CHAPTER 4: TRIGONOMETRY (INTRO)

SECTION 4.1: (ANGLES); RADIAN AND DEGREE MEASURE

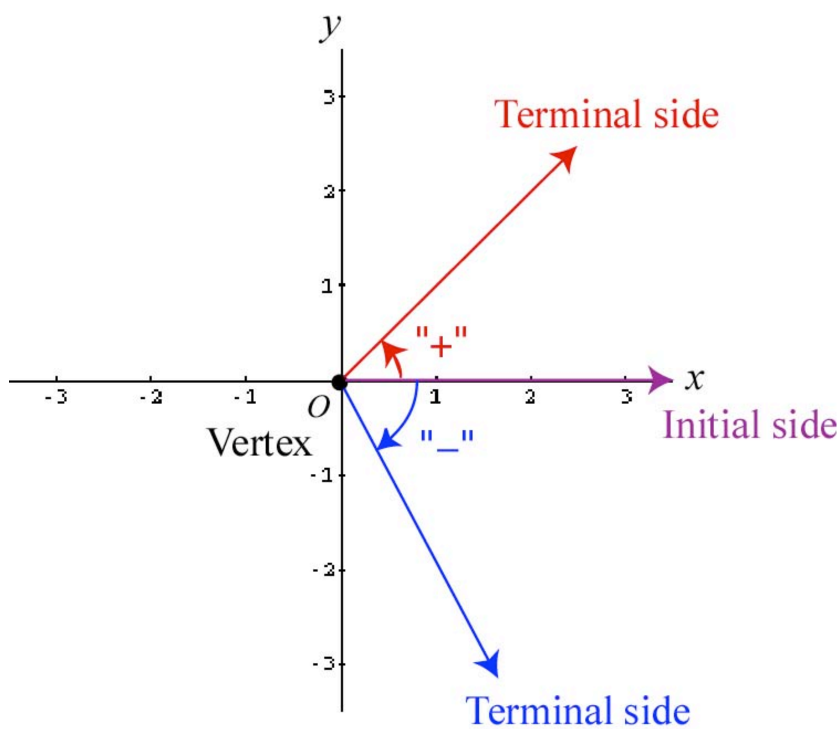
PART A: ANGLES

An angle is determined by rotating a ray (a "half-line") from an initial side to a terminal side about its endpoint, called the vertex.

A positive angle is determined by rotating the ray **counterclockwise**.

A negative angle is determined by rotating the ray **clockwise**.

A standard angle in standard position has the positive x-axis as its initial side and the origin as its vertex:



Angles are often denoted by capital letters (with maybe the $\angle$ symbol) and by Greek letters such as θ (theta), ϕ (phi), α (alpha), β (beta), and γ (gamma).

PART B: DEGREE MEASURE FOR ANGLES

We often associate angles with their rotational measures.

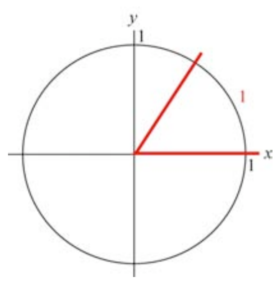
There are 360° (360 degrees) in a full (counterclockwise) revolution.
This is something of a cultural artifact; ancient Babylonians operated on a base-60 number system.

We sometimes use DMS (Degree-Minute-Second) measure instead of decimal degrees.
There are 60 minutes in 1 degree (Think: "hour"), and there are 60 seconds in 1 minute.
For example, $34^\circ 30' 20''$ denotes 34 degrees, 30 minutes, and 20 seconds.

PART C: RADIAN MEASURE FOR ANGLES

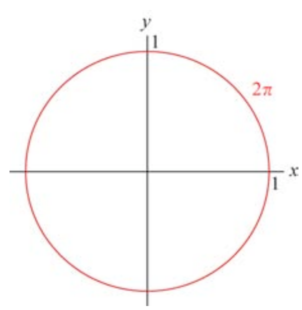
Radian measure is more "mathematically natural," and it is typically assumed in calculus.
In fact, radian measure is assumed if there are no units present.

Consider the unit circle (centered at the origin). 1 radian is defined to be the measure of a central angle (i.e., an angle whose vertex coincides with the center of the circle) that intercepts an arc length of 1 unit along the unit circle.



1 radian is about 57.3°; in fact, it is exactly $\left(\frac{180}{\pi}\right)^\circ$.

There are 2π radians in a full (counterclockwise) revolution, because the entire circle (which has circumference 2π) is intercepted exactly once by such an angle.



PART D: CONVERTING BETWEEN DEGREES AND RADIANS

2π radians is equivalent to 360°. Therefore, π radians is equivalent to 180°. Either relationship may be used to construct conversion factors.

In any unit conversion, we effectively multiply by 1 in such a way that the old unit is canceled out.

Example

Convert 45° into radians.

Solution

$$45^\circ = \left(45^\circ\right)\overbrace{\left(\frac{\pi\left[\text{rad}\right]}{180^\circ}\right)}^{=1} = \left(\cancel{45^\circ}\right)\left(\frac{\pi\left[\text{rad}\right]}{\cancel{180^\circ}_{\,4}}\right) = \frac{\boldsymbol{\pi}}{\mathbf{4}}\left[\textbf{rad}\right]$$

Example

Convert $\dfrac{5\pi}{18}$ radians into degrees.

Solution

$$\frac{5\pi}{18} = \left(\frac{5\pi}{18}\left[\text{rad}\right]\right)\underbrace{\left(\frac{180^\circ}{\pi\left[\text{rad}\right]}\right)}_{=1} = \left(\frac{5\cancel{\pi}}{\cancel{18}}\right)\left(\frac{\cancel{180^\circ}^{\,10^\circ}}{\cancel{\pi}}\right) = \mathbf{50^o}\text{, or}$$

$$\frac{5\pi}{18} = \frac{5}{18}\left(\pi\ \text{rad}\right) = \frac{5}{\cancel{18}}\left(\cancel{180^\circ}^{\,10^\circ}\right) = \mathbf{50^o}$$

Warning: Always make sure what mode your calculator is in (DEG vs. RAD) whenever you evaluate trig functions such as sin, cos, and tan.

PART E: ARC LENGTH

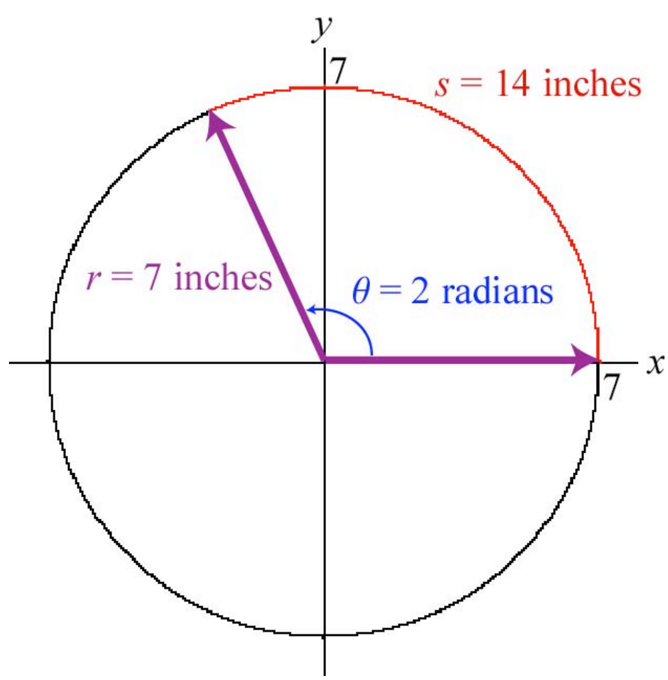
Consider a circle of radius r $(r > 0)$.
The arc length s of the arc intercepted by a central angle measuring θ radians is:

$$s = r\theta$$

Example

The arc length of an arc along a circle of radius 7 inches intercepted by a central angle measuring 2 radians is given by:

$$\begin{aligned} s &= r\theta \\ &= (7)(2) \\ &= \mathbf{14\ inches} \end{aligned}$$



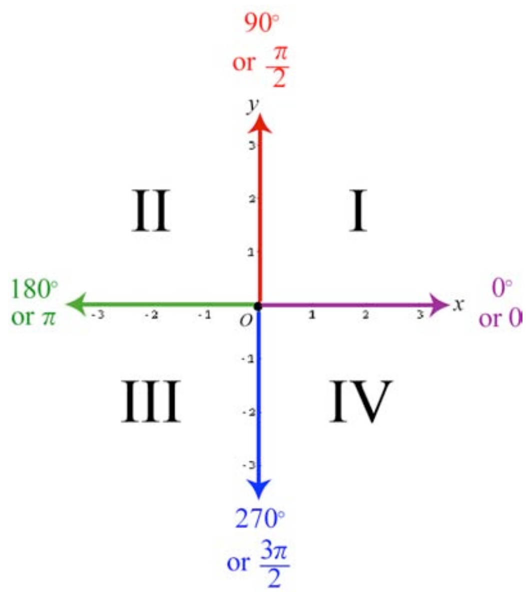
Note: If r is measured in inches, then s is measured in inches. Since θ is measured in radians, it may be seen as "unit-less." We can treat θ as simply a real value.

PART F: QUADRANTS AND QUADRANTAL ANGLES

The x- and y-axes divide the xy-plane into 4 quadrants.
Quadrant I is the upper right quadrant; the others are numbered in counterclockwise order.

A standard angle whose terminal side lies on the x- or y-axis is called a quadrantal angle.
Quadrantal angles correspond to "integer multiples" of 90° or $\frac{\pi}{2}$ radians.

The quadrants and some quadrantal angles:
(For convenience, we may label a standard angle by labeling its terminal side.)



Example

What quadrant does $\frac{5\pi}{6}$ lie in? (Through what quadrant does the terminal side pass when the angle is in standard position?)

Solution

Observe: $\underbrace{\frac{3\pi}{6}}_{=\frac{\pi}{2}} < \frac{5\pi}{6} < \underbrace{\frac{6\pi}{6}}_{=\pi}$, so $\frac{5\pi}{6}$ is in **Quadrant II**.

You could also consider degree measures: $90^\circ < 150^\circ < 180^\circ$

PART G: CLASSIFYING ANGLES

Type	Degree Measure	Radian Measure	Quadrant
Acute	in $\left(0^{\circ}, 90^{\circ}\right)$	in $\left(0, \dfrac{\pi}{2}\right)$	I
Right	90°	$\dfrac{\pi}{2}$	(Quadrantal)
Obtuse	in $\left(90^{\circ}, 180^{\circ}\right)$	in $\left(\dfrac{\pi}{2}, \pi\right)$	II

Complementary angles are a pair of positive angles that add up to 90°.
Supplementary angles are a pair of positive angles that add up to 180°.

Warning: It is easy to confuse these. Remember that "C" comes before "S" in the dictionary. Similarly, $90 < 180$.

PART H: COTERMINAL ANGLES

Standard angles that share the same terminal side are called coterminal angles. They differ by, at most, an integer number of full revolutions counterclockwise or clockwise.

If the angle θ is measured in **radians**, then its coterminal angles are of the form: $\theta + 2\pi n$, where *n* is any integer.

If the angle θ is measured in **degrees**, then its coterminal angles are of the form: $\theta + 360n^{\circ}$, where *n* is any integer.

Note: Since *n* could be negative, the "+" sign is sufficient in the above forms, as opposed to "$\pm$."

Example

The angles coterminal to $\frac{\pi}{3}$ are of the form $\frac{\pi}{3}+2\pi n$, where n is any integer.

To obtain some of these angles, we may take $\frac{\pi}{3}$ and successively add or subtract 2π, which equals $\frac{6\pi}{3}$ $\left(\text{Think: } 2=\frac{6}{3}\right)$:

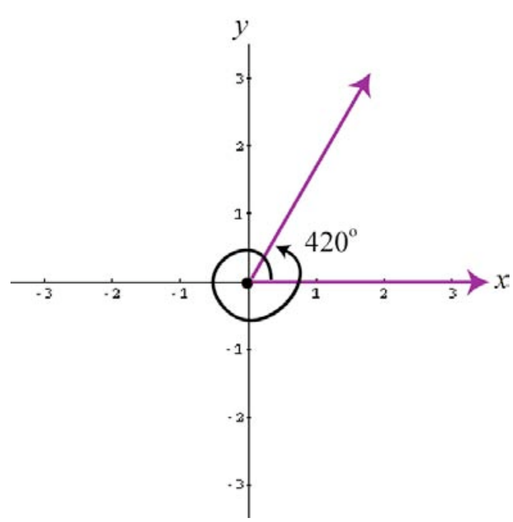
$$\ldots \quad -\frac{11\pi}{3} \xleftarrow{-\frac{6\pi}{3}} -\frac{5\pi}{3} \xleftarrow{-\frac{6\pi}{3}} \boxed{\frac{\pi}{3}} \xrightarrow{+\frac{6\pi}{3}} \frac{7\pi}{3} \xrightarrow{+\frac{6\pi}{3}} \frac{13\pi}{3} \quad \ldots$$

In degrees …

The angles coterminal to 60° are of the form $60^\circ+360n^\circ$, where n is any integer. To obtain some of these angles, we may take 60° and successively add or subtract 360°:

$$\ldots \quad -660^\circ \xleftarrow{-360^\circ} -300^\circ \xleftarrow{-360^\circ} \boxed{60^\circ} \xrightarrow{+360^\circ} 420^\circ \xrightarrow{+360^\circ} 780^\circ \quad \ldots$$

Below is a 420° angle:



Warning: If different scales are used for the x- and y-axes, there may be visual distortions. For example, a 45° angle may not "look like" a 45° angle, and perpendicular lines may not appear perpendicular. (Remember the "This is a square!" problem from Notes P.27?)

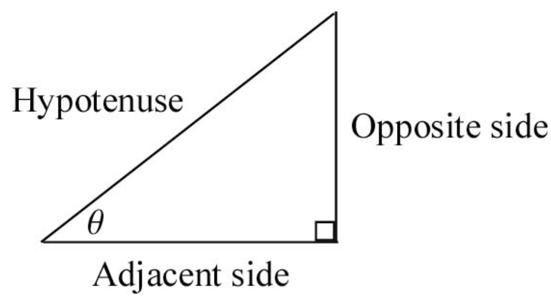
SECTIONS 4.2-4.4: TRIG FUNCTIONS (VALUES AND IDENTITIES)

We will consider two general approaches: the Right Triangle approach, and the Unit Circle approach.

PART A: THE RIGHT TRIANGLE APPROACH

The Setup

The acute angles of a right triangle are complementary. Consider such an angle, θ. Relative to θ, we may label the sides as follows:

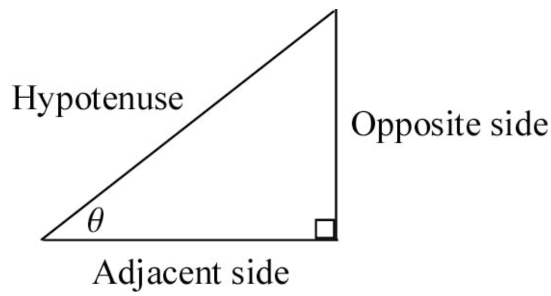


The hypotenuse always faces the right angle, and it is always the longest side.

The other two sides are the legs. The opposite side (relative to θ) faces the θ angle. The other leg is the adjacent side (relative to θ).

We sometimes use the terms "hypotenuse," "leg," and "side" when we are actually referring to a length.

<u>Defining the Six Basic Trig Functions</u> (where θ is acute)



<u>The Ancient Curse (or "How to Define Trig Functions")</u>

SOH-CAH-TOA

$$\text{Sine } \theta = \sin\theta = \frac{\text{Opp.}}{\text{Hyp.}}$$

$$\text{Cosine } \theta = \cos\theta = \frac{\text{Adj.}}{\text{Hyp.}}$$

$$\text{Tangent } \theta = \tan\theta = \frac{\text{Opp.}}{\text{Adj.}}$$

<u>Reciprocal Identities (or "How to Define More Trig Functions")</u>

$$\text{Cosecant } \theta = \csc\theta = \frac{1}{\sin\theta} \quad \left(= \frac{\text{Hyp.}}{\text{Opp.}} \right)$$

$$\text{Secant } \theta = \sec\theta = \frac{1}{\cos\theta} \quad \left(= \frac{\text{Hyp.}}{\text{Adj.}} \right)$$

$$\text{Cotangent } \theta = \cot\theta = \frac{1}{\tan\theta} \quad \left(= \frac{\text{Adj.}}{\text{Opp.}} \right)$$

<u>Warning</u>: Remember that the reciprocal of $\sin\theta$ is $\csc\theta$, not $\sec\theta$.

<u>Note</u>: We typically treat "0" and "undefined" as reciprocals when we are dealing with trig functions. Your algebra teacher will not want to hear this, though!

Quotient Identities

We may also define $\tan\theta$ and $\cot\theta$ as follows:

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \text{and} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Why is this consistent with SOH-CAH-TOA?

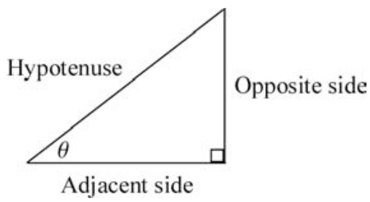
$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{\text{Opp.}}{\text{Hyp.}}}{\frac{\text{Adj.}}{\text{Hyp.}}} = \frac{\text{Opp.}}{\text{Adj.}} = \tan\theta$$

$\cot\theta$ is the reciprocal of $\tan\theta$.

We will discuss the Cofunction Identities soon ….

The Pythagorean Theorem

Given two sides, you can find the length of the third by using the Pythagorean Theorem (see p.349 for one of many proofs):



$$\boxed{(\text{Opp.})^2 + (\text{Adj.})^2 = (\text{Hyp.})^2}$$

Pythagorean Triples

Pythagorean triples are a set of three integers that can represent the side lengths of a right triangle.

The most famous Pythagorean triples are:

3-4-5
5-12-13
8-15-17

Some less famous ones are:

7-24-25
9-40-41

Warning: Remember that the hypotenuse must be the longest side. If the two legs of a right triangle have lengths 3 and 5, the hypotenuse is **not** 4.

Similar triangles have the same shape (but perhaps not the same size), and they share the same three angles. Their corresponding side lengths are in the same proportion. By considering similar triangles, we can get some more Pythagorean triples (though these are not "primitive," as the ones above were). For example, 6-8-10 triangles, 9-12-15 triangles, and so forth, are similar to 3-4-5 triangles.

Technical Note: There are infinitely many primitive Pythagorean triples. This has been determined in the field of number theory.

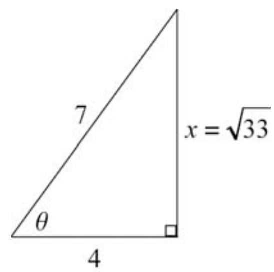
The more Pythagorean triples you know, the less frequently you will have to crank through the Pythagorean Theorem.

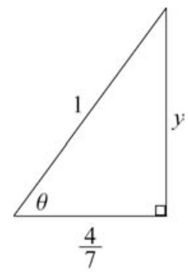
Example

Let θ be an acute angle such that $\cos\theta = \frac{4}{7}$. Find all six basic trig values for θ.

Solution

We will sketch a right triangle model. Either of the two models below would work, but the one on the left will probably be easier to work with.





A Pythagorean triple is not evident here, so we will use the Pythagorean Theorem to find x.

$$
\begin{aligned}
x^2 + (4)^2 &= (7)^2 \\
x^2 + 16 &= 49 \\
x^2 &= 33 \\
x &= \pm\sqrt{33} \\
\text{Take } x &= \sqrt{33}
\end{aligned}
$$

If θ is acute, then we always label our sides with positive numbers. Later, we will sometimes use negative numbers.

Find the six basic trig values for θ:

$\Rightarrow$ Reciprocals

$$\sin\theta = \frac{\text{Opp.}}{\text{Hyp.}} = \frac{\sqrt{33}}{7} \qquad \csc\theta = \frac{7}{\sqrt{33}} \cdot \frac{\sqrt{33}}{\sqrt{33}} = \frac{7\sqrt{33}}{33}$$

$$\cos\theta = \frac{\text{Adj.}}{\text{Hyp.}} = \frac{4}{7} \ (\text{Given}) \qquad \sec\theta = \frac{7}{4}$$

$$\tan\theta = \frac{\text{Opp.}}{\text{Adj.}} = \frac{\sqrt{33}}{4} \qquad \cot\theta = \frac{4}{\sqrt{33}} \cdot \frac{\sqrt{33}}{\sqrt{33}} = \frac{4\sqrt{33}}{33}$$

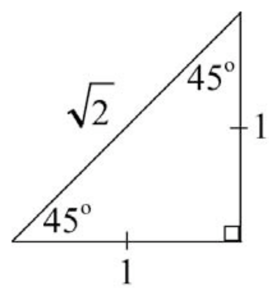
Special Triangles

We can use these triangles to find the basic trig values for 30°, 45°, and 60°. Later, we will efficiently summarize these values in a table.

45°-45°-90° Triangles

These are also known as isosceles right triangles, because they have two congruent (same-length) sides. They represent a family of similar triangles.

The most famous such triangle is:



(The tick marks on the legs indicate that they are congruent.)

The $\sqrt{2}$ is obtained from the Pythagorean Theorem.

Historical Note: The fact that the hypotenuse of this triangle is irrational really freaked out the cult of ancient Pythagoreans. One member who leaked the secret was done in by the cult.

Observe that both $\sin 45^\circ$ and $\cos 45^\circ$ are $\dfrac{1}{\sqrt{2}}$, or $\dfrac{\sqrt{2}}{2}$.

Then, both $\csc 45^\circ$ and $\sec 45^\circ$ are $\sqrt{2}$.

Note: It is sometimes easier if we do **not** rationalize a denominator before we take a reciprocal.

Observe that both $\tan 45^\circ$ and $\cot 45^\circ$ are 1.

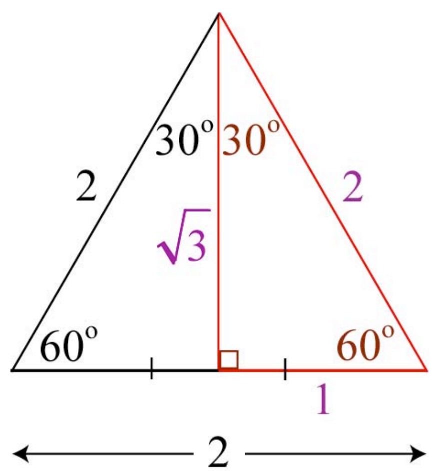
In general, the hypotenuse of any 45°-45°-90° triangle is $\sqrt{2}$ times either leg.

<u>30°-60°-90° Triangles</u>

This is another family of similar triangles.

An <u>equilateral triangle</u> is a triangle in which all three sides are congruent and all three interior angles are 60°.

We will start with an equilateral triangle of side length 2. We will draw a perpendicular from one vertex to the opposing side, thus bisecting (cutting in half) both the angle at the vertex and the opposing side.



The $\sqrt{3}$ is obtained from the Pythagorean Theorem.

Remember that, in a triangle, longer sides face larger angles.
Here, the shortest side must face the 30° angle.

In general, the hypotenuse is twice as long as the shortest side of any 30°-60°-90° triangle, and the medium side is $\sqrt{3}$ times as long as the shortest side.

Observe that both $\sin 30^\circ$ and $\cos 60^\circ$ are $\frac{1}{2}$. This is explained by the …

Cofunction Identities

Cofunctions are short for "**co**mplementary [trig] **functions**."
sin and cos are a pair of cofunctions, as are tan and cot, and also sec and csc.
Read out their full names!

Cofunctions of complementary angles are equal.

Remember that the acute angles of a right triangle are complementary.

For example, why is $\sin 30^\circ = \cos 60^\circ$? When we shift our perspective from one acute angle of a right triangle to the other, the "adjacent side" and the "opposite side" are switched.

In fact, we can look beyond right triangles ….

Cofunction Identities

If θ is measured in radians, then for **all** real values of θ:

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

If θ is measured in degrees, then for **all** angles θ:

$$\sin\theta = \cos\left(90^\circ - \theta\right)$$

$$\cos\theta = \sin\left(90^\circ - \theta\right)$$

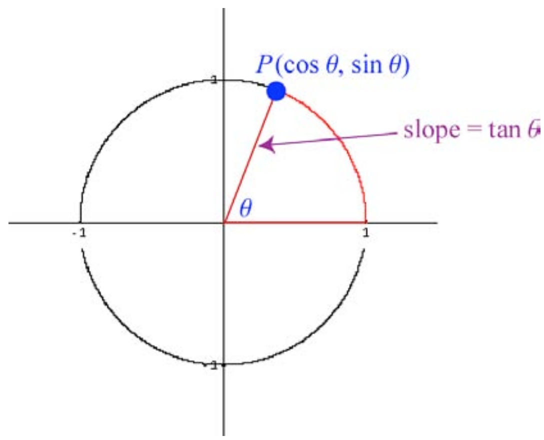
We have analogous relationships for tan and cot, and for sec and csc.

PART B: THE UNIT CIRCLE APPROACH

The Setup

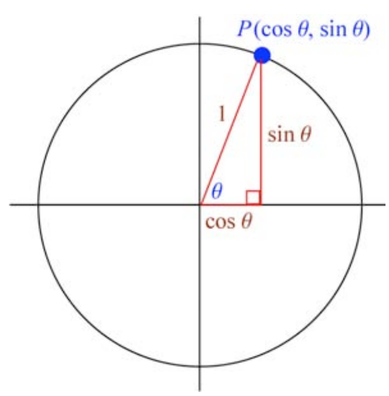
Consider a standard angle θ measured in radians (or, equivalently, let θ represent a real number).

The point $P(\cos\theta, \sin\theta)$ is the intersection point between the terminal side of the angle and the unit circle. The slope of the terminal side is, in fact, $\tan\theta$.



Note: The intercepted arc along the circle (in red) has arc length θ.

The figure below demonstrates how this is consistent with the SOH-CAH-TOA (or Right Triangle) approach. Observe: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\text{rise}}{\text{run}} = \text{slope of terminal side}$



For all real θ, $-1 \le \cos\theta \le 1$ $-1 \le \sin\theta \le 1$ $\tan\theta$ can be any real number, or it can be undefined

<u>"THE Table"</u>

We will use our knowledge of the 30°-60°-90° and 45°-45°-90° special triangles to construct THE Table below. The unit circle approach is used to find the trig values for quadrantal angles such as 0° and 90°.

Let P be the intersection point between the terminal side of the standard angle θ and the unit circle. P has coordinates $(\cos\theta, \sin\theta)$.

Key Angles θ: Degrees, (Radians)	$\sin\theta$	$\cos\theta$	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	Intersection Point $P(\cos\theta, \sin\theta)$
0°, (0)	$\dfrac{\sqrt{0}}{2} = \mathbf{0}$	$\mathbf{1}$	$\dfrac{0}{1} = \mathbf{0}$	$(1, 0)$
30°, $\left(\dfrac{\pi}{6}\right)$	$\dfrac{\sqrt{1}}{2} = \dfrac{\mathbf{1}}{\mathbf{2}}$	$\dfrac{\sqrt{\mathbf{3}}}{\mathbf{2}}$	$\dfrac{1/2}{\sqrt{3}/2} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{\mathbf{3}}}{\mathbf{3}}$	$\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$
45°, $\left(\dfrac{\pi}{4}\right)$	$\dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{\mathbf{2}}}{\mathbf{2}}$	$\dfrac{\sqrt{\mathbf{2}}}{\mathbf{2}}$	$\dfrac{\sqrt{2}/2}{\sqrt{2}/2} = \mathbf{1}$	$\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$
60°, $\left(\dfrac{\pi}{3}\right)$	$\dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{\mathbf{3}}}{\mathbf{2}}$	$\dfrac{\mathbf{1}}{\mathbf{2}}$	$\dfrac{\sqrt{3}/2}{1/2} = \sqrt{\mathbf{3}}$	$\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$
90°, $\left(\dfrac{\pi}{2}\right)$	$\dfrac{\sqrt{4}}{2} = \mathbf{1}$	$\mathbf{0}$	$\dfrac{1}{0}$ is **undefined**	$(0, 1)$

<u>Warning</u>: $\dfrac{\pi}{5}$ is not a "special" angle.

The values for the reciprocal functions, $\csc\theta$, $\sec\theta$, and $\cot\theta$, are then readily found. Remember that it is sometimes better to take a trig value where the denominator is **not** rationalized before taking its reciprocal.

For example, because $\tan 30^\circ = \dfrac{1}{\sqrt{3}}$, we know immediately that

$\cot 30^\circ = \sqrt{3}$.

<u>Observe</u>:

• The pattern in the "sin" column

<u>Technical Note</u>: An explanation for this pattern appears in the Sept. 2004 issue of the <u>College Mathematics Journal</u> (p.302).

• The fact that the "sin" column is flipped (or reversed) to form the "cos" column. This is due to the Cofunction Identities (or the Pythagorean Identities, which we will cover).

• As θ increases from 0° to 90° (i.e., from 0 to $\frac{\pi}{2}$ radians),

•• $\sin\theta$ (the y-coordinate of P) increases from 0 to 1.

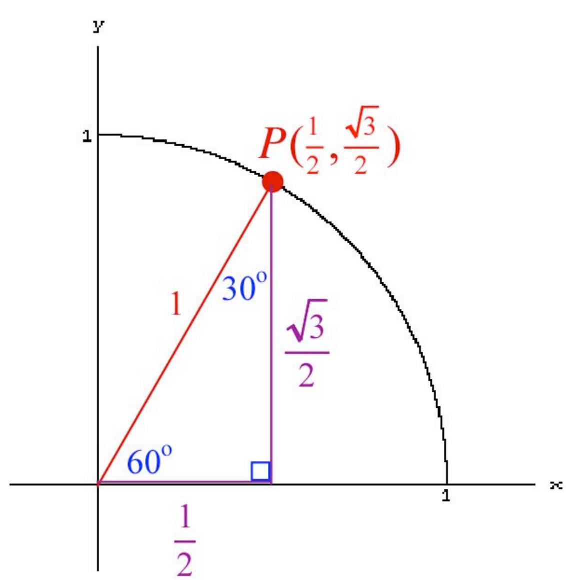
<u>Note</u>: This is more obvious using the Unit Circle approach instead of the Right Triangle approach.

•• $\cos\theta$ (the x-coordinate of P) decreases from 1 to 0.

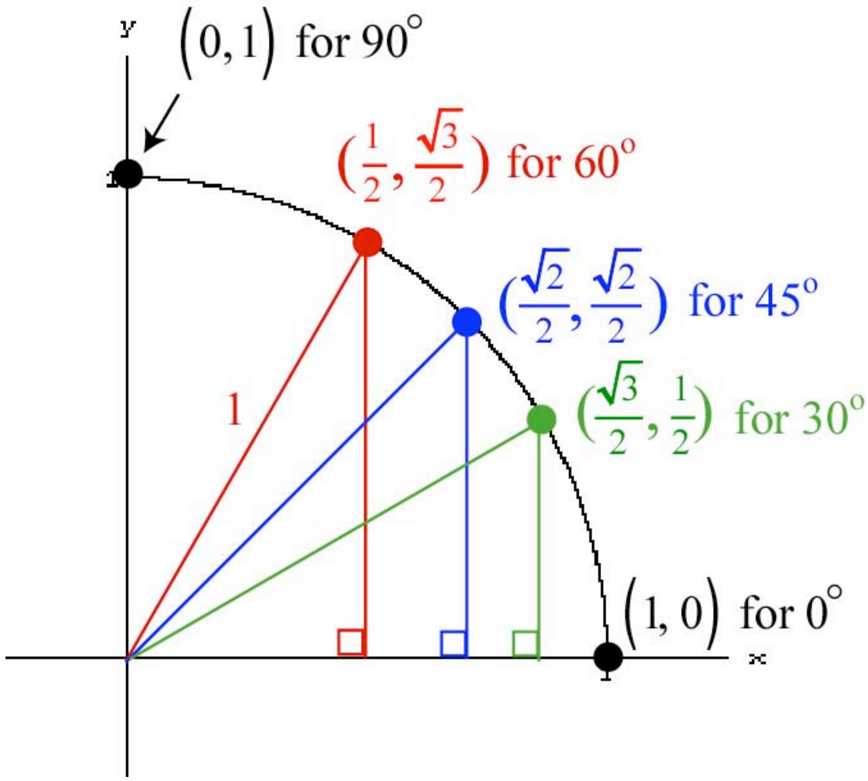
•• $\tan\theta$ (the slope of the terminal side of the standard angle θ) starts at 0, increases, and approaches ∞.

• The table is consistent with our observations based on the special triangles.

For example, consider $\theta = 60^\circ \left(\text{or } \frac{\pi}{3} \right)$:



• Here is the "Big Picture." Remember that each intersection point is of the form $P(\cos\theta, \sin\theta)$.



•• The blue point for the 45° angle has coordinates $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$; it lies on the line $y = x$, which has slope (i.e., tangent) 1.

•• Quick – what is $\cos 60^\circ$? It is $\dfrac{1}{2}$, as opposed to $\dfrac{\sqrt{3}}{2}$.

The red point for the 60° angle has coordinates $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$;

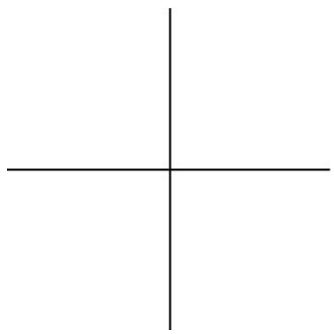
it is "high" and "to the left" relative to the blue and green points. Its *x*-coordinate (which corresponds to cos) must be relatively low, then. Granted, $\dfrac{1}{2}$, which is equal to 0.5, is not really **that** low, but it is lower than $\dfrac{\sqrt{3}}{2}$.

Other Tricks:

• You may remember that $\tan 45^\circ = 1$, and that $\dfrac{\sqrt{3}}{3}$ and $\sqrt{3}$ are other key tan values. Which is $\tan 30^\circ$, and which is $\tan 60^\circ$? Remember that $\tan\theta$ corresponds to the slope of the terminal side, so the higher slope $\left(\sqrt{3}\right)$ must be $\tan 60^\circ$.

• Here is a nice way of remembering key trig values for 90° (or $\dfrac{\pi}{2}$ radians); thanks to some clever students for this:

SIN
COS
TAN



$\sin\dfrac{\pi}{2} = 1$, whose Roman numeral is I.

$\cos\dfrac{\pi}{2} = 0$, which resembles the letter "O."

$\tan\dfrac{\pi}{2}$ is undefined, so there will be a vertical "A"symptote at

$x = \dfrac{\pi}{2}$ when we graph $f\left(\theta\right) = \tan\theta$.

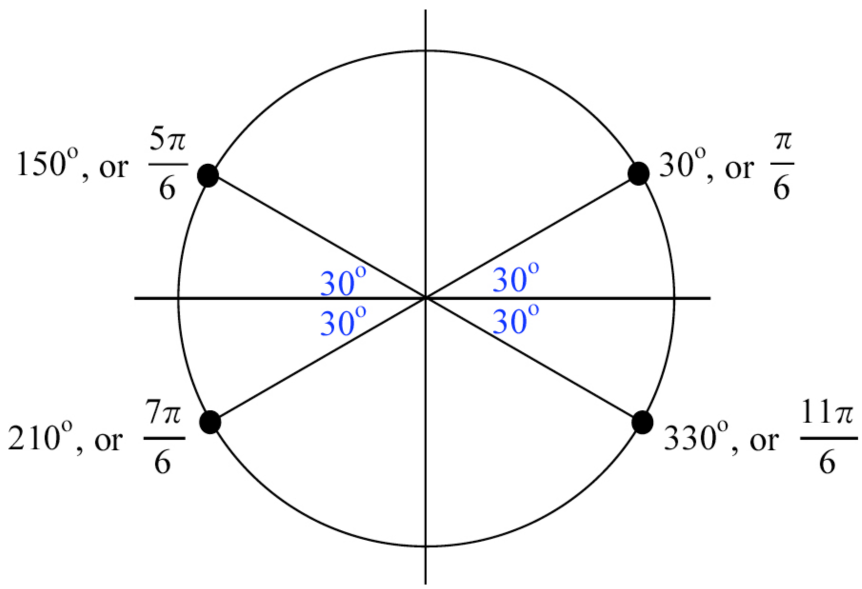
PART C: EXTENDING FROM QUADRANT I TO OTHER QUADRANTS

Reference angles

The reference angle for a non-quadrantal standard angle is the acute angle that its terminal side makes with the *x*-axis.

Brothers (the author's term) are angles that have the same reference angle.

For example, the angles below are brothers; they all have the same reference angle, namely 30°, or $\frac{\pi}{6}$ radians.



Brothers include coterminal "twins." For example, -30° (or $-\frac{\pi}{6}$ radians) is a "twin" for the 330° (or $\frac{11\pi}{6}$ radian) angle.

Let's try to discover patterns from the four "famous" positive brothers of the $\frac{\pi}{6}$ angle (including $\frac{\pi}{6}$, itself) noted in the figure on the previous page.

Quadrant II: $\pi - \frac{\pi}{6} = \frac{6\pi}{6} - \frac{1\pi}{6} = \boldsymbol{\frac{5\pi}{6}}$

We get this angle by making a half-revolution counterclockwise (corresponding to π radians) and then backtracking (going clockwise) by $\frac{\pi}{6}$ radians.

Trick: 5 is 1 less than 6.

Quadrant III: $\pi + \frac{\pi}{6} = \frac{6\pi}{6} + \frac{1\pi}{6} = \boldsymbol{\frac{7\pi}{6}}$

We get this angle by making a half-revolution counterclockwise (corresponding to π radians) and then proceeding counterclockwise by another $\frac{\pi}{6}$ radians.

Trick: 7 is 1 more than 6.

Quadrant IV: $2\pi - \frac{\pi}{6} = \frac{12\pi}{6} - \frac{1\pi}{6} = \boldsymbol{\frac{11\pi}{6}}$

We get this angle by making a full revolution counterclockwise (corresponding to 2π radians) and then backtracking (going clockwise) by $\frac{\pi}{6}$ radians.

Trick: 11 is 1 less than twice 6.

In fact, these patterns apply to any reference (acute) angle of the form $\frac{\pi}{k}$ radians, where k is an integer greater than 2.

Here are the "famous" positive brothers of the $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ angles:

We've already seen the situation with $\frac{\pi}{6}$:

(The boxes correspond to Quadrants.)

$\frac{5\pi}{6}$	$\frac{\pi}{6}$
$\frac{7\pi}{6}$	$\frac{11\pi}{6}$

Now, $\frac{\pi}{4}$:

$\frac{3\pi}{4}$	$\frac{\pi}{4}$
$\frac{5\pi}{4}$	$\frac{7\pi}{4}$

Now, $\frac{\pi}{3}$:

$\frac{2\pi}{3}$	$\frac{\pi}{3}$
$\frac{4\pi}{3}$	$\frac{5\pi}{3}$

<u>Coterminal</u> "twins" have the same trig values, including the signs.

This is true, because coterminal standard angles correspond to the same intersection point $P(\cos\theta, \sin\theta)$ on the unit circle.

For example, $-\frac{\pi}{6}$ and $\frac{11\pi}{6}$ have the same sin, cos, tan, csc, sec, and cot values. Their common intersection point is $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

Why is it useful to deal with brothers?

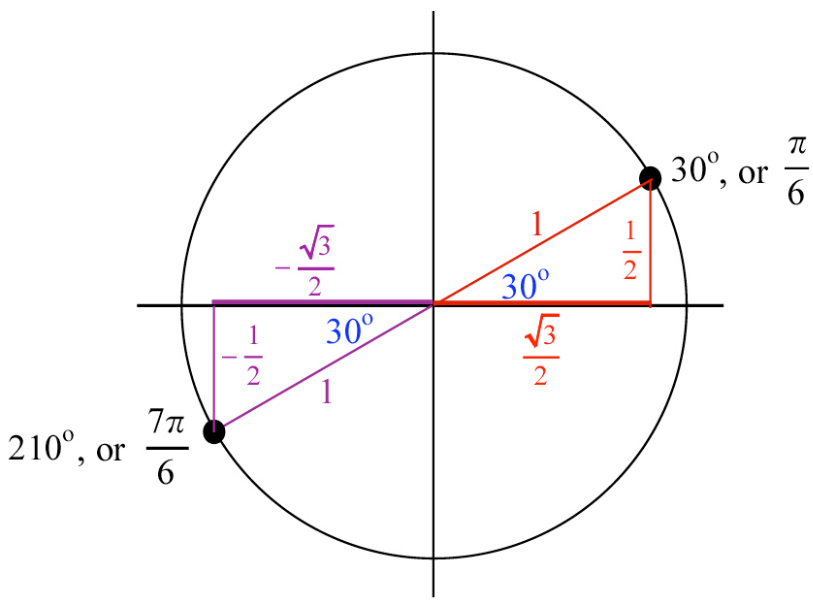
<u>Brothers</u> have the same basic trig values up to (i.e., except maybe for) the signs.

In other words, the basic trig values between two brothers are the same in magnitude, or absolute value.

For example, consider the 30° (or $\frac{\pi}{6}$) angle and its famous brother in Quadrant III, the 210° (or $\frac{7\pi}{6}$) angle.

From the Unit Circle perspective, witness the symmetry about the origin. (In other cases, we have symmetry about the y-axis if we consider Quadrant II, or about the x-axis if we consider Quadrant IV.)

With the Right Triangle perspective, we note that we have to use **negative** numbers (representing "signed lengths") to label horizontal legs that extend left from the y-axis and vertical legs that extend down from the x-axis. The hypotenuse of a right triangle is always labeled with a positive number.



Observe that $\sin\frac{\pi}{6} = \frac{1}{2}$, whereas $\sin\frac{7\pi}{6} = -\frac{1}{2}$.

Both $\tan\frac{\pi}{6}$ and $\tan\frac{7\pi}{6}$ equal $\frac{\sqrt{3}}{3}$.

<u>What is a quick way of finding the signs of trig values for angles in a particular Quadrant?</u>

Remember that reciprocal values have the same sign.

<u>"ASTC"</u>

Think: "**A**ll **S**tudents **T**ake **C**alculus"

Start in Quadrant I and progress counterclockwise through the Quadrants:

S	A
T	C

All of the six basic trig functions are **positive** in Quadrant I.
(They are all positive for acute angles.)
Sin and its reciprocal, Csc, are positive in Quadrant II.
(The other four functions are negative.)
Tan and its reciprocal, Cot, are positive in Quadrant III.
Cos and its reciprocal, Sec, are positive in Quadrant IV.

This should be evident from the Unit Circle approach. Remember that points on the unit circle (centered at the origin) correspond to $P(\cos\theta, \sin\theta)$.

(C, S) $(-,+)$	(C, S) $(+,+)$
(C, S) $(-,-)$	(C, S) $(+,-)$

Observe that $\cos\theta > 0$ in Quadrants I and IV (where $x > 0$), and $\sin\theta > 0$ in Quadrants I and II (where $y > 0$).

Remember that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. Also, it corresponds to the slope of the terminal side. $\tan\theta > 0$ in Quadrants I and III.

$-$	$+$
$+$	$-$

Example

Find the six basic trig values for $\theta = \frac{19\pi}{6}$.

Solution

(There are many possible approaches, including converting to degrees.)

First, observe that $\frac{19\pi}{6}$ is not in the interval $[0, 2\pi)$.
Let's find a coterminal angle that is in that interval.
Note: You may be satisfied with an angle in the interval $\left[-\frac{\pi}{2}, 0\right)$, because it may be a "nice" negative angle.

$$\frac{19\pi}{6} \xrightarrow{-2\pi} \frac{19\pi}{6} - \frac{12\pi}{6} = \frac{7\pi}{6}$$

$\frac{19\pi}{6}$ has the same basic trig values as $\frac{7\pi}{6}$.

The reference angle for $\frac{7\pi}{6}$ is $\frac{\pi}{6}$, and we know their family well.
We recognize that "7 is 1 more than 6," and this is simply the famous brother of $\frac{\pi}{6}$ that lies in Quadrant III.

Warning: Although the denominator often helps in determining the reference angle, watch out for simplifications! For example, $\frac{8\pi}{6} = \frac{4\pi}{3}$, and its reference angle is $\frac{\pi}{3}$, not $\frac{\pi}{6}$.

From THE Table for Quadrant I, we know that:

$$\sin\frac{\pi}{6}=\frac{1}{2}$$
$$\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$$
$$\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

By the Reciprocal Identities, we then find that:

$$\csc\frac{\pi}{6}=2$$
$$\sec\frac{\pi}{6}=\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$$
$$\cot\frac{\pi}{6}=\sqrt{3}$$

The trig values for $\frac{7\pi}{6}$ may differ by at most the signs.

We know $\frac{7\pi}{6}$ lies in Quadrant III.

S	A
T	C

We know tan and cot are positive here, but the other four are negative.

Since $\frac{7\pi}{6}$ was coterminal with $\frac{19\pi}{6}$, we have that:

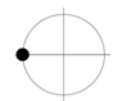
$$\sin\frac{19\pi}{6}=\sin\frac{7\pi}{6}=\mathbf{-\frac{1}{2}}$$
$$\cos\frac{19\pi}{6}=\cos\frac{7\pi}{6}=\mathbf{-\frac{\sqrt{3}}{2}}$$
$$\tan\frac{19\pi}{6}=\tan\frac{7\pi}{6}=\mathbf{\frac{\sqrt{3}}{3}}$$
$$\csc\frac{19\pi}{6}=\csc\frac{7\pi}{6}=\mathbf{-2}$$
$$\sec\frac{19\pi}{6}=\sec\frac{7\pi}{6}=\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\mathbf{-\frac{2\sqrt{3}}{3}}$$
$$\cot\frac{19\pi}{6}=\cot\frac{7\pi}{6}=\mathbf{\sqrt{3}}$$

Example: Evaluate $\csc(7\pi)$.

Solution

Observe that 7π is coterminal with π. Therefore:

$\csc(7\pi) = \csc\pi$, or they are both undefined.

Ordinarily, we could write $\csc\pi = \dfrac{1}{\sin\pi}$, but $\sin\pi = 0$; this is because the intersection point for the π angle on the unit circle is $\left(\underbrace{-1}_{\cos\pi}, \underbrace{0}_{\sin\pi}\right)$. 

Therefore, $\csc\pi$ and $\csc(7\pi)$ must both be **undefined**.

Finding Reference Angles using Degrees

First, find a coterminal angle α in the interval $\left(0^\circ, 360^\circ\right)$ or $\left(-90^\circ, 360^\circ\right)$. Determine which "famous flat" angle associated with the *x*-axis is closest: 0°, 180°, or 360°. The reference angle for α (and, therefore, the original angle) is the absolute value of the difference between this "famous flat" angle and α.

Note: This trick works with radians, provided you find a coterminal angle α in the interval $(0, 2\pi)$ or $\left(-\dfrac{\pi}{2}, 2\pi\right)$ and compare it with 0, π, or 2π. This comes in handy if you are dealing with "non-special" angles.

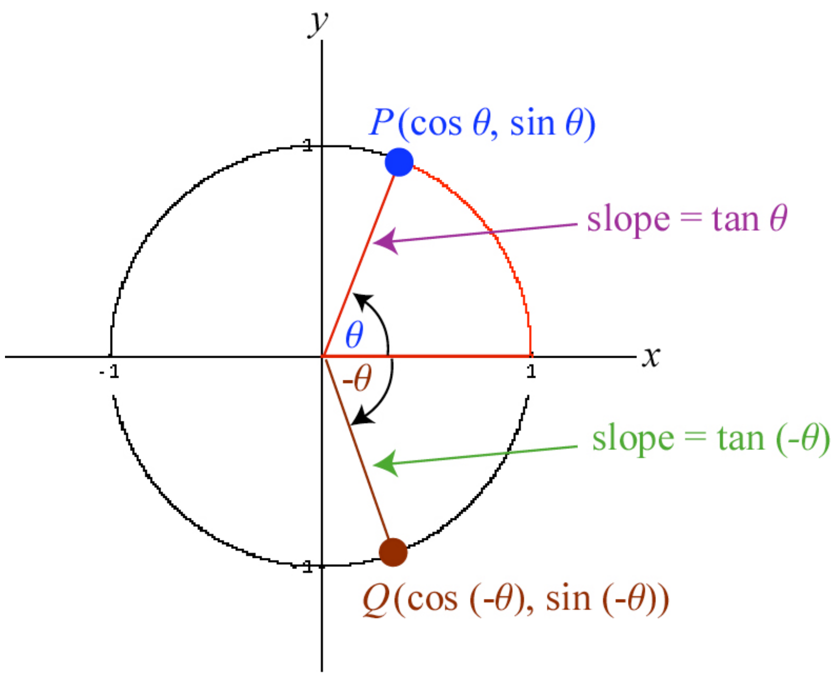
Example: Evaluate $\tan\left(-225^\circ\right)$.

Solution

Observe that $-225^\circ + 360^\circ = 135^\circ$, which is our desired coterminal angle (α). The reference angle is $\left|180^\circ - 135^\circ\right| = 45^\circ$. We know that $\tan\left(45^\circ\right) = 1$, and 135° is in Quadrant II, since $90^\circ < 135^\circ < 180^\circ$. The ASTC trick gives us a sign flip, so $\tan\left(-225^\circ\right) = \tan\left(135^\circ\right) = \mathbf{-1}$.

PART D: EVEN/ODD IDENTITIES

Consider an angle θ and its opposite, $-\theta$, measured in radians.



Observe that intersection points *P* and *Q* have the same *x*-coordinates but opposite *y*-coordinates and slopes. Therefore …

Even/Odd Identities

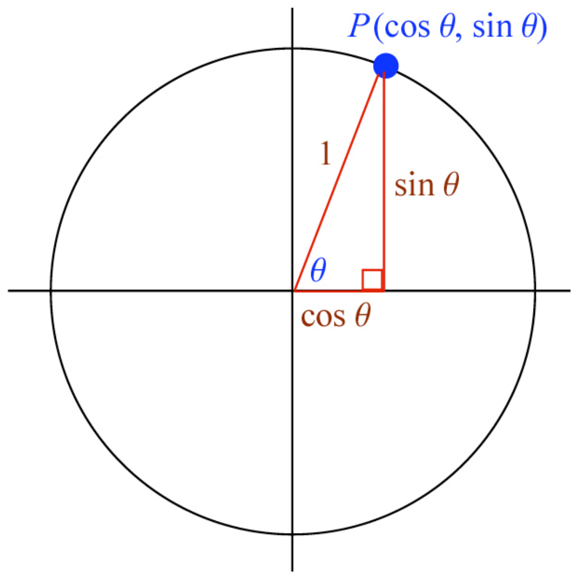
For any angle θ,

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta, \text{ or they are both undefined}$$

In other words, cos (and its reciprocal, sec) are even, but the other four (sin and tan, and their reciprocals, csc and cot) are odd.

PART E: PYTHAGOREAN IDENTITIES

Consider the Unit Circle approach:



From the Pythagorean Theorem, we get the "basic" Pythagorean Identity:

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

(This holds for quadrantal angles θ, even though a right triangle can't be drawn in those cases. It also holds for other Quadrants because of the symmetry properties of brother angles.)

If we divide both sides of the "basic" identity by $\sin^2\theta$ $\left(\text{where } \sin\theta \neq 0\right)$, we obtain:

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\boxed{1 + \cot^2\theta = \csc^2\theta}$$

If we divide both sides of the "basic" identity by $\cos^2\theta$ $\left(\text{where } \cos\theta \neq 0\right)$, we obtain:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

PART F: INVERSE TRIG FUNCTIONS (INTRO)

Example

Let θ be an acute angle such that $\sin\theta = 0.4$. What is θ?

Solution

Instead of asking, "What is the sine of this angle?", we are asking, "What is the acute angle whose sine is this number?"

You want to use the $\sin^{-1}$ ("inverse sine") button on your calculator.

If you are in degree mode, you should get about $\mathbf{23.6^\circ}$. Observe that this is less than 30°, which is $\sin^{-1}(0.5)$, because $\sin(30^\circ) = 0.5$, and 30° is an acute angle.

If you are in radian mode, you should get about **0.412 radians**, which is less than $\dfrac{\pi}{6} \approx 0.523$ radians.

Note: There are infinitely many non-acute angles that have a sine of 0.4.
For example, consider the supplementary "brother" angle in Quadrant II.
In degrees, this angle is about 156.4°, since $180^\circ - 23.6^\circ = 156.4^\circ$.
In radians, this angle is about 2.730 radians, since $\pi - 0.412 \approx 2.730$.
There are also the coterminal twins for the 23.6° angle and the 156.4° angle.

We will discuss inverse trig functions in further detail in Section 4.7.

Warning: When evaluating trig or inverse trig functions using a calculator, make sure you are in the right mode: DEG vs. RAD.

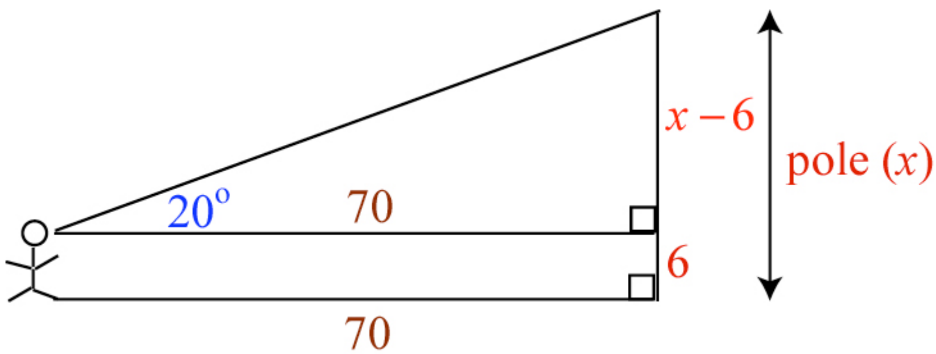
PART G: A WORD PROBLEM!

Example

A six-foot tall man standing on a flat surface looks up at a 20° angle of elevation at the top of a flagpole. His feet are 70 feet away from the base of the pole. How tall is the flagpole?

Solution

It often helps to set up unknowns and draw pictures.



Warning: Make sure your calculator is in degree mode.

$$\tan 20^\circ = \frac{x-6}{70}$$
$$70\tan 20^\circ = x-6$$
$$x = 6+70\tan 20^\circ$$
$$x \approx \textbf{31.5 feet}$$

Warning: Don't forget units in your final answer, if appropriate.

Warning: Avoid approximating anything until the end. You don't want to compromise the accuracy of your final answer by introducing inappropriate roundoff errors. The memory key on your calculator may be helpful. If you do round off intermediate results, use many significant digits.