CHAPTER 3: EXPONENTIAL AND LOG FUNCTIONS

SECTION 3.1: EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

PART A: THE LEGEND OF THE CHESSBOARD

The original story takes place in the Middle Ages and involves grains of wheat. Instead, we shall transport ourselves to the distant realm of Seattle, where a smart programmer is haggling with King Bill. The programmer agrees to work for King Bill for 63 days, starting tomorrow. After seeing a large chessboard engraved into King Bill's floor, the programmer comes up with a scheme for his salary. For now, the programmer tells King Bill to place a check for \$1 on "Square 0" on his chessboard. With each new workday, King Bill is to place twice as much money on the corresponding square as the day before. The chortling King Bill, who has forgotten all of his math, agrees. What will happen?

The chessboard:

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
8	**9**	**10**	**11**	**12**	**13**	**14**	**15**
16	**17**	**18**	**19**	**20**	**21**	**22**	**23**
24	**25**	**26**	**27**	**28**	**29**	**30**	**31**
32	**33**	**34**	**35**	**36**	**37**	**38**	**39**
40	**41**	**42**	**43**	**44**	**45**	**46**	**47**
48	**49**	**50**	**51**	**52**	**53**	**54**	**55**
56	**57**	**58**	**59**	**60**	**61**	**62**	**63**

The amount of money (in dollars) placed on square x is given by $f(x) = 2^x$.

Here are some sample values:

Square x	$f(x)$ (in \$)
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024
20	Over 1 million (i.e., 10^6)
30	Over 1 billion (i.e., 10^9)
40	Over 1 trillion (i.e., 10^{12})
63	Over 9 quintillion (i.e., 9×10^{18})

The amount of money on Square $(x+10)$ will be over 1000 times the amount of money on Square x $(0 \leq x \leq 53)$, because the multiplier is $2^{10} = 1024$.

Challenge: How much money should be on the entirety of the chessboard after Day 63? Hint: Experiment with the first few days. We will see a relevant formula in Chapter 9, when we get to finite geometric series.

Remember that our national debt is "only" in the trillions.

No wonder we associate this kind of exponential growth with "rapid growth" in our language!

PART B: BASIC EXPONENTIAL GRAPHS

We call b a "nice base" if $b > 0$ and $b \neq 1$.

Basic exponential functions have the form $f(x) = b^x$, where b is nice.

Example

Graph $f(x) = 2^x$, our "payment" function from Part A.

Solution

The table on the previous page gives some sample points (x, y).

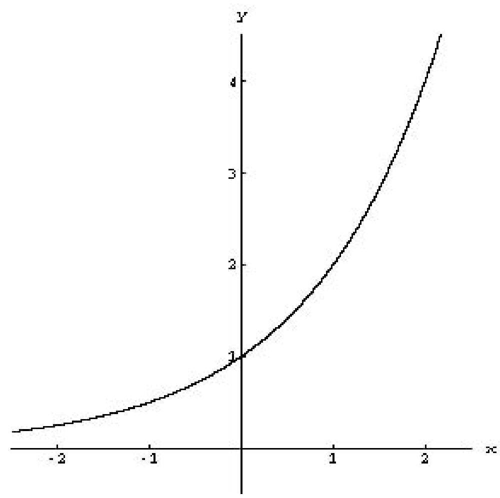
However, the domain of 2^x is assumed to be **R**, not just the nonnegative integers.

Technical Note: Let's look at $2^{3/4}$, for example. We may interpret $2^{3/4}$ as $\sqrt[4]{2^3}$, or $\sqrt[4]{8}$. It is the real number whose fourth power equals 8. The idea makes sense, although the number, which is irrational, may be time-consuming to approximate by "trial-and-error" on a calculator. You will encounter helpful methods and tools such as Newton's Method in Calculus I: Math 150 and series in Calculus II: Math 151 at Mesa. The explanation for values of 2^x for **irrational** values of x is actually a calculus idea, in and of itself! See p.198 of the textbook.

What about when $x < 0$? Observe the pattern:

x	$f(x) = 2^x$
3	8
2	4
1	2
0	1
−1	$2^{-1} = \frac{1}{2}$
−2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
−3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

Here is the graph of $f(x) = 2^x$:



In general, if $b > 1$ …

> The graph of $f(x) = b^x$, where $b > 1$, will resemble the "J" graph above.
> Think: <u>Exponential growth</u>.

For $f(x) = b^x$, where b is **any** nice base:

- The domain is **R**.
- The range is $(0, \infty)$.
- The x-axis is a horizontal asymptote for the graph.
- The y-intercept is 1, because $b^0 = 1$.

The various transformations from Section 1.6 apply here, as well.

What about if $0 < b < 1$?

Example

Graph $g(x) = \left(\frac{1}{2}\right)^x$ by first considering $f(x) = 2^x$.

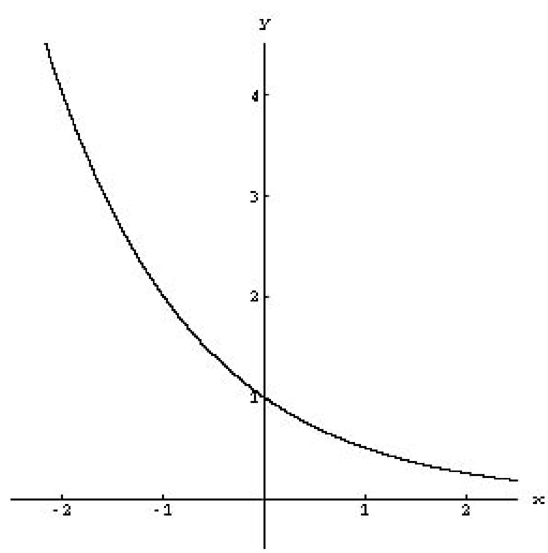
Solution

Observe:

$$\begin{aligned} f(-x) &= 2^{-x} \\ &= \left(2^{-1}\right)^x \\ &= \left(\frac{1}{2}\right)^x \\ &= g(x) \end{aligned}.$$

We reflect the old graph about the *y*-axis to obtain the new graph.

Here is the graph of $g(x) = \left(\frac{1}{2}\right)^x$:



In general, if $0 < b < 1$ …

The graph of $f(x) = b^x$, where $0 < b < 1$, will resemble the "curvy L" graph above. Think: Exponential decay.

What happens if b is **not** a nice base? (Optional discussion)

What happens if $b = 1$?

$f(x) = 1^x = 1$ is a constant function, not an exponential function.

What happens if $b = 0$?

Observe that $f(x) = 0^x = 0$, if $x \neq 0$.

We sometimes have to conveniently define 0^0 ourselves, depending on our problem. Observe that 0^2, for example, is 0, yet 2^0 is 1. What would 0^0 be?

What happens if $b < 0$?

We have a real problem here. Literally. Think about the fact that $(-2)^2 = 4$, a positive real number, while $(-2)^3 = -8$, a negative real number. Meanwhile, $(-2)^{5/2} = \left(\sqrt{-2}\right)^5$ is not even a real number.

PART C: *e*

$e \approx 2.718$
Like π, it is an irrational number. They're pretty close in value, too!
There are different ways of defining *e*. Here's a limit definition for *e*:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Technical Note: In Math 151: Calculus II at Mesa, we say that we are dealing here with the limit form 1^∞, because the base of $\left(1 + \frac{1}{n}\right)^n$ is approaching 1, while the exponent is approaching ∞. It's as though *e* is a "long-term compromise" between these two countervailing forces. The form 1^∞ is called an indeterminate limit form, because it is not immediately clear what the limit should be in such cases, if there even is one. Different expressions with this limit form may have different limits. Other examples of indeterminate limit forms: 0^0, $0 \cdot \infty$, $\frac{0}{0}$, and $\frac{\infty}{\infty}$.

$f(x) = e^x$ gives us the natural exponential function, and *e* is referred to as the natural base. The graph of $f(x) = e^x$ resembles the J-graph for 2^x.

Technical Note: This function has many nice properties. For example, its derivative function is itself. That is not true of, say, the 2^x function.

PART D: EXPONENTIAL MODELS

Many applications employ the model $f(x) = a \cdot b^x$, where $a > 0$ and b is a nice base.

If $b > 1$ ("J" graphs), we obtain exponential growth models used in such applications as population growth (the Malthusian model) and compound interest, as we will see in Parts E and F.

> Historical Note: Thomas Malthus (1766-1834) was a famed economist who believed that populations would grow exponentially, but that food supplies would only grow linearly. His bleak views and proposed social remedies led people to call economics the "dismal science." (Microsoft® Encarta® Encyclopedia)

If $0 < b < 1$ ("curvy L" graphs), we obtain exponential decay models. For example, radioactive decay models are used in such applications as carbon-14 dating of ancient objects. If $b = \frac{1}{2}$, we deal with "half-life" models. See p.205.

PART E: COMPOUND INTEREST

Consider a banking account with compound interest where:

$P =$ principal deposited (in dollars)
$r =$ annual interest rate (as a decimal)
$n =$ number of compoundings (i.e., number of times interest is paid) per year
$t =$ time elapsed (in years) since the deposit

(P, r, n, and t must always be positive in value.)

Then, after t years, the account has:

$$f(t) = P\left(1 + \frac{r}{n}\right)^{nt} \text{ dollars}$$

I will give you this formula on exams, if you need it.

This assumes that there are no withdrawals or deposits after the principal is deposited. We also assume that r stays constant for the time being. This may not be realistic!

Observe that the formula takes on a basic exponential form: $f(t) = a \cdot b^t$, where t is the independent variable, $a = P$, and the nice base $b = \left(1 + \frac{r}{n}\right)^n > 1$.

Technical Note: We assume that t is always an integer, or at least that t represents a time at which interest is being compounded. Otherwise, if we allow t to represent any positive real number, we need to set up something that resembles a piecewise-defined function with a step graph.

Note: Simple interest is always applied to the principal only. Compound interest is applied to the combined total of the principal and the earned interest to date. The classic simple interest model is given by the formula $f(t) = P + Prt$, which is **linear** in t. Our compound interest models are **exponential** in t.

Note: We need the "1" term in the base. Otherwise, you are given interest, but then the rest is taken away!

Example

We initially deposit \$10,000 in an account that earns 6% annual interest compounded monthly. How much money will be in the account after 5 years, if no withdrawals or deposits are made in the meantime?

Solution

We have:

$$P = 10{,}000\,(\$)$$
$$r = 0.06$$
$$n = 12\,(\text{because there are 12 months in a year})$$
$$t = 5\,(\text{years})$$

Then,

$$f(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$f(5) = 10{,}000\left(1 + \frac{0.06}{12}\right)^{(12)(5)} \quad (\text{See Warning below!})$$
$$= 10{,}000\left(1 + \frac{0.06}{12}\right)^{60} \quad (\text{We have 60 compoundings in 5 years.})$$
$$\approx \mathbf{\$13{,}488.50}$$

Warning: You should simplify the exponent immediately. Otherwise, you must use grouping symbols around the exponent when you use your calculator. For example, if we want to compute $2^{(3)(4)}$, it is **incorrect** to input 2 (exponent) 3 (times) 4 (equals) on your calculator. Because of the order of operations, that would give us $2^3 \cdot 4$, which is not correct. This is a **very** common type of error made by students!

Note: Observe that $f(0) = P$, the initial amount.

PART F: CONTINUOUS COMPOUND INTEREST

What happens as we let $n \to \infty$ in our compound interest formula? We do **not** earn infinitely many dollars in finite time. We are now dealing with continuous compound interest, in which case our account is always growing "continuously" (i.e., at each moment) over time.

Consider a banking account with continuous compound interest where (P, r, and t are defined as before):

$P =$ principal deposited (in dollars)

$r =$ annual interest rate (as a decimal)

$t =$ time elapsed (in years) since the deposit

(P, r, and t must always be positive in value.)

Then, after t years, the account has:

$$A = Pe^{rt} \quad \text{dollars}$$

Know this formula for exams.

Proof / Derivation (Optional)

Take the compound interest formula $f(t) = P\left(1+\frac{r}{n}\right)^{nt}$.

Let $k = \frac{n}{r} \quad \Rightarrow \quad n = kr$. Observe: $k \to \infty \quad \Leftrightarrow \quad n \to \infty$.

$$\begin{aligned}
&\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} \\
&= \lim_{k\to\infty} P\left(1+\frac{r}{kr}\right)^{krt} \\
&= \lim_{k\to\infty} P\left[\underbrace{\left(1+\frac{1}{k}\right)^{k}}_{\to e}\right]^{rt} \qquad \text{(See the limit definition of } e \text{ in Part C.)} \\
&= Pe^{rt}
\end{aligned}$$

Example (similar to our previous one)

We initially deposit \$10,000 in an account that earns 6% annual interest compounded continuously. How much money will be in the account after 5 years, if no withdrawals or deposits are made in the meantime?

Solution

We have:

$$P = 10{,}000\,(\$)$$
$$r = 0.06$$
$$t = 5\,(\text{years})$$

Then,

$$\begin{aligned} A &= Pe^{rt} \\ &= 10{,}000e^{(0.06)(5)} \\ &= 10{,}000e^{0.3} \quad \left(\text{Warning: Simplify the exponent now!}\right) \\ &\approx \mathbf{\$13,498.59} \end{aligned}$$

Compare this to the **\$13,488.50** obtained from **monthly** compounded interest.

SECTION 3.2: LOGARITHMIC (LOG) FUNCTIONS AND THEIR GRAPHS

PART A: LOGS ARE EXPONENTS

Example

Evaluate: $\log_3 9$

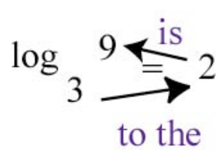
Solution

The question we ask is: "3 to what exponent gives us 9?"

$$\underbrace{\log_3 9 = \boxed{2}}_{\text{logarithmic form}} \text{, because } \underbrace{3^{\boxed{2}} = 9}_{\text{exponential form}}$$

We say: "Log base 3 of 9 is 2."

Think "Zig-zag":



Answer: 2.

More Examples

Log Form	Exponential Form
$\log_5\left(\frac{1}{5}\right) = -1$	$5^{-1} = \frac{1}{5}$
$\log_9 3 = \frac{1}{2}$	$\underbrace{9^{1/2}}_{=\sqrt{9}} = 3$
$\log_{10} 10^7 = 7$	$10^7 = 10^7$

PART B: COMMON LOGS

$f(x) = \log_{10} x$ gives the <u>common log</u> function.
It is also written as simply: $f(x) = \log x$
(A missing log base is implied to be 10.)

Your calculator should have the LOG button.

Common logs are used in the Richter scale for measuring earthquakes and the pH scale for measuring acidity. Bear in mind that an earthquake measuring a “7” on the Richter scale is 10 times as powerful as one measuring a “6” and 100 times as powerful as one measuring a “5.” Negative Richter numbers are also possible.

PART C: NATURAL LOGS

$f(x) = \log_{e} x$ gives the <u>natural log</u> function.
It is almost always written as: $f(x) = \ln x$

Your calculator should have the LN button.

<u>In Calculus</u>: This function is very useful, especially because its derivative is $\frac{1}{x}$.

<u>Example</u>: $\ln e^5 = \log_e e^5 = 5$

PART D: BASIC LOG PROPERTIES

Let b be any nice base.

Property	Because…	Special case $(b = e) \Rightarrow$	Memorize!
$\log_b 1 = 0$	$b^0 = 1$	$\log_e 1 = 0$	$\mathbf{ln\ 1 = 0}$
$\log_b b = 1$	$b^1 = b$	$\log_e e = 1$	$\mathbf{ln}\ \boldsymbol{e} = \mathbf{1}$
$\log_b b^x = x$	$b^x = b^x$	$\log_e e^x = x$	$\mathbf{ln}\ \boldsymbol{e^x = x}$
$b^{\log_b x} = x$ $(\text{if } x > 0)$	$\log_b x$ is the exponent that "takes us from b to x"	$e^{\log_e x} = x$	$\boldsymbol{e^{\ln x} = x}$

We need the restriction $(x > 0)$ for the last property, because:

> The log of a nonpositive number is not real.
> In this class, we only take logs of **positive** real numbers.

Example: $\log_2(-1)$ is not real, because, if we set up $\log_2(-1) = \square$,
we see that $2^{\square} = -1$ has no real solution for $\square$.
This is because the range of the 2^x function is $(0, \infty)$, which excludes -1.

Example: Similarly, $\log_2 0$ is not real (in fact, it is undefined), because
$2^{\square} = 0$ has no real solution.

Technical Note: In a course on complex variables, you will see that it is possible to take the log of a negative number (but not 0) in that setting.

Technical Note: We didn't need the restriction $(x > 0)$ for the other properties, because b and therefore b^x are presumed to be positive in value, anyway.

The last two properties are called inverse properties, because they imply that …

PART E: $f(x) = b^x$ AND $f^{-1}(x) = \log_b x$ REPRESENT INVERSE FUNCTIONS

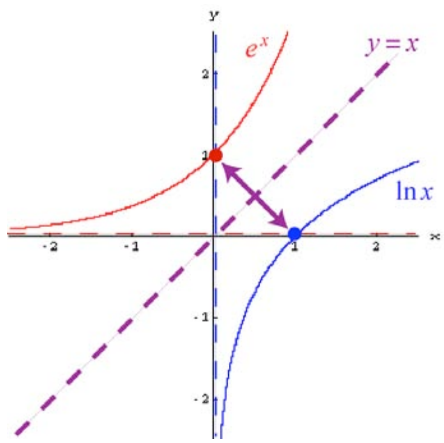
Let b be any nice base. The above exponential and log functions "undo" each other in that their composition in either order yields the identity function. (If the initial input is x, then the final output is x, at least if $x > 0$. See the last two properties in Part D.)

Example

$f(x) = e^x$ and $f^{-1}(x) = \log_e x$, or $\ln x$ represent a pair of inverse functions.

We know that the graph of $f(x) = e^x$ is a "J graph" similar to the one for 2^x. Observe that it passes the Horizontal Line Test (HLT), so f is one-to-one and therefore invertible.

We reflect this graph about the line $y = x$ to obtain the graph of the inverse function $f^{-1}(x) = \log_e x$, or $\ln x$.



Observe that the domain of one function is the range of the other, and vice-versa.

	Domain (x)	**Range (y)**	**Asymptote for graph**
e^x	**R**	$(0, \infty)$, the positive reals	x-axis
$\ln x$	$(0, \infty)$, the positive reals	**R**	y-axis

The graph for e^x has a y-intercept at $(0, 1)$, which reflects the fact that $e^0 = 1$.

The graph for $\ln x$ has an x-intercept at $(1, 0)$, which reflects the fact that $\ln 1 = 0$.

PART F: DOMAINS OF LOG FUNCTIONS

Example

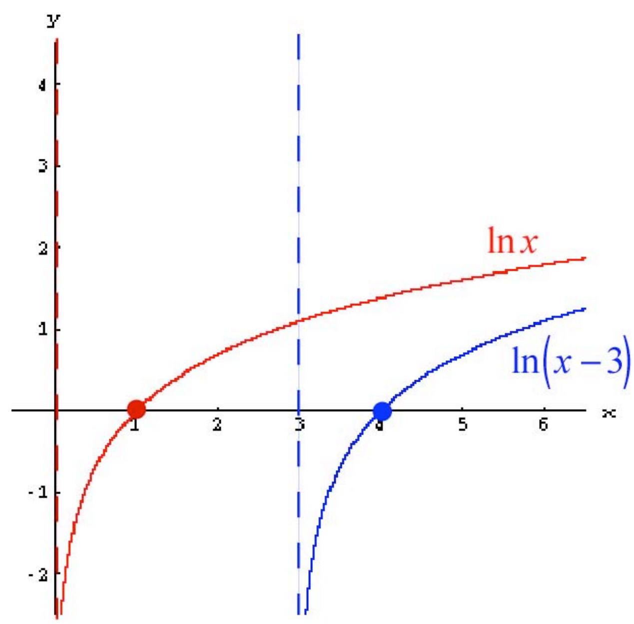
Write the domain of $f(x) = \ln(x-3)$ in interval form.

Solution

$$\ln(x-3) \text{ is real } \Leftrightarrow x-3>0$$
$$\Leftrightarrow x>3$$

The domain is: $(3, \infty)$

The graph of f is simply the graph for $\ln x$ shifted 3 units to the right:



SECTION 3.3: (MORE) PROPERTIES OF LOGS

PART A: READING LOG EXPRESSIONS

We will use grouping symbols as a means of clarifying the order of operations in expressions.

Often, grouping symbols are omitted when they could have helped.
How do we read log expressions in those cases?

We use ln for convenience, but any log function with any nice base is dealt with similarly.

Example: Read $\ln 3x+7$ as: $\ln(3x)+7$

Example: Read $\ln 3x^4$ as: $\ln(3x^4)$

Example: Read $\ln x^4$ as: $\ln(x^4)$, **not** as $(\ln x)^4$

Generally speaking, absent any grouping symbols, if we see "ln" or "log" followed by a product, quotient, power, or mixture of the above, the "ln" applies to the whole expression that follows. However, + and $-$ signs that introduce new terms tend to terminate the ln expression.

PART B: LOG PROPERTIES BASED ON LAWS OF EXPONENTS

Remember, logs are exponents. The laws for exponents imply laws for logs.

We will state these properties using ln, though they apply to any log function with a nice base.

If we use the rules from left-to-right, we are "expanding" the expression.
If we use the rules from right-to-left, we are "condensing" the expression.

Assume $A > 0$ and $B > 0$.
A and *B* may represent constants or variable expressions.

Product Rule

$$\ln(AB) = \ln A + \ln B$$

Think: The log of a **product** equals the **sum** of the logs.
(We go one step down in the order of operations if we read left-to-right.)

Related Exponent Law: When multiplying powers of *e*, the exponent on the product equals the sum of the exponents (of the factors): $e^A e^B = e^{A+B}$

Quotient Rule

$$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$$

Think: The log of a **quotient** equals the **difference** of the logs.

Related Exponent Law: When dividing powers of *e*, the exponent on the quotient equals the difference of the exponents: $\frac{e^A}{e^B} = e^{A-B}$

Power Rule

$$\ln A^p = p\ln A$$

Think: Smackdown Rule (from left-to-right);
Basketball Rule (from right-to-left)

Related Exponent Law: When raising a power of e to a power, the exponents are multiplied: $\left(e^k\right)^p = e^{pk}$

Proofs (Optional): See p.257 in the textbook.

Warning: We do **not** have nice rules for the log of a sum or a difference: $\ln\left(A \pm B\right)$

PART C: WHAT IF YOU FORGET THE RULES?

Experimenting with different powers of e may help you verify a guess to a rule you're not 100% confident about.

Example

Test the Product Rule: $\ln\left(AB\right) = \ln A + \ln B$

Let $A = e^2$ and $B = e^3$.

$$\begin{aligned} \ln\left(AB\right) &= \ln A + \ln B \\ \ln\left(e^2e^3\right) &= \ln e^2 + \ln e^3 \\ \ln\left(e^5\right) &= \ln e^2 + \ln e^3 \\ 5 &= 2+3 \\ 5 &= 5 \qquad \left(\text{Checks out}\right) \end{aligned}$$

This is not a proof, but it should be encouraging.
If your experiment does **not** work out, then you may have guessed the wrong rule!
Don't just make up your own rules!

PART D: MORE ON THE POWER RULE

The Power Rule for Logs is the most commonly abused rule among the three we've introduced thus far in this section.

In order for the rule, $\ln A^p = p \ln A$, to apply, we require the exponent, p, to apply to the **entire** base A, and **not** to the "log."

Examples

Yes or No: Are the two expressions equivalent, according to the Power Rule for Logs? (Assume $x > 0$ and $y > 0$.)

Expression #1	Expression #2	Equivalent?	Comments
$\ln xy^3$	$3 \ln xy$	**No**	The 3 does not apply to the x. Use the Product Rule, first.
$\ln (xy)^3$	$3 \ln xy$	**Yes**	The 3 applies to the entire xy base.
$(\ln x)^3$	$3 \ln x$	**No**	The 3 applies to the "ln," as well.
$\ln x^3$	$3 \ln x$	**Yes**	

Technical Note: In the rule $\ln A^p = p \ln A$, what if we allow $A < 0$? Then, it really matters what kind of number p is. For example, what if p is an even integer, say 2? Then, we have: $\ln A^2 = 2 \ln |A|$.

PART E: EXPANDING LOG EXPRESSIONS

Assume that all variables are restricted to positive values.

Example

Expand (i.e., completely expand, but evaluate expressions where appropriate): $\ln\sqrt[3]{ex}$

Solution

$$
\begin{aligned}
\ln\sqrt[3]{ex} &= \ln(ex)^{1/3} \\
&= \frac{1}{3}\ln(ex) \qquad (\text{By the Power or "Smackdown" Rule}) \\
&= \frac{1}{3}(\ln e + \ln x) \quad (\text{By the Product Rule}) \\
&= \frac{1}{3}(1 + \ln x) \qquad (\text{Evaluation}) \\
&\text{or} \quad \frac{1}{3} + \frac{1}{3}\ln x
\end{aligned}
$$

Example

Expand: $\log_2 \dfrac{8x^3}{y}$

Solution

$$
\begin{aligned}
\log_2 \frac{8x^3}{y} &= \log_2\left(8x^3\right) - \log_2 y \\
&\qquad (\text{You may skip the above step if you write the next step.}) \\
&= \log_2 8 + \log_2 x^3 - \log_2 y \\
&= \quad 3 \quad + 3\log_2 x - \log_2 y \\
&\qquad (\text{Evaluation and Power / "Smackdown" Rule})
\end{aligned}
$$

Example

Expand: $\log \dfrac{a^2 b^3}{c^4 d}$

Solution

$$\log \frac{a^2 b^3}{c^4 d} = \left(\log a^2 + \log b^3\right) - \left(\log c^4 + \log d\right)$$

(You may skip the above step if you write the next step.)

$$= \log a^2 + \log b^3 - \log c^4 - \log d$$

(Warning: Watch out for that last minus sign!
Remember that d was a factor of the **denominator**
of the log argument.)

$$= 2\log a + 3\log b - 4\log c - \log d$$

In Calculus: These expansion techniques come in very handy when we do logarithmic differentiation (used to find the derivative of a complicated function) and when we differentiate complicated log functions. In the latter case, we apply the log rules to tear apart the log expression, and then we differentiate the pieces term-by-term. This is often easier than directly differentiating the given expression.

PART F: CONDENSING LOG EXPRESSIONS

Assume that all variables are restricted to positive values.

Example

Condense (i.e., completely condense): $\ln x + 3\ln y$

Solution

$$\begin{aligned} \ln x + 3\ln y &= \ln x + \ln y^3 \quad \left(\text{"Reverse" Power / "Basketball" Rule}\right) \\ &= \ln\left(xy^3\right) \end{aligned}$$

In the last step, we applied the Product Rule "in reverse":
The sum of the logs equals the log of the product.

Example

Condense (i.e., completely condense): $\frac{1}{2}\log 3x + \log y - 3\log z$

Solution

$$\frac{1}{2}\log 3x + \log y - 3\log z = \log\underbrace{(3x)^{1/2}}_{=\sqrt{3x}} + \log y - \log z^3$$

$\left(\text{"Reverse" Power / "Basketball" Rule}\right)$

(Warning: Notice the grouping symbols around the $3x$.)

$$= \log\left(\sqrt{3x}\cdot y\right) - \log z^3$$

$\left(\text{"Reverse" Product Rule}\right)$

$$= \log\frac{y\sqrt{3x}}{z^3}$$

(Warning: $\sqrt{3x}y$ may be confusing.)

In the last step, we applied the Quotient Rule "in reverse":
The difference of the logs equals the log of the quotient.

These tools will help us solve logarithmic equations in Section 3.4.

PART G: CHANGE OF BASE FORMULA

Example

We know that $\log_2 8 = 3$.

How can we approximate $\log_2 9$? We do not need a LOG_2 button on our calculators. It turns out the following formulas work:

$$\log_2 9 = \frac{\ln 9}{\ln 2} \approx 3.1699$$

$$\log_2 9 = \frac{\log 9}{\log 2} \approx 3.1699$$

In fact, $\log_2 9 = \dfrac{\log_a 9}{\log_a 2}$ for **any** nice base a.

In general,

Change of Base Formula
If b is a nice base, and if $x > 0$, then
$\log_b x = \dfrac{\log_a x}{\log_a b}$, where a is **any** nice base.

Of course, we normally choose $a = e$ or $a = 10$ so that we can use the LN or the LOG button on our calculators.

In Calculus: The Change of Base Formula comes in very handy when differentiating log functions with various bases. We can then lean on the fact that the derivative of $\ln x$ is $\frac{1}{x}$.

Technical Note: In algorithm analysis, computer scientists often get very lazy when dealing with functions that model running time or space/memory requirements. Often, "a log is a log" to them.

Observe: the binary-friendly expression $\log_2 x = \frac{\ln x}{\ln 2} \quad \Rightarrow \quad \ln x = \underbrace{(\ln 2)}_{\text{constant}}(\log_2 x)$.

By this reasoning, the log functions for various bases are simply constant multiples of each other. These functions are lumped together in the category $O(\log x)$, where O (called "big-O") is referred to as "order."

Exponential functions such as $f(n) = 2^n$ grow much faster than log functions in the long run. If the running time (or space/memory requirements) of an algorithm is approximately exponential in n (the size of the input), then that is very bad news in the long run compared to an algorithm that is approximately logarithmic in n.

Proof of the Change of Base Formula (Optional):

Let $y = \log_b x$, where b is nice and $x > 0$.

$$
\begin{aligned}
\log_b x &= y \\
b^y &= x \\
\log_a b^y &= \log_a x \quad (\text{where } a \text{ is any nice base}) \\
y \log_a b &= \log_a x \\
y &= \frac{\log_a x}{\log_a b} \\
\log_b x &= \frac{\log_a x}{\log_a b}
\end{aligned}
$$

SECTIONS 3.4 AND 3.5: EXPONENTIAL AND LOG EQUATIONS AND MODELS

PART A: ONE-TO-ONE PROPERTIES

(Assume that b is nice.)

The b^x and $\log_b x$ functions are one-to-one. (Their graphs pass the HLT.) Therefore,

1) $\left(b^M = b^N\right) \Leftrightarrow \left(M = N\right)$, and
2) $\left(\log_b M = \log_b N\right) \Leftrightarrow \left(M = N\right)$, where M and N are positive in value

Informally: We can insert or delete the same base or the same log on both sides in order to obtain an equivalent equation, provided that we only take logs of positive values.

Technical Note: The $\Leftarrow$ directions are immediate in both 1) and 2).
The $\Rightarrow$ directions require the one-to-one properties of the two functions.

PART B: SOLVING EXPONENTIAL EQUATIONS

Exponential equations contain variable powers of constant bases, such as 3^{x+1}.

Example

Solve $2^x - 10 = 0$. Approximate your answer to 4 decimal places.

Solution

We can easily isolate the basic exponential expression, 2^x, on one side.

$$\begin{aligned} 2^x - 10 &= 0 \\ 2^x &= 10 \end{aligned}$$

We'd like to bring variable exponents "down to earth."
We can do so by inserting logs.

Method 1

Take $\log_2$ of both sides.

$$\log_2 2^x = \log_2 10$$

We can then exploit the Inverse Properties.

$$x = \log_2 10$$

If we would like a decimal approximation for *x*, we need to use the Change of Base Formula.

$$\begin{aligned} x &= \frac{\ln 10}{\ln 2} \\ &\approx 3.3219 \end{aligned}$$

Method 2 (more common)

Since we usually prefer calculating with ln instead of $\log_2$, we may prefer taking ln of both sides of $2^x = 10$.

$$\begin{aligned} 2^x &= 10 \\ \ln 2^x &= \ln 10 \\ x \underbrace{\ln 2}_{\text{Just a number}} &= \ln 10 \quad \left(\text{Power / "Smackdown" Rule for Logs}\right) \\ x &= \frac{\ln 10}{\ln 2} \\ x &\approx 3.3219 \end{aligned}$$

The solution set is: $\{\textbf{about 3.3219}\}$

Warning: Some instructors prefer a solution set as your final answer.

Example

Solve $5^{3x-1} = 125$.

Solution

Although we could take $\log_5$ or ln of both sides, it may be easier to recognize immediately that $125 = 5^3$.

$$5^{3x-1} = 5^3$$

We can then delete the 5 base on both sides.
In general, we can equate exponents on the same base;
i.e., $\left(b^M = b^N\right) \Rightarrow \left(M = N\right)$.

$$\begin{aligned} 3x - 1 &= 3 \\ 3x &= 4 \\ x &= \frac{4}{3} \end{aligned}$$

The solution set is: $\left\{ \mathbf{\frac{4}{3}} \right\}$

PART C: EXPONENTIAL MODELS; MORE ON SOLVING EQUATIONS

The Malthusian Population Growth Model

$$P = P_0 e^{rt}\text{, where}$$

P = population at time t

P_0 = initial population (i.e., population at time $t = 0$)

r = a parameter indicating population growth (as a decimal)

(See Notes 3.33.)

(t is measured in years.)

Notice the similarities between this formula and the formula for continuous compound interest. Be aware that the P_0 here takes on the role of the P from the interest formulas.

If $r < 0$, then we actually have a decay model.

Example

Use the Malthusian model $P = P_0 e^{0.0138t}$ to model the population of Earth, where $t = 0$ corresponds to January 1, 2000. If the population of Earth on January 1, 2000 was about 6.083 billion, in what year will the population reach 20 billion?

Solution

The population at $t = 0$ is (about) 6.083 billion, so $P_0 \approx 6.083\text{ billion}$. We first need to find the value for t that yields a population (P) of 20 billion. Solve for t:

$$P = P_0 e^{0.0138t}$$

$$20\ \cancel{(\text{billion})} = 6.083\ \cancel{(\text{billion})}\ e^{0.0138t}$$

$$20 = 6.083 e^{0.0138t}$$

We will isolate the basic exponential, $e^{0.0138t}$.

$$\frac{20}{6.083} = e^{0.0138t}$$
$$e^{0.0138t} = \frac{20}{6.083}$$

Warning: Try to keep exact values until the end of the problem. Since we ultimately need to round off our answer, anyway, it may be reasonable to round off here. Try to round off to many significant digits so as to not compromise the accuracy of your final answer.

To bring the variable exponent down to earth, take ln of both sides.

$$\ln e^{0.0138t} = \ln\left(\frac{20}{6.083}\right)$$
$$0.0138t = \ln\left(\frac{20}{6.083}\right)$$
$$t = \frac{\ln\left(\frac{20}{6.083}\right)}{0.0138} \quad \underline{\leftarrow \text{Process entire N before "} \div 0.0138\text{"}}$$
$$t \approx 86.2\left[\text{years}\right]$$

Warning: Remember units in your final answer when appropriate. However, we do not yet have our final answer!

Remember that $t = 0$ corresponded to January 1, 2000.

The population of Earth will reach 20 billion in the year 2086.

<u>Doubling Time Formula for Malthusian Growth</u>

In the growth model $P = P_0 e^{rt}$ $(r > 0)$, the time it takes for the population to double is $t = \dfrac{\ln 2}{r}$ years. This is independent of the initial population, P_0.

This formula also applies to continuous compound interest, which has an analogous growth model.

<u>Proof</u>

At what time t will the population $P = 2P_0$, twice the initial population?

$$\begin{aligned} P &= P_0 e^{rt} \\ 2\cancel{P_0} &= \cancel{P_0} e^{rt} \quad \left(\text{This is why the formula is indep. of } P_0.\right) \\ 2 &= e^{rt} \\ e^{rt} &= 2 \\ \ln e^{rt} &= \ln 2 \\ rt &= \ln 2 \\ t &= \frac{\ln 2}{r} \text{ years} \end{aligned}$$

In our "Earth" model, we had $r \approx 0.0138$. The doubling time here is:

$$t \approx \frac{\ln 2}{0.0138} \approx 50.2 \text{ years}$$

In general, the time it takes for the population to be multiplied by a factor of M $(M \geq 1)$ is given by: $t = \dfrac{\ln M}{r}$ years. The proof is similar to the above proof.

<u>Note</u>: If $r < 0$, we use the same formula, but we consider M where $0 < M \leq 1$.

Technical Note: In the formula $P = P_0 e^{rt}$, the interpretation of *r* is awkward, because we normally don't talk about the "continuous compounding" of people. Let's say we are told that a country's population is growing by 3% per year; this is called the "effective yield." When we were working with interest rates, you were usually given the "nominal" interest rate, maybe with various compounding schemes, and the effective yield, which was the overall percent growth in your balance after one year, was something you usually had to find yourself.

If the effective yield is 3% per year, what is the value for *r* that we use?
Let *k* be the effective yield as a decimal; here, $k = 0.03$. The formula is:

$$r = \ln(1+k)$$

because it solves $Pe^{rt} = P(1+k)^t$, which equates the formulas for continuous compounding and annual compounding, which corresponds to effective yield.

We obtain:

$$\begin{aligned} r &= \ln(1+0.03) \\ &\approx 0.02956 \end{aligned}$$

In a continuous compound interest problem, we would need a nominal annual interest rate of about 2.956% in order for our account to actually grow by an effective yield of 3% per year.

Conversely, if we are given *r* (as in the nominal interest rate for continuous compound interest), we can obtain the effective yield by using:

$$k = e^r - 1$$

Logistic Models: "S" Curves (Optional Discussion)

A sample model is:

$$f(t) = \frac{L}{1 + ke^{-rt}}, \text{ where constants } L, k, r > 0$$

These are more realistic models for population growth that incorporate the problem of overpopulation. The idea is that there is a limiting capacity L for a population given finite resources. The line $y = L$ serves as a horizontal asymptote for the population graph (can you see this from the formula above?). In the 1960s, scientists determined that the carrying capacity for Earth was about 10 billion. That may have been rendered obsolete by such advances as those in food technology.

See p.241 in the textbook. Logistic curves are "S-shaped," as opposed to the "J-shaped" curves modeling exponential growth. The point at which the population changes from increasing at an increasing rate (the "India" phase) to increasing at a decreasing rate (the "China" phase) is called an inflection point. It is a point at which a continuous graph changes from concave up to concave down, or vice-versa. In Calculus, you will locate these points.

Example 5 on p.241 deals with epidemiology, the study of the spread of diseases.

Gaussian Models: "Bell" Curves (Optional Discussion)

These aren't used as often in models of population growth, but they are often used to model distributions of characteristics of populations, such as human height, weight, and IQ. Example 4 on p.240 gives an application to Math SAT scores.

A normal ("bell-shaped") distribution with mean μ ("mu") and standard deviation σ ("lowercase sigma") has as its probability density function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The mean μ is a measure of center, and the standard deviation σ is a measure of spread. In fact, μ is where the bell curve peaks, and σ gives the distance between the "mean-line" and either inflection point on the bell curve. In Example 4 on p.240, $\mu = 214$ points and $\sigma = 113$ points.

PART D: SOLVING LOG EQUATIONS

Case 1: Equations with One Log

Example

Solve $3\ln x = 12$.

Solution

First, isolate the log.

$$\ln x = 4$$

Method 1: Exponentiate by "inserting" the same base (e)

$$e^{\ln x} = e^4$$
$$x = e^4$$

by the Inverse Properties.

Method 2: Go to Exponential Form

$$\log_e x = 4$$
$$e^4 = x \quad \left(\text{by "Zig-zagging"}\right)$$
$$x = e^4$$

Warning: Check to ensure that your tentative solutions do not yield logs of nonpositive numbers in the original equation.

Here, $x = e^4$ checks out in the original equation $\left(3\ln x = 12\right)$.

Solution set: $\left\{ \boldsymbol{e^4} \right\}$

Example

Solve $\log(x-1)^4 = -8$. Assume $x > 1$.

Solution

Use the Power / "Smackdown" Rule:

$$4\log(x-1) = -8$$

Then, isolate the log, and identify the base as 10.

$$\log_{10}(x-1) = -2$$

By either exponentiating with base 10, or by converting to Exponential Form, we obtain:

$$\begin{aligned} x-1 &= \underbrace{10^{-2}}_{=0.01} \\ x &= 1+0.01 \\ x &= 1.01 \qquad \text{(This checks out.)} \end{aligned}$$

Solution set: $\{\mathbf{1.01}\}$

Note: What if we had not had the restriction $x > 1$? Because 4 is an even exponent, we need to modify the Power / "Smackdown" Rule:

$$\begin{aligned} \log(x-1)^4 &= -8 \\ 4\log|x-1| &= -8 \\ \log_{10}|x-1| &= -2 \\ |x-1| &= 10^{-2} \\ |x-1| &= 0.01 \end{aligned}$$

$$\begin{aligned} x-1 &= 0.01 \\ x &= 1.01 \quad \text{(Checks out)} \end{aligned}$$

or

$$\begin{aligned} x-1 &= -0.01 \\ x &= 0.99 \quad \text{(Checks out)} \end{aligned}$$

Solution set: $\{\mathbf{0.99, 1.01}\}$

Case 2: Equations with Two Logs and No Constant Term

Example

Solve $2\log x - \log(3x) = 0$.

Solution

First, use the Power / "Basketball" Rule.
We tend to condense rather than expand at first when we are dealing with an equation with two logs.

$$\log\left(x^2\right) - \log(3x) = 0$$

Method 1: Separate the logs.

$$\log\left(x^2\right) = \log(3x)$$

We may now delete the logs on both sides by the One-to-One Properties.

$$\begin{aligned} x^2 &= 3x \\ x^2 - 3x &= 0 \\ x(x-3) &= 0 \end{aligned}$$

$$\cancel{x=0} \quad \text{or} \quad x = 3$$

The 3 checks out in the original equation, but the 0 does not.

Solution set: $\{\mathbf{3}\}$

Note: An exponential or a log equation may have no solutions, one solution, or more than one solution.

<u>Method 2</u>: Condense the left side of the equation

$$\log\left(x^2\right) - \log\left(3x\right) = 0$$

$$\log\left(\frac{x^2}{3x}\right) = 0 \quad \left(\text{by the Quotient Rule for Logs}\right)$$

Observe that 0 cannot be a solution to the original equation. Assume x is not zero.

$$\log_{10}\left(\frac{x}{3}\right) = 0$$

$$10^{\log_{10}\left(\frac{x}{3}\right)} = 10^0 \quad \left(\text{or go directly to Exponential Form}\right)$$

$$\frac{x}{3} = 10^0$$

$$\frac{x}{3} = 1$$

$$x = 3$$

The 3 checks out in the original equation.

Solution set: $\left\{\mathbf{3}\right\}$

Case 3: Equations with Two Logs and a Nonzero Constant Term

Example

Solve $\log_{15} x + \log_{15}(x-2) = 1$.

Solution

First, condense the left side by using the Product Rule "in reverse."

$$\log_{15}\left[x(x-2)\right] = 1$$

Exponentiate with base 15 or go to Exponential Form:

$$x(x-2) = 15^1$$

Warning: Do not set the factors equal to 0 now. We need to isolate 0 on one side if we are to apply the Zero Factor Property (ZFP).

Solve the resulting quadratic equation:

$$\begin{aligned} x^2 - 2x &= 15 \\ x^2 - 2x - 15 &= 0 \\ (x-5)(x+3) &= 0 \end{aligned}$$

$x = 5$ or $\cancel{x = -3}$

Does $x = 5$ check out?

$$\log_{15} \underbrace{5}_{"+"} + \log_{15} \underbrace{(5-2)}_{"+"} = 1$$

Yes.

Does $x = -3$ check out?

$$\log_{15}\underbrace{(-3)}_{"-"} + \log_{15}\underbrace{(-3-2)}_{"-"} = 1$$

No. If even one of the log arguments had been nonpositive, then we would have had to reject $x = -3$.

Solution set: $\{\mathbf{5}\}$

Newton's Law of Cooling

See #63 in the exercise set for Section 3.5.