $(x \neq 2)$.

The graph of f is the same as the second graph from Part B: Notes 2.67:



We have a VA and not a hole at $x = 2$, because not all of the $(x-2)$ factors in the denominator were canceled (divided) out in the simplification process. In our previous Example, all of the $(x-3)$ factors were canceled (divided) out in the denominator.

PART F: COMMENTS ON GRAPHING RATIONAL FUNCTIONS (BONUS TOPIC)

Our prior observations, in conjunction with the "Zoom Out" Dominance Property for polynomials, tell us that, in the "long run," graphs of rational functions look like lines, bowls, or snakes.

See Notes 2.22-2.24 on Section 2.2, Part H.
When considering the graph of a rational function f, we make the following modifications to those Notes:

Determine the domain of f. Find any VAs, HAs (see #4 below), and holes.

Remember that graphs of rational functions have no cusps or sharp corners (such as for $|x|$).

1) Find the *y*-intercept, if any.

   If 0 is not in the domain of f, then there is no *y*-intercept.

2) Find the *x*-intercept(s), if any.

   When determining the real zeros of f, make sure that your zeros are, in fact, in the domain of f. Look for real zeros of $N(x)$ that are not zeros of $D(x)$.

3) Exploit symmetry, if possible.

   Is f even? Odd?

4) Determine the "long-run" behavior of the graph as $x \to \infty$ and as $x \to -\infty$.

   Use Part C to find any HAs. Use Part D and maybe basic algebra or Long or Synthetic Division to find SAs or other "long-run polynomial behaviors"; remember the "Zoom Out" Properties for both rational and polynomial functions.

5) Find where $f(x) > 0$ and where $f(x) < 0$.

Let's modify our comments on the Test Interval (or "Window") Method from Section 2.2:

A rational function can only change sign …

- **… at its zeros, or**

- **… where it is undefined (i.e., where there is a hole or a VA).**

Note: A rational function can only change sign at a hole if it lies on the *x*-axis.

6) Maybe do some point-plotting (for a more accurate graph).

For many more examples, see the figures in the textbook.

SECTION 2.7: NONLINEAR INEQUALITIES

We solved linear inequalities to find domains, and we discussed intervals in Section 1.4: Notes 1.24 to 1.30.

In this section, we will solve nonlinear inequalities to find domains.

Example 1

Let $f(x)=\sqrt{x^2-9}$.

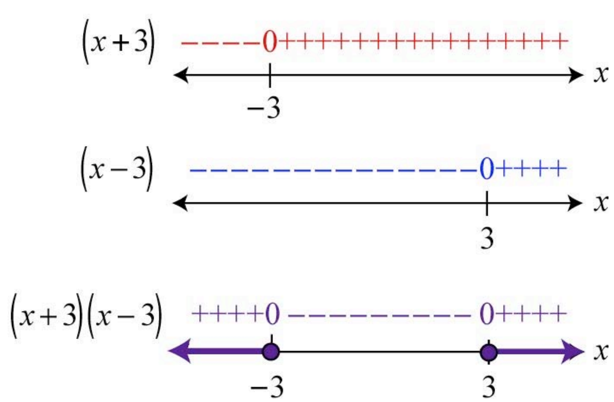
We get real outputs $\Leftrightarrow$ $x^2-9\geq 0$. There are different ways to solve this inequality; its solution set is the domain of f.

Method 1: Sign Chart Method;
we are solving $x^2-9\geq 0$

The key idea here is that we'd rather perform a sign analysis on products of factors as opposed to sums of terms. (For example, the product of a positive real number and a negative real number is guaranteed to be negative; however, there is no such guarantee regarding their sum.) Factoring can be a key tool.

$$x^2-9\geq 0$$
$$(x+3)(x-3)\geq 0$$

We need to determine where each of the factors on the left side is negative, 0, and positive in value. We ultimately want to know where their product is 0 or positive.



The domain of f is: $(-\infty,-3]\cup[3,\infty)$.

Method 2: Parabola Method (for Quadratic Inequalities);
we are solving $x^2 - 9 \geq 0$

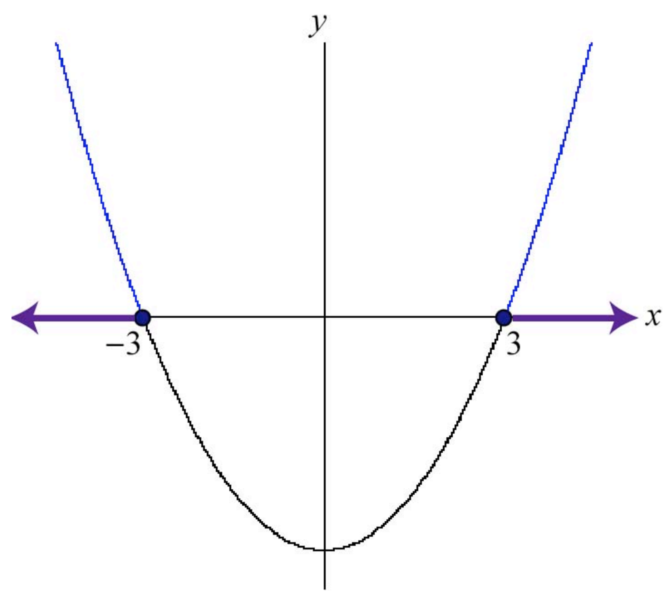
The real zeros of $x^2 - 9$ are the x-intercepts of the corresponding parabola:

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

Warning: Here, the $\pm$ symbol means "take **both** the $+3$ and the -3."
See Warning 1 in Section 1.5: Notes 1.46.

The leading coefficient of $x^2 - 9$ is positive, so the parabola opens up.

This is enough information for us to sketch the parabola to our satisfaction.



Given an input x, the y-coordinate of the corresponding point gives the output (or function value).

Because of the "≥ 0" in our inequality, we need the values of x, if any, that correspond to the parts of the parabola that lie above or on the x-axis.

Again, the domain of f is: $(-\infty, -3] \cup [3, \infty)$.

See also the bottom of p.198.

Method 3: Test Value or Test Interval Method;
we are solving $x^2 - 9 \geq 0$

This method presented in Larson, Section 2.7 may be the most straightforward one for rational inequalities in general. We discussed these ideas for continuous, particularly polynomial, functions in Section 2.2: Notes 2.23, and we extended them to rational functions in Section 2.6: Notes 2.76:

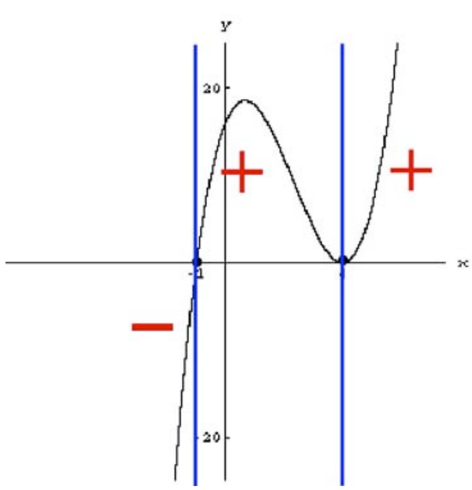
> A rational function can only change sign …
>
> - … at its zeros, or
> - … where it is undefined (i.e., where the graph has a hole or a VA).

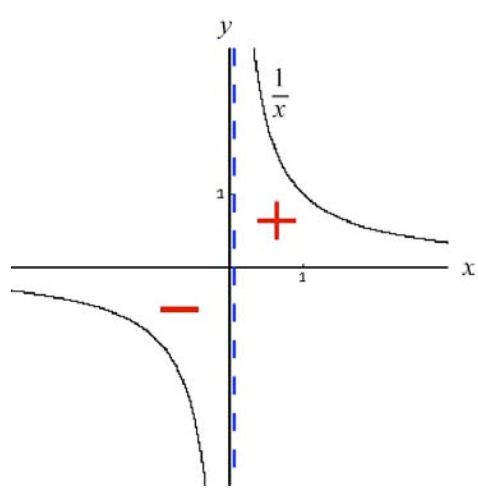
Warning: These places are called critical numbers in Larson, but this term has a different meaning in Calculus. In Calculus I, they are numbers in the domain of a function where the function's derivative (see Notes 1.63-1.66) is 0 or undefined; see Notes 1.49 to get a hint as to why we care ….

Warning: The function may or may not change sign at these places.

For example, look at the graphs of $y = x^3 - 7x^2 + 8x + 16$ and of $y = \frac{1}{x}$ below.

Note: The blue window separators marking the zeros in the first graph are not asymptotes, but the blue dashed line marking where $\frac{1}{x}$ is undefined in the second graph is.





Remember, we are solving $x^2 - 9 \geq 0$. Now, $x^2 - 9$ is a polynomial in x, so it is never undefined, and its zeros are 3 and -3, as we have seen in Method 2.

We use 3 and -3 as breakpoints (or fence posts) along the real number line. They break up the number line into three open test intervals, excluding 3 and -3, themselves. We test an x-value in each of the three intervals, and we at least determine the **sign** of $x^2 - 9$ at that test x-value; the sign there must be the common sign throughout the entire interval (see the box on the previous page).

		-3		**3**	
Test x-values	-4		0		4
Value of $x^2 - 9$	$(-4)^2 - 9 = 7$	0	$(0)^2 - 9 = -9$	0	$(4)^2 - 9 = 7$
Sign of $x^2 - 9$	+	0	−	0	+

We want the values of x for which $x^2 - 9$ is either "+" or "0."

Again, the domain of f is: $(-\infty, -3] \cup [3, \infty)$.

Note: Instead of testing 4 and -4, you may want to test extreme values such as 100 and -100. The corresponding signs may be easier to figure out. Bear in mind that you do not need to find the corresponding numerical values of $x^2 - 9$; only signs matter. In fact, the following variation is often an improvement ….

Method 4: Test Value or Test Interval Method using Factored Forms;
we are solving $x^2 - 9 \geq 0$

The Study Tip on p.199 of Larson suggests the sign analysis of factored forms, which can make our work for Method 3 more efficient, especially for more complicated inequalities. This method is a hybrid of Methods 1 and 3. Remember that $x^2 - 9 = (x+3)(x-3)$. We revise the chart from Method 3 as follows:

		-3		**3**	
Test x-values	-4		0		4
Signs: $(x+3)(x-3)$	$(-)(-)$	0	$(+)(-)$	0	$(+)(+)$
Sign of Product, $x^2 - 9$	+	0	−	0	+

Example 2

Let $f(x) = \dfrac{1}{\sqrt{x^2 - 9}}$.

This is similar to Example 1, except that we get real outputs $\Leftrightarrow$ $x^2 - 9 > 0$.
Note that we exclude 0, itself, here.

In our graphs for the first two methods in Example 1, we replace filled-in circles with hollow ones.

In our charts for the last two methods, the "0"s in the bottom lines are no longer in red.

We also replace brackets with parentheses in our answer, because we must exclude the endpoints -3 and 3 from the domain.

The domain of f is: $(-\infty, -3) \cup (3, \infty)$.

Example 3

Let $f(x) = \sqrt[3]{x^2 - 9}$.

The domain of f is **R**, because:

• $x^2 - 9$ is a polynomial with unrestricted domain, and

• (Warning!) The taking of **odd** roots (such as cube roots) does **not** impose any new restrictions on the domain. Remember that the cube root of a negative real number is a negative real number. This is different from **even** roots (such as square roots); we do not permit even roots of negative numbers when we find a domain.

For more on domains of radical functions, see Notes P.19.

Warning 1: When dealing with inequalities, if you multiply or divide both sides by a negative quantity, you must **reverse** the direction of the inequality symbol.
For example, $-x < -2 \quad \Leftrightarrow \quad x > 2$. You must also reverse the direction if you switch the left side and the right side. For example, $a < b \quad \Leftrightarrow \quad b > a$.

Warning 2: Do not multiply or divide both sides of an inequality by a variable expression, unless you take the time to consider cases when the expression is positive, zero, and negative in value.

Warning 3: When solving a nonlinear inequality such as $x^2 - 9 \geq 0$, make sure 0 is isolated on one side if you are going to use one of the methods from Example 1. This is because sign analyses are based on comparisons with 0.

Example (Warnings 2 and 3)

When solving the inequality $x^2 > x$, do not divide both sides by x.
Instead, isolate 0 on one side by subtracting x from both sides:

$$\begin{aligned} x^2 &> x \\ x^2 - x &> 0 \end{aligned}$$

Then, use one of the methods given in Example 1.
The solution set turns out to be: $(-\infty, 0) \cup (1, \infty)$.
These are the numbers whose squares are greater than themselves.

Warning 4: Let's say we have $f(x) = \sqrt{9 - x^2}$, and we want to use the Sign Chart Method (Method 1). Remember that $9 - x^2$ factors as $(3 + x)(3 - x)$, which is the **opposite** of $(x + 3)(x - 3)$ because of the "Switch Rule for Subtraction."

Example 3 on p.200 in Larson deals with "Unusual Solution Sets."