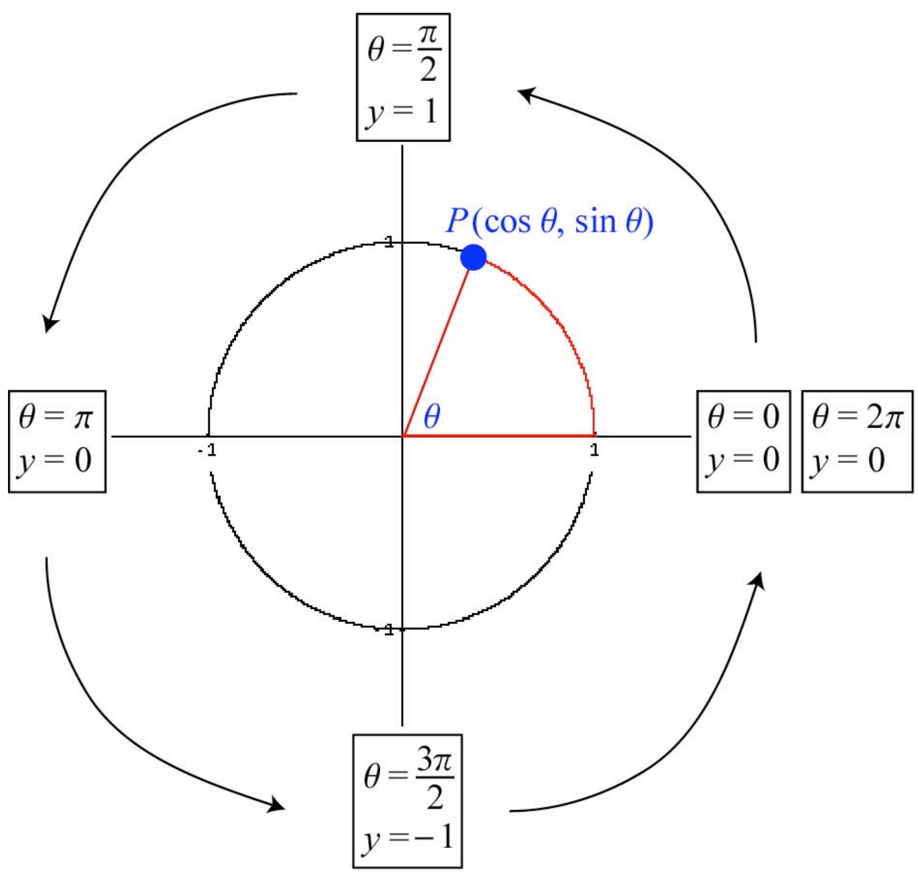
SECTION 4.5: GRAPHS OF SINE AND COSINE FUNCTIONS

PART A : GRAPH $f(\theta) = \sin\theta$

Note: We will use θ and $f(\theta)$ for now, because we would like to reserve x and y for discussions regarding the Unit Circle.

We use radian measure (i.e., real numbers) when we graph trig functions.

To analyze $\sin\theta$, begin by tracing the y-coordinate of the blue intersection point as θ increases from 0 to 2π.



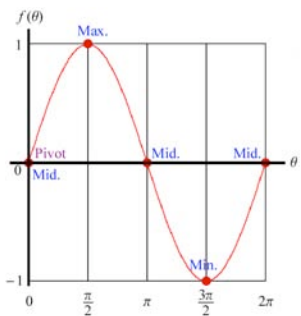
We will draw one cycle of the graph of $f(\theta) = \sin\theta$. A cycle is the smallest part of a graph (on some bounded θ-interval) whose repetition yields the entire graph; think of wallpaper. Because such a cycle can be found here, f is called a periodic function.

The θ-length of the cycle is the period of f. Here, the period for $\sin\theta$ is 2π. This is because coterminal angles have the same values for sin, cos, etc. (Think: Retracing the Unit Circle), and the *y*-coordinate of the blue intersection point never exhibits the same behavior twice as we trace the motion of the point in one revolution along the Unit Circle.

To construct the entire graph of $f(\theta) = \sin\theta$, draw one cycle for every 2π units along the θ-axis.

“Framing” One Cycle

(Inspired by Tom Teegarden and loosely by Karl Smith’s trigonometry texts)



(enlarged on the next page)

We can use a frame to graph one or more cycles of the graph of $f(\theta) = \sin\theta$, including the cycle from $\theta = 0$ to $\theta = 2\pi$.

• The five “key points” on the graph that lie on the gridlines correspond to the maximum points, the minimum points, and the “midpoints” (here, the θ-intercepts). The midpoints may also be thought of as inflection points, points where the graph changes curvature (you will locate these kinds of points in Calculus). For example, observe that the “curvature” of the cycle changes from concave down to concave up at the central point of the frame.

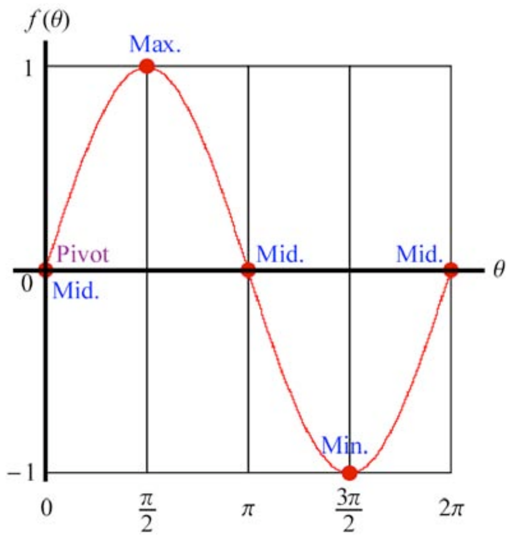
• Our vertical gridlines will break a cycle of the graph into fourths.

• Let’s call the “left-center” point of the frame the “pivot.” When we graph a sine function, the pivot of the frame will be a point on the graph. This will not be true of cosine functions.

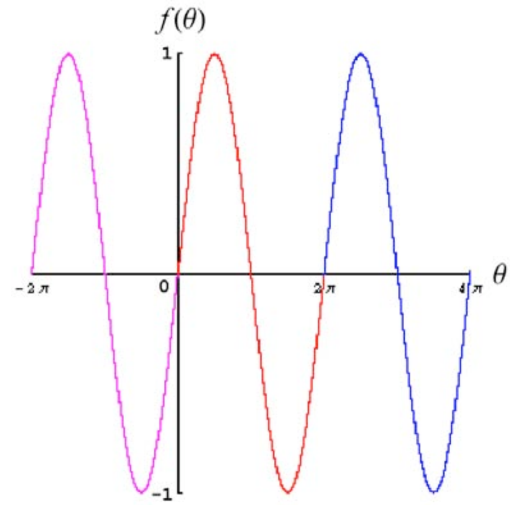
• It looks cleaner if we place coordinates on the bottom and to the left of the overall frame. (We don’t want our numbers to interfere with the actual graph.)

• (Debatable.) We will superimpose the coordinate axes on our graph, even if they would ordinarily be far from the frame; in this sense, we may not be drawing to scale. For the purposes of analysis, it is the relative positions of the coordinate axes to the frame that matter.

One cycle:



Three cycles (not framed):

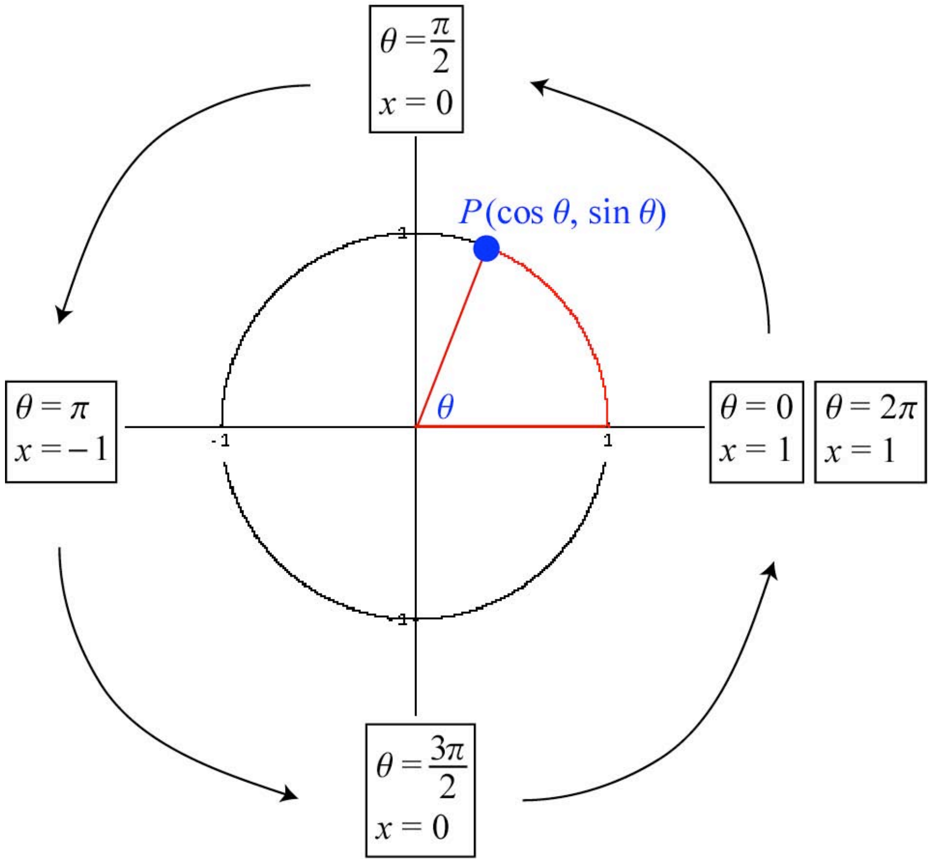


From the graph on the right, you can see that the sine function, given by $f(\theta) = \sin\theta$, is odd due to the symmetry about the origin.

Warning: In the graph on the left, the key θ-coordinates correspond to our most "famous" quadrantal angles. This is not always the case for more general sine functions, however!

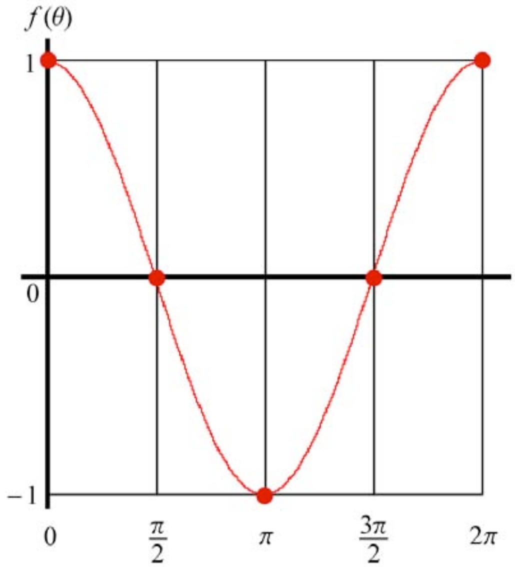
PART B : GRAPH $f(\theta) = \cos\theta$

This time, we trace the x-coordinate of the blue intersection point as θ increases from 0 to 2π.

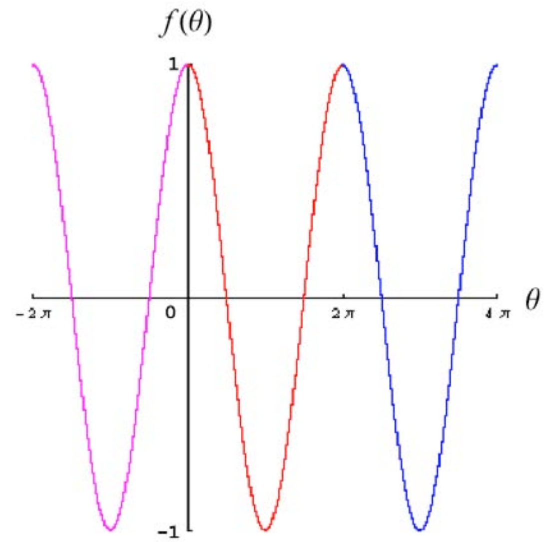


We obtain one cycle of the graph of $f(\theta) = \cos\theta$. Again, the period is 2π.

One cycle ("curvy V"):



Three cycles (not framed):



From the graph on the right, you can see that the cosine function, given by $f(\theta)=\cos\theta$, is even due to the symmetry about the vertical coordinate axis.

In fact, the graph of $f(\theta)=\cos\theta$ is simply a horizontally shifted version of the $\sin\theta$ graph. Any cycle of the $\cos\theta$ graph looks exactly like some cycle of the $\sin\theta$ graph if you graph them in the same coordinate plane.

Technical Note: In fact, $\cos\theta=\sin\left(\theta+\frac{\pi}{2}\right)$, which means that the $\cos\theta$ graph can be obtained by simply taking the $\sin\theta$ graph and shifting it to the left by $\frac{\pi}{2}$ units. Consider the behavior of the *y*-coordinate (corresponding to $\sin\theta$) of the blue intersection point on the Unit Circle if you start at $\theta=0$ and increase θ. The *x*-coordinate (corresponding to $\cos\theta$) of the point exhibits the same behavior if you start at $\theta=-\frac{\pi}{2}$ and increase θ. Also remember the Cofunction Identity $\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$, which can be rewritten as: $\cos\theta=-\sin\left(\theta-\frac{\pi}{2}\right)$. Our cosine graph can also be obtained by taking our sine graph, shifting it to the right by $\frac{\pi}{2}$ units, and reflecting it about the θ-axis.

PART C: DOMAIN AND RANGE

From both the Unit Circle and the graphs we've just seen, observe that:

> If $f(\theta) = \sin\theta$ or $\cos\theta$, then:
>
> The domain of f is **R**, and
> The range of f is $[-1, 1]$.

PART D: GRAPH $f(\theta) = a\sin\theta,\ f(\theta) = a\cos\theta$

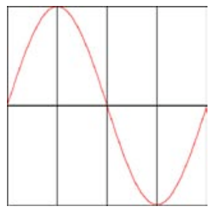
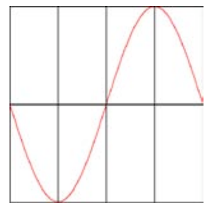
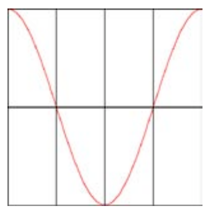
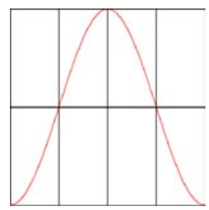
Recall the issue of transformations in Section 1.7.
(It may be more profitable to look at examples as opposed to reading the following blurb.)

In Sections 4.5 and 4.6, we assume that a is a nonzero real number.

> Let G be the graph of $y = f(x)$.
>
> If $a > 0$, then the graph of $y = a \cdot f(x)$ is:
>
> a **vertically stretched** version of G if $a > 1$
> a **vertically squeezed** version of G if $0 < a < 1$
>
> (See Notes 1.86-1.87.)
>
> If $a < 0$, then we take the graph of $y = |a| \cdot f(x)$ and reflect it about the x-axis.
>
> (See Notes 1.83-1.84.)

Cycle Shape and our Frame Method

If we want to draw one cycle of the graph of $f(x) = a \sin x$ or $f(x) = a \cos x$ using the Frame Method we will discuss later, then we will draw the "cycle shape" given to us by the following Cycle Grid:

	$a > 0$	$a < 0$
sin	 “Good morning”	 “Good evening”
cos	 “valley”	 “hill”

All four cycle shapes above appear on the complete graph of f, but we want the one suited for our Frame Method.

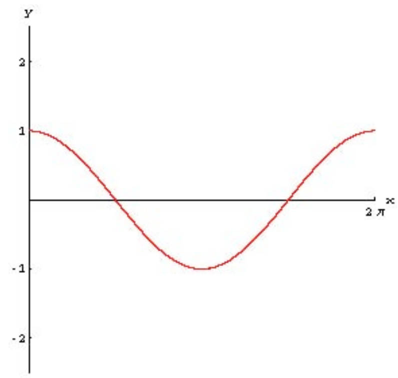
Amplitude

If $f(x) = a \sin x$ or $f(x) = a \cos x$, then $|a|$ = the amplitude of its graph. It is half the "height" of the graph.

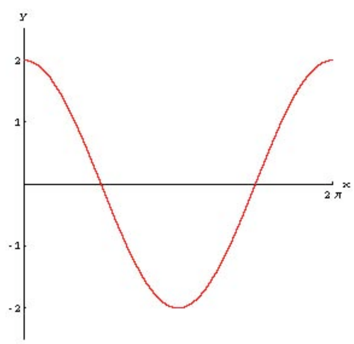
<u>Examples</u>

The graphs below are not framed, but they are well suited for comparative purposes.

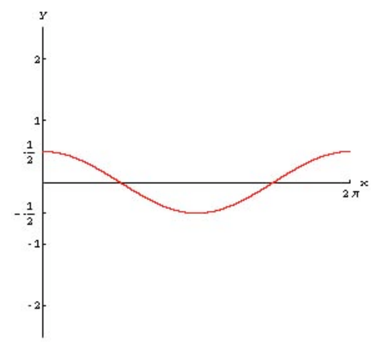
Graph of [one cycle of] $y = \cos x$:



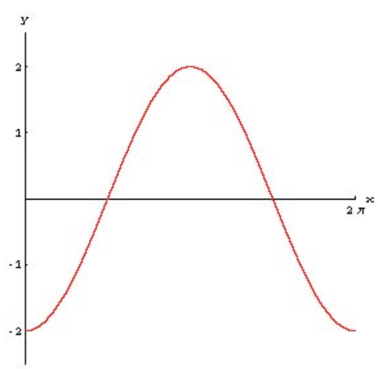
Graph of $y = 2\cos x$:



Graph of $y = \frac{1}{2}\cos x$:



Graph of $y = -2\cos x$:



Observe that the amplitude for both $y = 2\cos x$ and $y = -2\cos x$ is 2, and the range of the corresponding functions is $\left[-2, 2\right]$; the domain of both is still **R**.

PART E : PERIOD; GRAPH $f(x) = a\sin(bx)$, $f(x) = a\cos(bx)$

We now consider the forms $f(x) = a\sin(bx)$ and $f(x) = a\cos(bx)$.
In Sections 4.5 and 4.6, we assume that b is a positive real number.

<u>"Accordion Effects"</u>

Recall from Section 1.7: Notes 1.86-1.87 that, if x replaced by (bx), then the corresponding graph is:

horizontally squeezed if $b > 1$
horizontally stretched if $0 < b < 1$

<u>Warning</u>: This is the "opposite" of how $|a|$ affects **vertical** stretching and squeezing.

The period of $y = a\sin x$ or $y = a\cos x$ is 2π. Observe that a has no effect on the period.

More generally, the period of $y = a\sin(bx)$ or $y = a\cos(bx)$ is $\frac{2\pi}{b}$.

We assume $b > 0$; otherwise, the period is $\frac{2\pi}{|b|}$.

<u>Technical Note</u>: Observe that we obtain exactly one cycle on the x-interval $\{x \mid 0 \le bx \le 2\pi\}$, or, equivalently, $\left\{x \mid 0 \le x \le \frac{2\pi}{b}\right\}$.

<u>Interpretations of b</u>

Again, we assume $b > 0$.

We may think of b as an "aging factor." (Thanks to Peter Doyle for this idea.)
The higher b is, the more rapidly the graph changes (or "ages").

Also, b is the number of cycles one finds on an x-interval of length 2π.

What if $b < 0$?

We apply the Even/Odd Properties of the basic trig functions and write a new equation in which $b > 0$. Identify a after applying the properties, not before. See the Example below.

If we fail to apply the Even/Odd Properties, we may be in danger of having to draw our cycle shapes backwards (i.e., from right-to-left). This is something of a "time reversal" idea.

Example

Graph $y = -7\sin(-3x)$.

Solution

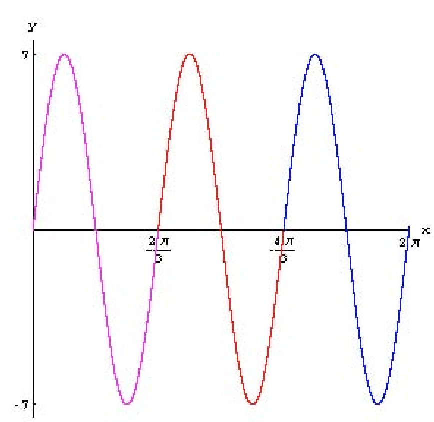
Because $b = -3 < 0$ at present, we use the fact that sine is an odd function.

$$y = -7\sin(-3x)$$
$$y = -7\left[-\sin(3x)\right]$$
$$y = 7\sin(3x)$$

We will focus on our new equivalent equation, which has $b = 3 > 0$.

Amplitude $= |a| = |7| = 7$

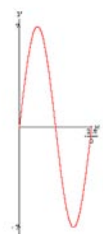
$$\text{Period} = \frac{2\pi}{b} = \frac{2\pi}{3}$$



Observe that we find $b = 3$ complete cycles on an x-interval of length 2π. "3" is the aging factor.

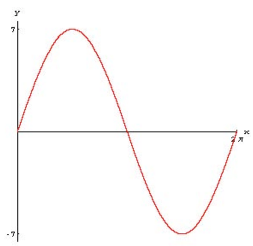
Examples (for comparative purposes)

Graph of [one cycle of] $y = 7\sin(3x)$, as in the previous Example:



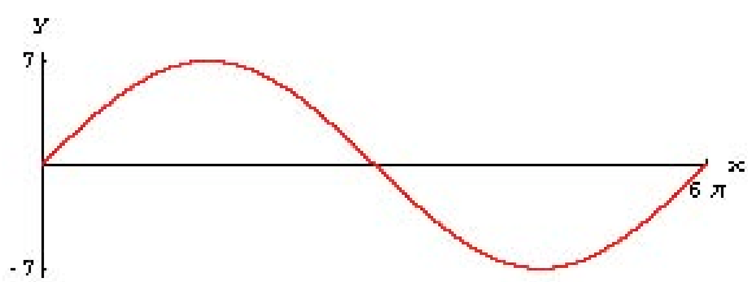
Period $= \dfrac{2\pi}{3}$

Graph of $y = 7\sin x$:



Period $= 2\pi$

Graph of $y = 7\sin\left(\dfrac{1}{3}x\right)$, or $y = 7\sin\left(\dfrac{x}{3}\right)$:



Period $= \dfrac{2\pi}{1/3} = 2\pi \cdot 3 = 6\pi$

PART F: THE "FRAME METHOD" (PCAPI)

We can use the "Frame Method" to graph one or more cycles of the graph of $y = a\sin(bx)$ or $y = a\cos(bx)$. The ingredients are:

Pivot

This is the left-center point on the frame. It is $(0,0)$ for now.

Later, when we do translations, it is the point that $(0,0)$ is "moved to".

Cycle shape

This is determined by our Cycle Grid in Notes 4.39.
Clearly mark the key points.

Amplitude

This is $|a|$. It is half the height of the graph.

Period

This is $\dfrac{2\pi}{b}$; we assume $b > 0$. If not, it is $\dfrac{2\pi}{|b|}$.

Increment

This equals $\dfrac{1}{4}(\text{Period})$. It is the horizontal distance between our vertical gridlines. (A cycle can be logically broken up into quarter-pieces.)

We will superimpose the coordinate axes, perhaps not "to scale."

One mnemonic (memory) device we can use is PCAPI.

Example

Use the Frame Method to graph one cycle of the graph of $y = -3\cos(-4x)$. (There are infinitely many possible cycles.)

In this class, you will be expected to know how to use the Frame Method on exams, so you should practice it on your homework.

Solution

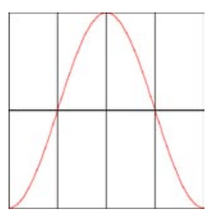
Because $b = -4 < 0$ at present, we use the fact that cosine is an even function.

$$y = -3\cos(-4x)$$
$$y = -3\cos(4x)$$

We will focus on our new equivalent equation, which has $b = 4 > 0$.

Pivot $= (0, 0)$

Cycle shape: We have a cos graph with $a = -3 < 0$, so we will use



Amplitude $= |a| = |-3| = 3$

Period $= \dfrac{2\pi}{b} = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

Increment $= \dfrac{1}{4}(\text{Period}) = \dfrac{1}{4}\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{8}$

<u>What are the horizontal lines on the frame?</u>

Let's say the pivot is (p, d). We will discuss p and d later.

In this Example, we know the pivot is $(0, 0)$.

The middle line is: $y = d$, the y-coordinate of the pivot.

In this Example, it is $y = 0$ (i.e., the x-axis).

The top line is: $y = d + \text{Amplitude}$.

In this Example, it is $y = 3$.

The bottom line is: $y = d - \text{Amplitude}$.

In this Example, it is $y = -3$.

<u>What are the vertical lines on the frame?</u> (These take longer to find!)

Let's say the pivot is (p, d).

In this Example, we know it is $(0, 0)$.

The leftmost line is: $x = p$, the x-coordinate of the pivot.

In this Example, it is $x = 0$ (i.e., the y-axis).

We will successively add the Increment ("Inc.") in order to find the x-coordinates of the other four vertical lines. They will be:

$$x = p + \text{Inc.}$$

$$x = p + (2 \cdot \text{Inc.})$$

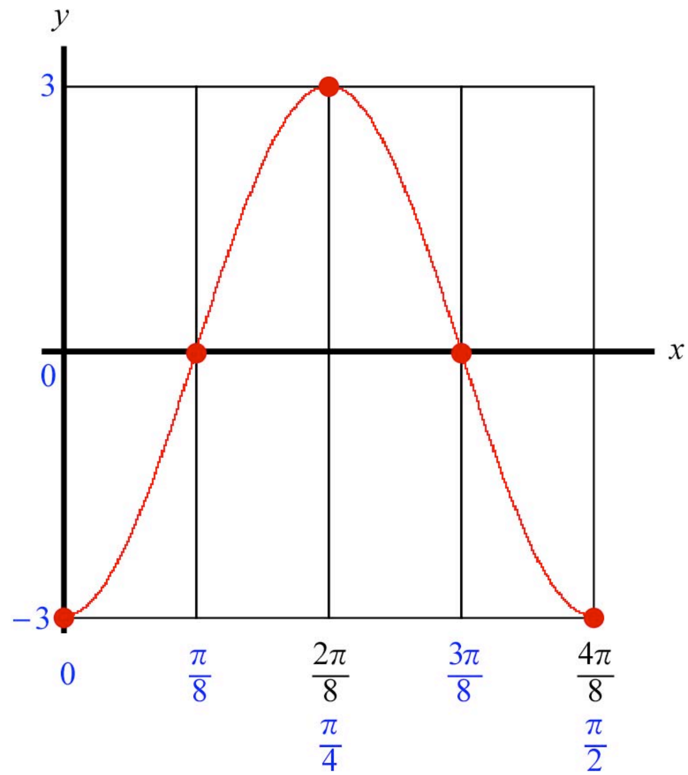
$$x = p + (3 \cdot \text{Inc.})$$

$$x = p + \left(\underbrace{4 \cdot \text{Inc.}}_{= \text{Period}} \right)$$

In this Example, the Increment is $\dfrac{\pi}{8}$.

Make sure you simplify the x-coordinates as appropriate.
Remember that common denominators are helpful when adding.

<u>The Frame</u>



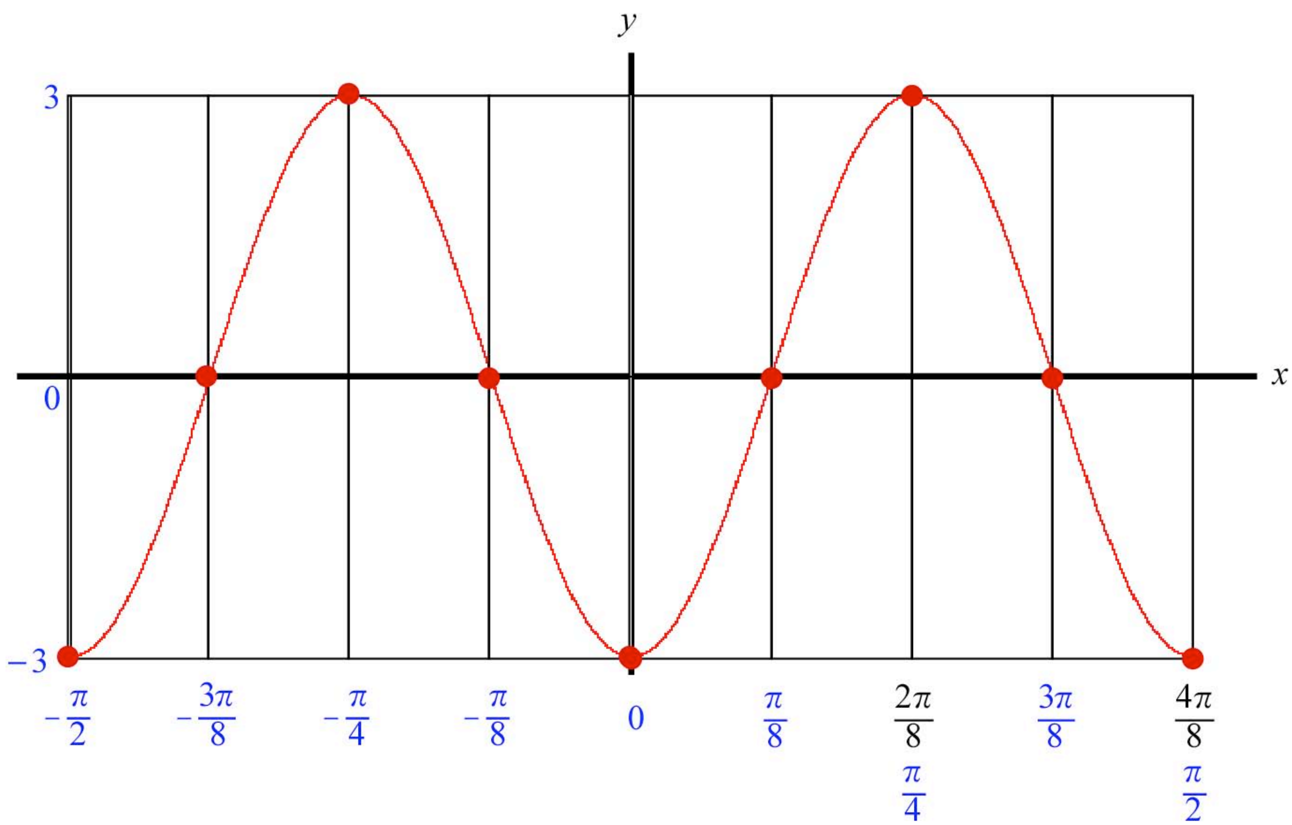
When drawing the vertical lines of the frame, you may want to first divide the box in half and then divide the smaller boxes in half. It's tricky to divide a box equally into fourths if you don't do this.

You may want to clearly indicate the five key points on the cycle. Observe that the coordinates of each key point are easily read off.

Because we are dealing with a cos graph, the pivot of the frame is not a point on the graph.

<u>Warning</u>: Don't forget to superimpose the coordinate axes. It is important to know where they are relative to your frame. They may or may not be "to scale."

Note: If we would like to graph two cycles, we can attach another frame of one cycle to the right (or to the left) of our current one. We would successively add (or subtract) the increment to obtain the additional x-coordinates we need. In our Example, it may be easier to graph the second cycle to the left of our first cycle, because there is a nice symmetry between the negative x-coordinates and the positive x-coordinates on the frame.



PART G: TRANSLATIONS ("SHIFTS")

We have thus far considered the forms:

$$y = a\sin(bx), \text{ and}$$
$$y = a\cos(bx)$$

In these cases, the point $(0, 0)$ could be used as the pivot for our frame.

What if we want to translate a cycle (and, for that matter, the entire graph) so that the pivot is moved from $(0, 0)$ to the point (p, d)? In other words, what if we want a horizontal shift of p units and a vertical shift of d units?

We now consider the forms:

$$y = a\sin\left[b(x - p)\right] + d, \text{ and}$$
$$y = a\cos\left[b(x - p)\right] + d$$

Review Section 1.7: Notes 1.83-1.84 if you don't see why.

p is called the phase shift (really, "a" phase shift; there are infinitely many possibilities); it indicates horizontal shifting.

Technical Note: Some people prefer to define phase shift so that $\left|\text{phase shift}\right| < \text{period}$. If that is the case and you want to give the phase shift, you may want to take p and add or subtract the period repeatedly until $\left|\text{phase shift}\right| < \text{period}$.

Technical Note: Larson uses the forms:

$$y = a\sin(bx - c) + d, \text{ or}$$
$$y = a\cos(bx - c) + d$$

Then, the phase shift $p = \dfrac{c}{b}$, because $(bx - c) = \left[b\left(x - \underbrace{\frac{c}{b}}_{=p}\right)\right]$.

As far as PCAPI goes, the pivot is the only item that can change as a result of a translation. (Although, in principle, we could change the cycle shape if we employ a different method.)

Example

Use the Frame Method to graph one cycle of the graph of $y = -3\cos(4x + \pi) + 2$.

Solution

We first factor the coefficient of x out of the argument of cos. Remember that we divide when we factor. You may get fractions.

$$y = -3\cos(4x + \pi) + 2$$
$$y = -3\cos\left[4\left(x + \frac{\pi}{4}\right)\right] + 2$$

If $b < 0$ then use the Even/Odd Properties. That is not the case here.

To clearly indicate the phase shift, we may rewrite this as:

$$y = -3\cos\left[4\left(x - \underbrace{\left(-\frac{\pi}{4}\right)}_{=p}\right)\right]\underbrace{+2}_{=d}$$

The phase shift: $p = -\frac{\pi}{4}$

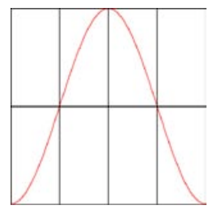
The phase shift can also be obtained by setting the argument of cos equal to 0 and solving for x; remember that we want to know where the point $(0, 0)$ is being "moved to". Observe that:

$$4x + \pi = 0 \quad \Leftrightarrow \quad x = -\frac{\pi}{4}$$

Pivot: $(p, d) = \left(-\frac{\pi}{4}, 2\right)$

The rest of PCAPI is the same as for our previous Example, which dealt with $y = -3\cos(-4x)$, or, equivalently, $y = -3\cos(4x)$.

<u>**C**ycle shape</u>: We have a cos graph with $a = -3 < 0$, so we will use



<u>**A**mplitude</u> $= |a| = |-3| = 3$

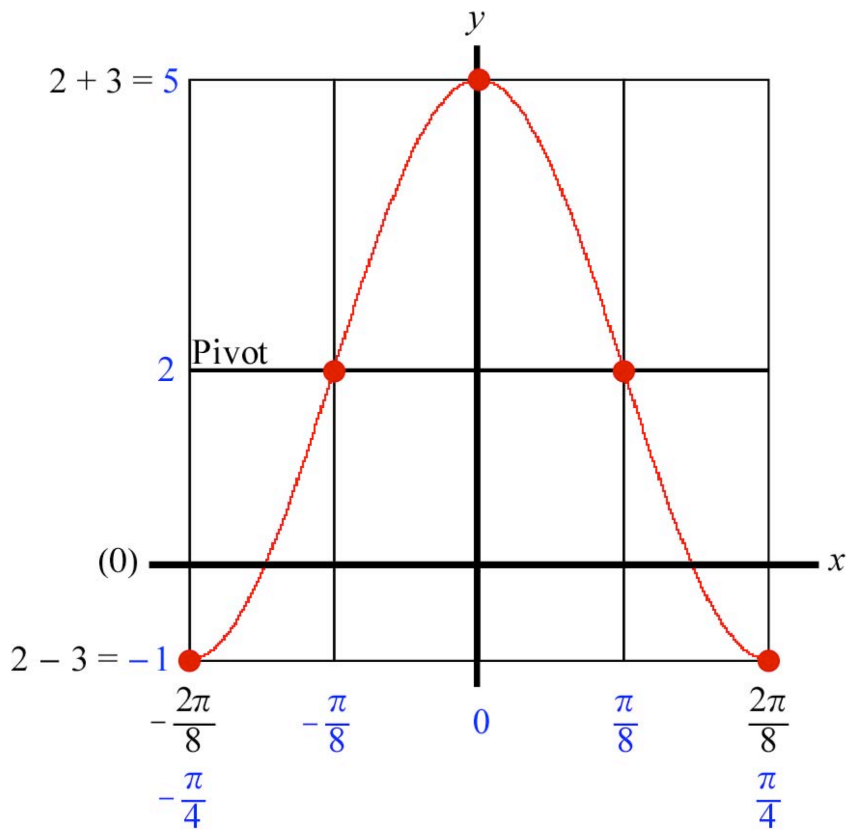
<u>**P**eriod</u> $= \dfrac{2\pi}{b} = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

<u>**I**ncrement</u> $= \dfrac{1}{4}(\text{Period}) = \dfrac{1}{4}\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{8}$

<u>The Frame</u>

Because the phase shift p is $-\dfrac{\pi}{4}$ and the increment is $\dfrac{\pi}{8}$, we want to use 8 as our common denominator when we perform our additions. We recognize that: $-\dfrac{1}{4} = -\dfrac{2}{8}$, so $-\dfrac{\pi}{4} = -\dfrac{2\pi}{8}$.

Remember to superimpose the coordinate axes, perhaps not "to scale." Remember that the x-axis has equation $y = 0$ and the y-axis has equation $x = 0$.



Again, because we are dealing with a cos graph, the pivot of the frame is not a point on the graph.

Note: Because of the vertical shift, the "midpoints" (or inflection points) are no longer x-intercepts.

Note: Observe that the range for $f(x) = -3\cos(4x + \pi) + 2$ is $[-1, 5]$.

Technical Note: If p is extreme in value, you may want to add or subtract the period repeatedly to get a new phase shift, perhaps such that $|\text{phase shift}| < \text{period}$..

Challenge: Remember the Cofunction Identity $\sin x = \cos\left(\frac{\pi}{2} - x\right)$.

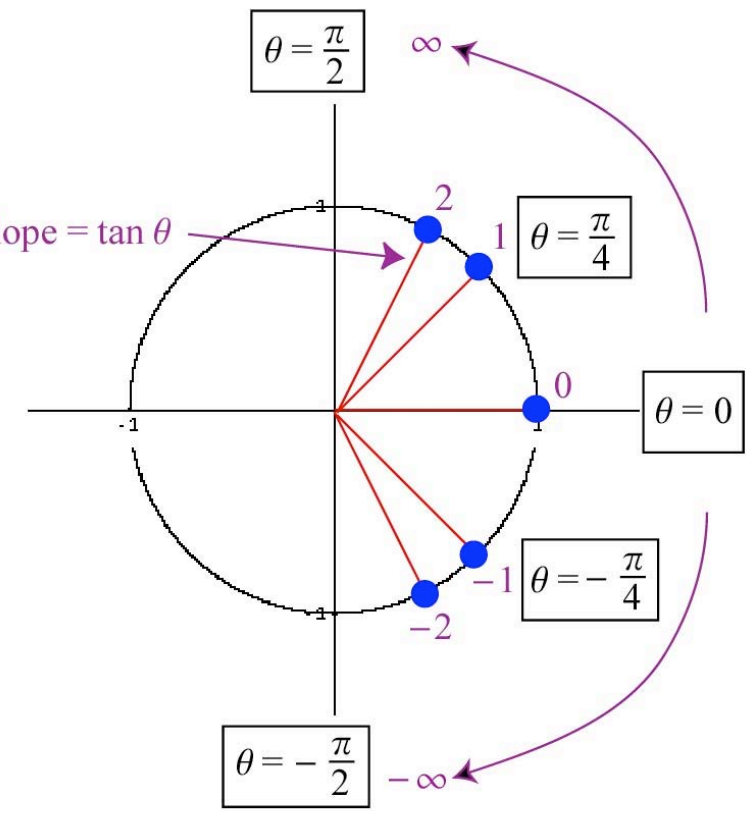
Can you see why the graph of $y = \cos\left(\frac{\pi}{2} - x\right)$ coincides with the graph of $y = \sin x$?

SECTION 4.6: GRAPHS OF OTHER TRIG FUNCTIONS

PART A: GRAPH $f(\theta) = \tan\theta$

We begin by tracing the slope of the terminal side of the standard angle θ as θ increases from 0 towards $\frac{\pi}{2}$ and as it decreases from 0 towards $-\frac{\pi}{2}$.

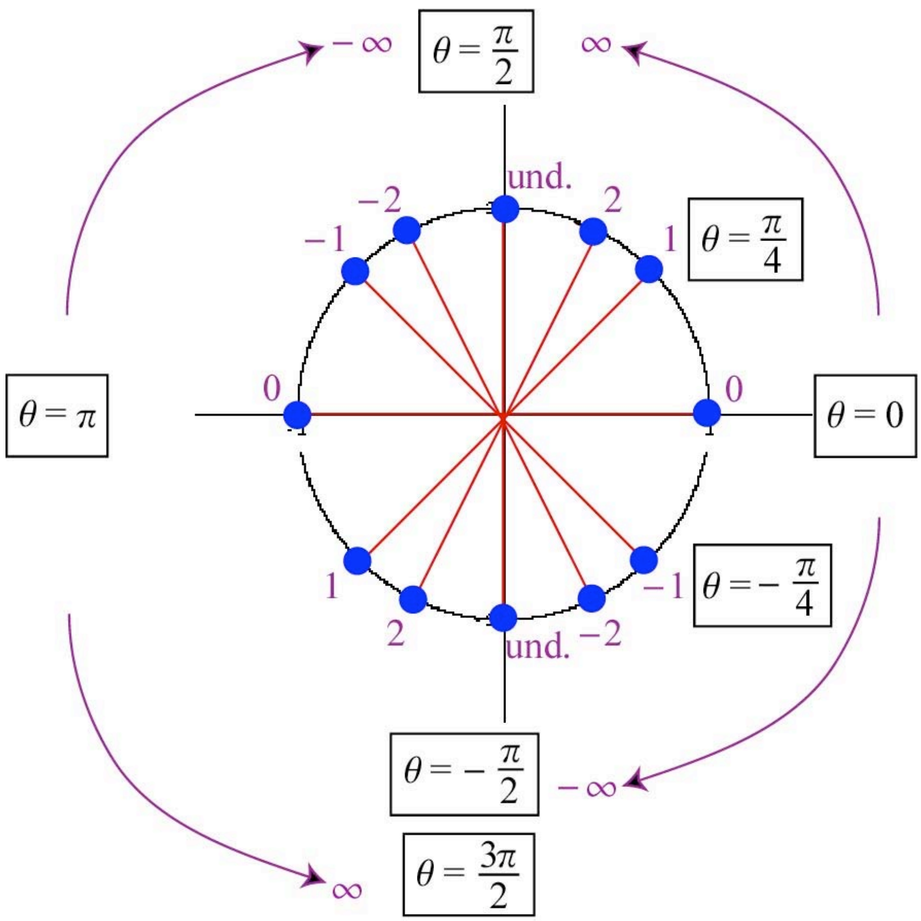
The information on slopes is in purple in the figure below.



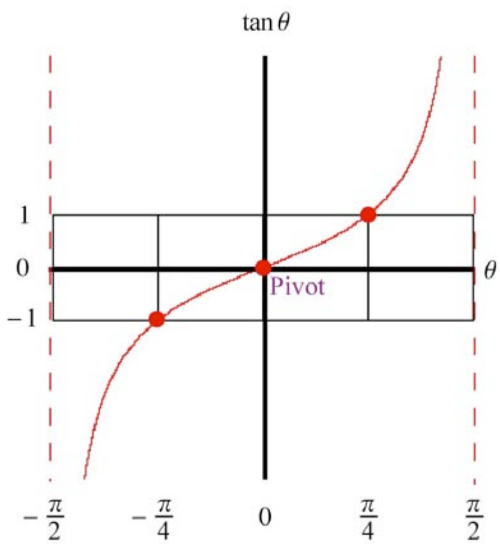
We obtain one cycle of the graph of $f(\theta) = \tan\theta$. The period is π, not 2π.

Why?

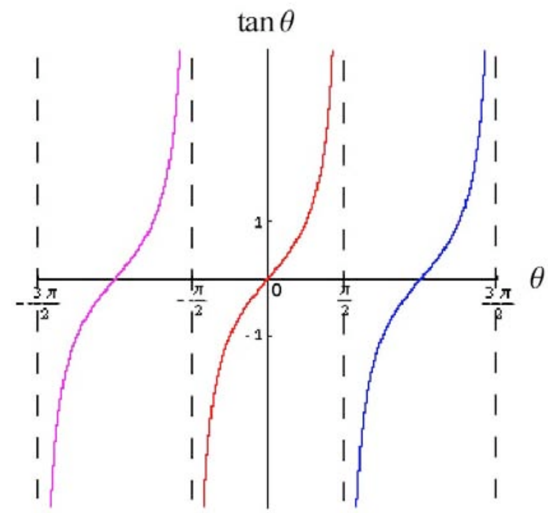
Observe that the behavior of f on the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is identical to its behavior on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:



One cycle:



Three cycles (not framed):



You can see that $f(\theta) = \tan\theta$ is odd due to the symmetry about the origin. Observe that the vertical asymptotes (VAs) naturally divide the graph into cycles.

“Framing” One Cycle

The setup for the frame of a tan or cot function has a number of differences from our setup in the last section:

• (Warning!) When we graph a tan function, the central point, not the “left-center” point, of the frame will be our pivot; it is typically a point on the graph. (When we graph a cot function, the “left-center” point will again be our pivot; it is typically not a point on the graph.) For now, the pivot is the point $(0, 0)$.

• The left and right edges of the frame correspond to vertical asymptotes (VAs).

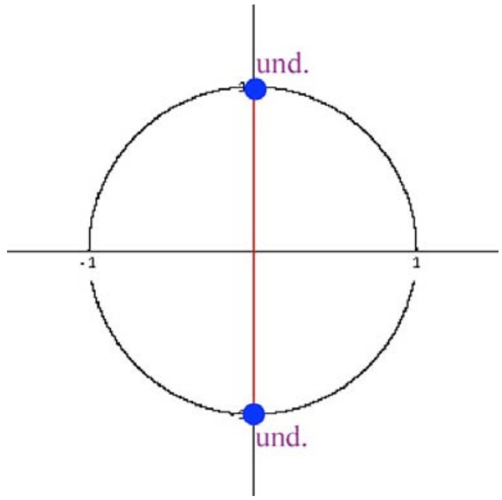
> Warning: When you are told to graph a trig function, you are expected to draw in the vertical asymptotes (if any) as dashed lines.

• The graph of a cycle will stretch beyond its frame and approach these asymptotes.

• The cycles are “snakes.” There are no maximum or minimum points. The “midpoint” of a cycle (which is the pivot here), is still an inflection point where the cycle changes curvature (concavity).

• There are only three “key points” on the graph (instead of five) that lie on the gridlines.

PART B: DOMAIN, RANGE, AND VERTICAL ASYMPTOTES (VAs) FOR $f(\theta) = \tan\theta$



From both the Unit Circle and the graphs we've just seen, observe that, if θ is real,

$$\tan\theta \text{ is undefined} \quad \Leftrightarrow \quad \theta = \frac{\pi}{2} + \pi n \quad \left(\text{i.e., } \theta = \underbrace{(2n+1)}_{\text{an odd integer}} \frac{\pi}{2} \right) \text{ for some integer } n.$$

When you see the πn, think "half revolutions."

The VAs of the graph appear at the values of θ mentioned above.

Remember that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. Observe that the values of θ mentioned above are the zeros of $\cos\theta$; the x-coordinate is 0 at the corresponding intersection points on the Unit Circle.

If $f(\theta) = \tan\theta$, then the domain of f is:

$$\left\{ \theta \;\middle|\; \theta \neq \frac{\pi}{2} + \pi n \quad \left(\text{i.e., } \theta \neq \underbrace{(2n+1)}_{\text{an odd integer}} \frac{\pi}{2} \right) \text{ for all integers } n \right\}$$

Because a slope can be any real number, …

$$\boxed{\text{If } f(\theta) = \tan\theta \text{, then the range of } f \text{ is } \mathbf{R}.}$$

This explains why our snake cycles are "infinitely tall."

PART C : GRAPH $f(\theta) = \cot\theta$

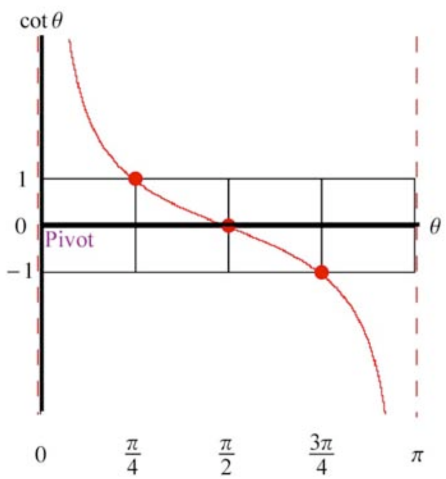
How can we use the graph for $\tan\theta$ to obtain the graph for $\cot\theta$?

$$\begin{aligned} \cot\theta &= \tan\left(\frac{\pi}{2} - \theta\right) && \text{(by the Cofunction Identities)} \\ &= \tan\left[-\left(\theta - \frac{\pi}{2}\right)\right] && \text{(by the "Switch Rule" for Subtraction)} \\ &= -\tan\left(\theta - \frac{\pi}{2}\right) && \text{(because tan is an odd function)} \end{aligned}$$

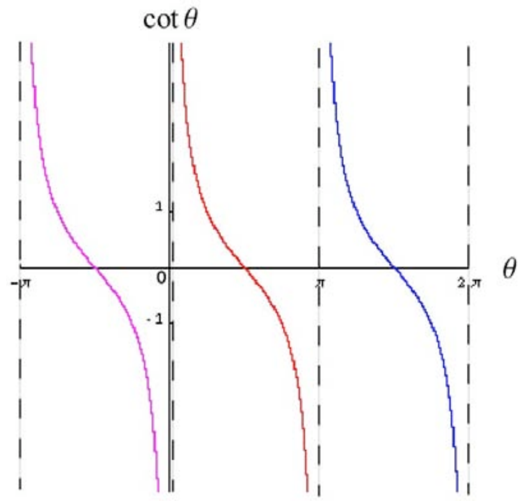
We can obtain the graph for $\cot\theta$ by:

- Taking the graph of $\tan\theta$,
- Shifting it to the right by $\frac{\pi}{2}$ units (it turns out that shifting to the left by $\frac{\pi}{2}$ units also works), and
- Reflecting the resulting graph about the θ-axis.

One cycle:



Three cycles (not framed):



You can see that $f(\theta) = \cot\theta$ is odd due to the symmetry about the origin.
We also know that the reciprocal of an odd function is also odd, and we know that the $\tan\theta$ function is odd.

Observe that the VAs naturally divide the graph into cycles.

Warning: When we are framing a cycle for a cot function, the pivot is the "left-center" point of the frame, just as when we were graphing sin and cos functions. The central point of the frame is the pivot only when we are graphing tan functions.

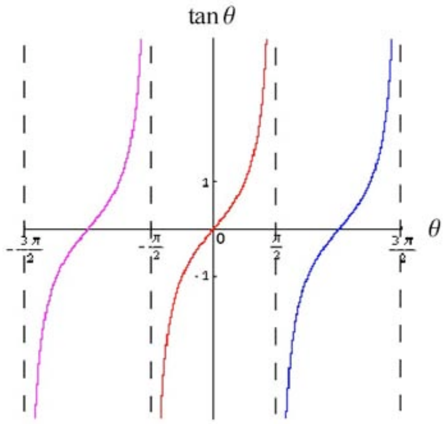
How does the graph for $\cot\theta$ compare with the graph for $\tan\theta$?

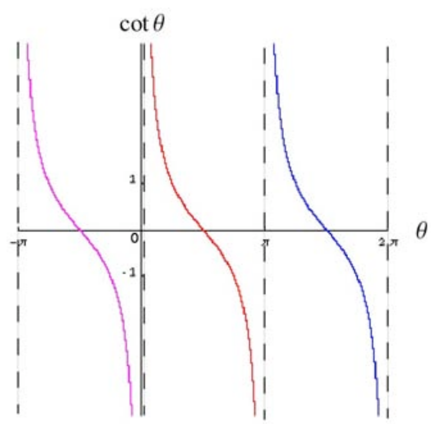
Because of the shift, the VAs are shifted $\frac{\pi}{2}$ units to the right (or to the left; either perspective works).

Because of the reflection, the snake cycles "fall" instead of "rise."

Properties of Graphs of Pairs of Basic Reciprocal Trig Functions

(See how these apply to the $\tan\theta$ (top) and $\cot\theta$ (bottom) functions.)





• One function is 0 in value $\Leftrightarrow$ The other is undefined.

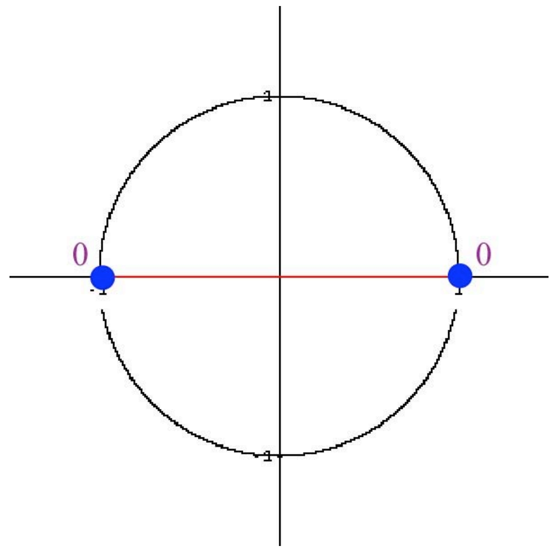
• Otherwise, their values have the same sign.
(In particular, they are positive in value in Quadrant I.)

• If you collect the asymptotes from both graphs, then, between them, one function increases $\Leftrightarrow$ The other decreases.

• They have the same period.

PART D: DOMAIN, RANGE, AND VERTICAL ASYMPTOTES (VAs) FOR $f(\theta) = \cot\theta$

It will help to notice when $\tan\theta = 0$:



$\cot\theta$ is undefined	$\Leftrightarrow$	$\tan\theta = 0$	
	$\Leftrightarrow$	$\theta = \pi n$	for some integer n

When you see the πn, think "half revolutions."

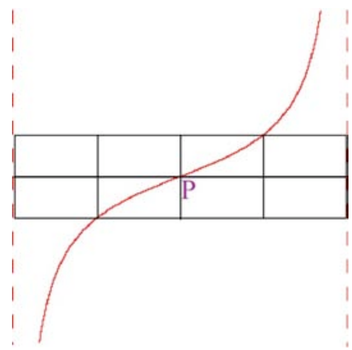
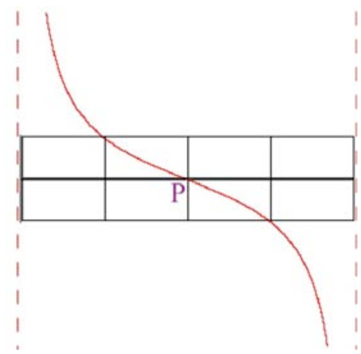
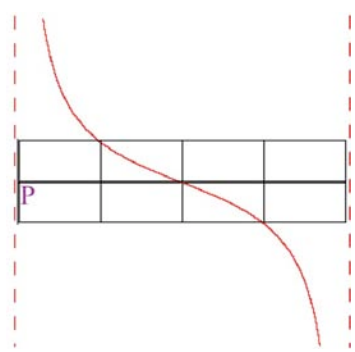
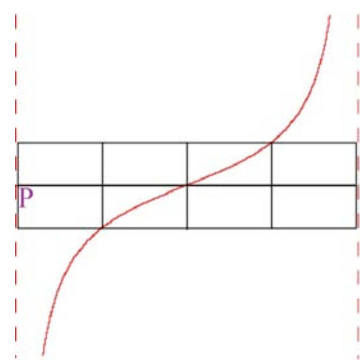
The VAs of the graph appear at the values of θ mentioned above.

Remember that $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$. Observe that the values of θ mentioned above are the zeros of $\sin\theta$; the y-coordinate is 0 at the corresponding intersection points on the Unit Circle.

> If $f(\theta) = \cot\theta$, then the domain of f is: $\left\{ \theta \,\middle|\, \theta \neq \pi n \text{ for all integers } n \right\}$,
>
> and the range of f is **R** (just as for $\tan\theta$).

PART E: CYCLE SHAPE AND OUR FRAME METHOD

Use the following Cycle Grid:

	$a > 0$	$a < 0$
tan		
cot		

The "P"s are reminders that the pivot for a tan frame is at the center (not the "left-center") of the frame.

PART F: EXAMPLES

We now consider the forms:

$$y = a\tan\left[b\left(x-p\right)\right]+d \text{, and}$$
$$y = a\cot\left[b\left(x-p\right)\right]+d$$

PCAPI still applies, but with the following modifications:

• Use the cycle shape determined by the Cycle Grid on the previous page.

• tan and cot graphs technically have no "amplitude." Nevertheless, we will informally say that "Amplitude" $= \left|a\right|$, because it helps us label the frame in an expected way.

• The period is given by $\dfrac{\pi}{b}$, not $\dfrac{2\pi}{b}$. We assume $b>0$; otherwise, it is $\dfrac{\pi}{\left|b\right|}$.

• If you are drawing a cycle for a tan graph, remember that the pivot $\left(p,d\right)$ is the central point of the frame. When writing the x-coordinates for the frame, you will be moving both to the right and to the left of the pivot. The x-coordinates you will label will be:

$$x = p-\left(2\cdot \text{Inc.}\right)$$
$$x = p-\text{Inc.}$$
$$x = p$$
$$x = p+\text{Inc.}$$
$$x = p+\left(2\cdot \text{Inc.}\right)$$

Example

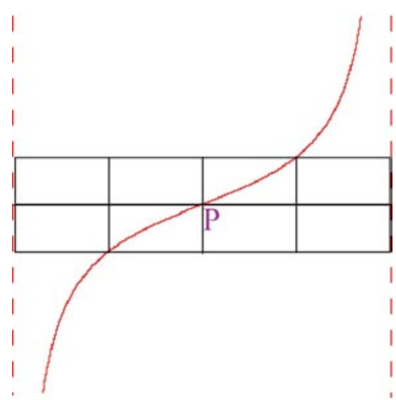
Use the Frame Method to graph one cycle of the graph of
$y = 2\tan\left(\frac{2}{5}x\right) - 3$. (There are infinitely many possible cycles.)

Solution

Fortunately, $b = \frac{2}{5} > 0$. If $b < 0$, we would need to use the Even/Odd Properties. Remember that both tan and cot are odd functions.

Pivot: $\left(p = 0, d = -3\right)$

Cycle shape: We have a tan graph with $a = 2 > 0$, so we will use:

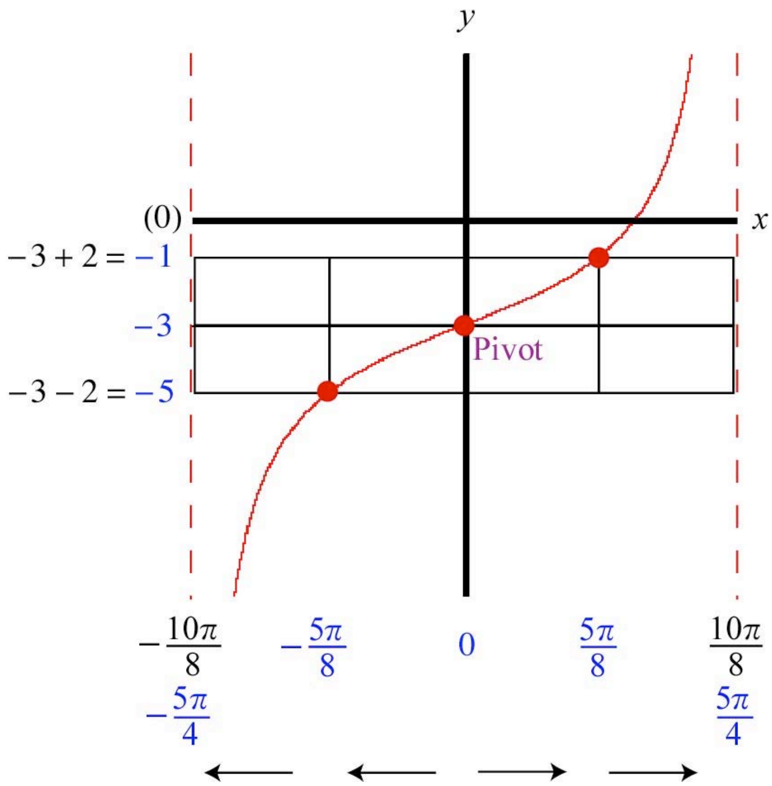


"**A**mplitude" $= |a| = |2| = 2$

Period $= \frac{\pi}{b} = \frac{\pi}{2/5} = \frac{5\pi}{2}$

Increment $= \frac{1}{4}\left(\text{Period}\right) = \frac{1}{4}\left(\frac{5\pi}{2}\right) = \frac{5\pi}{8}$

<u>The Frame</u>



Since there was no discernible phase shift, we see some nice symmetry between the positive and negative x-coordinates on this tan frame.

<u>Note</u>: If you would prefer to start the labeling process at the "left-center" point, just as for sin and cos cycles, you could find the x-coordinate of a VA by setting the argument of tan equal to $-\frac{\pi}{2}$ or $\frac{\pi}{2}$ (for example), which correspond to asymptotes for $y = \tan x$, and solving for x. Here:

$$\frac{2}{5}x = -\frac{\pi}{2}$$

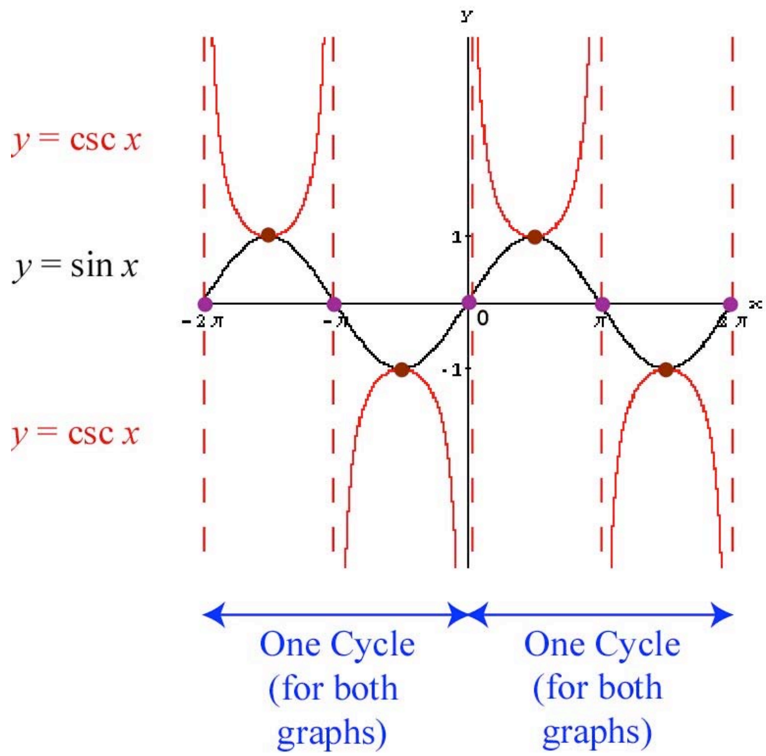
$$x = \left(\frac{5}{2}\right)\left(-\frac{\pi}{2}\right)$$

$$x = -\frac{5\pi}{4}$$

If you are dealing with a cot graph, then you would set the argument of cot equal to 0, just as for sin and cos graphs.

PART G: GRAPHS OF CSC AND SEC FUNCTIONS ("UP-U, DOWN-U" GRAPHS)

Remember that $\csc x = \dfrac{1}{\sin x}$.



How can we use the graph of $y = \sin x$ to obtain the graph of $y = \csc x$?

1) Draw VAs through the x-intercepts (in purple) of the $\sin x$ graph.

2) Between any consecutive pair of VAs:

If the $\sin x$ graph lies **above** the x-axis, then draw an **"up-U"** that has as its **minimum** point (in brown) the **maximum** point of the sin graph and that approaches both VAs.

If the $\sin x$ graph lies **below** the x-axis, then draw a **"down-U"** that has as its **maximum** point (also in brown) the **minimum** point of the sin graph and that approaches both VAs.

Warning: When graphing $\csc x$ and $\sec x$, the VAs separate the graphs into **half cycles**, not full cycles.

Why 1)?

Observe that $\csc x$ is undefined $\Leftrightarrow$ $\sin x = 0$.

Why 2)?

The reciprocal of 1 is 1, so: $\csc x = 1 \Leftrightarrow \sin x = 1$. This explains the brown intersection points between the "up-U"s and the $\sin x$ graph.

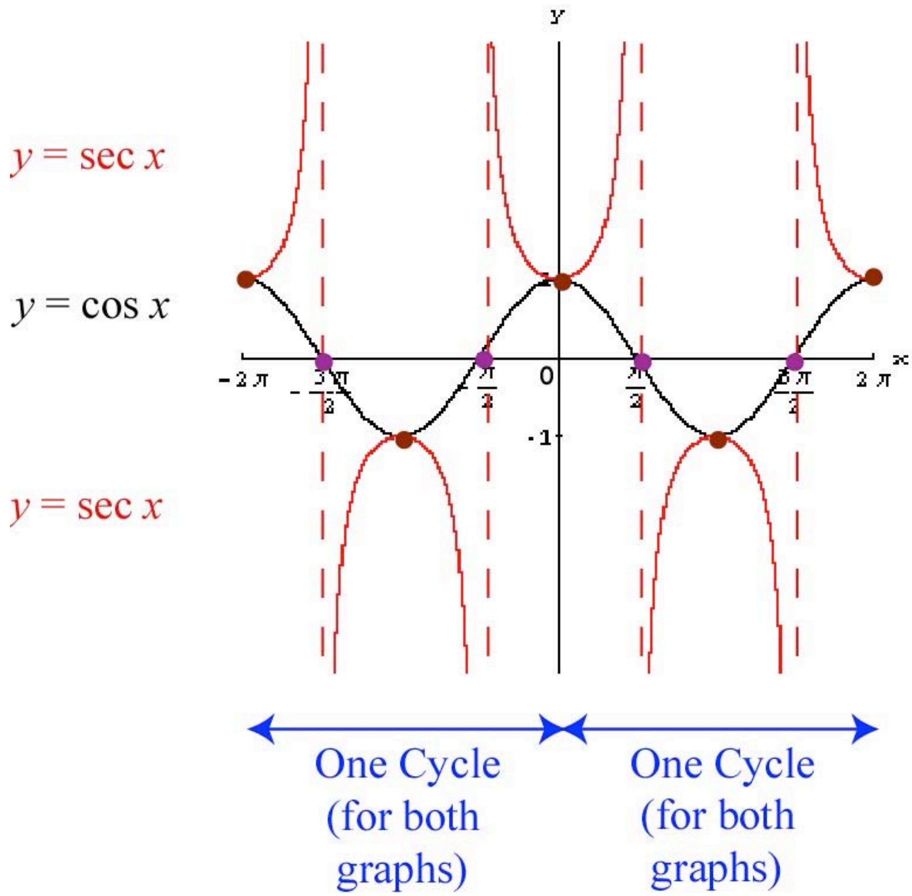
Likewise, the reciprocal of -1 is -1, so: $\csc x = -1 \Leftrightarrow \sin x = -1$. This explains the brown intersection points between the "down-U"s and the $\sin x$ graph.

Because $\sin x$ and $\csc x$ are reciprocal trig functions, we know that, between the asymptotes of the $\csc x$ graph, one increases $\Leftrightarrow$ the other decreases. For example, as $\sin x$ decreases from 1 to $\frac{1}{2}$, $\csc x$ actually increases from 1 to 2. As $\sin x$ decreases from 1 to 0, $\csc x$ increases without bound; it approaches ∞.

Technical Note: Another reason why we get "U"-shaped structures is that each "sine lump" between a pair of consecutive x-intercepts is symmetric about it maximum/minimum point. This leads to the symmetry of the "U" shapes.

The same procedure is applied to the graph of $y = \cos x$ to obtain the graph of $y = \sec x$.

Remember that $\sec x = \dfrac{1}{\cos x}$.



Because reciprocal periodic functions have the same period, the $\csc x$ and $\sec x$ functions have period 2π (because that is the period of the $\sin x$ and $\cos x$ functions).

One cycle of the $\csc x$ graph or the $\sec x$ graph must include exactly one "up-U," exactly one "down-U," and nearby VAs. One of the "U"s may be broken up into pieces at the left and right edges of the cycles; in other words, "wraparounds" are permissible when "counting" the "U"s.

Observe that:

- Reciprocal functions have the same sign (where both are defined), and this property is satisfied by the "up-U, down-U" structure of our $\csc x$ and $\sec x$ graphs.

- Because the $\sin x$ function is odd, the $\csc x$ function is odd, also. The $\csc x$ graph is symmetric about the origin.

- Because the $\cos x$ function is even, the $\sec x$ function is even, also. The $\sec x$ graph is symmetric about the y-axis.

- The graphs of $y = \cot x \left(= \dfrac{\cos x}{\sin x} \right)$ and $y = \csc x \left(= \dfrac{1}{\sin x} \right)$ have the same set of VAs (corresponding to where $\sin x$ is 0) and, therefore, the same domain for their corresponding functions. (See next page.)

- Similarly, the graphs of $y = \tan x \left(= \dfrac{\sin x}{\cos x} \right)$ and $y = \sec x \left(= \dfrac{1}{\cos x} \right)$ have the same VAs (corresponding to where $\cos x$ is 0) and the same domain for their corresponding functions. (See next page.)

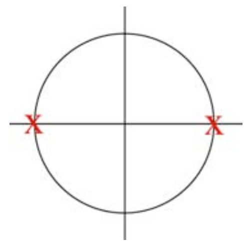
- $\csc x$ and $\sec x$ are never 0 in value. Their graphs have no x-intercepts.

PART H: DOMAIN AND RANGE

Just as for $\cot x$, the domain for the $\csc x$ function is:

$$\left\{ x \,\middle|\, x \neq \pi n \text{ for all integers } n \right\}.$$

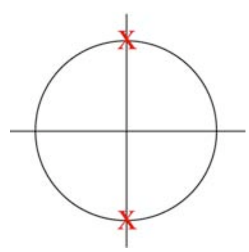
Remember: The zeros of $\sin x$ are excluded from the domain.



Just as for $\tan x$, the domain for the $\sec x$ function is:

$$\left\{ x \,\middle|\, x \neq \frac{\pi}{2} + \pi n \quad \left(\text{i.e., } x \neq \underbrace{(2n+1)}_{\substack{\text{an odd} \\ \text{integer}}} \frac{\pi}{2} \right) \text{ for all integers } n \right\}.$$

Remember: The zeros of $\cos x$ are excluded from the domain.



The range for both the $\csc x$ and the $\sec x$ functions is: $(-\infty, -1] \cup [1, \infty)$

Think: A "high-low" game where you "win" if you get a value that's low (i.e., -1 or lower) or high (i.e., 1 or higher).

Think: We're turning the interval $[-1, 1]$ (the range for the $\sin x$ and $\cos x$ functions) "inside out."

The above range does not contain 0.
The fact that $\csc x$ and $\sec x$ are never 0 in value proves helpful!

PART I: TRANSFORMATIONS

To graph $y = a\csc\left[b(x-p)\right] + d$, first graph $y = a\sin\left[b(x-p)\right] + d$, and apply the "up-U, down-U" trick to it.

To graph $y = a\sec\left[b(x-p)\right] + d$, first graph $y = a\cos\left[b(x-p)\right] + d$, and apply the "up-U, down-U" trick to it.

If there is a vertical shift (i.e., if $d \neq 0$), then the VAs will pass through the "midpoints" (or inflection points) of the corresponding sin or cos graph; those points are no longer x-intercepts, however.

Technical Note: The "up-U, down-U" trick works to get us from $y = a\sin(bx)$, say, to $y = a\csc(bx)$, because the observations we made earlier still hold. If both graphs are translated horizontally by p units and vertically by d units, we see that the same trick works to get us from $y = a\sin\left[b(x-p)\right] + d$ to $y = a\csc\left[b(x-p)\right] + d$. We can envision the "up-U"s and "down-U"s (and nearby VAs) following the sin graph to its new position. Basically, the transformations that affect the sin graph simultaneously affect the csc graph. Make sure that you consider the translations **after** the nonrigid vertical and horizontal transformations, though!

PART J: EXAMPLE

Use the Frame Method to graph one cycle of the graph of
$y = -3\sec\left(4x + \pi\right) + 2$.

Solution

Fortunately (OK OK, I set it up), we did $y = -3\cos\left(4x + \pi\right) + 2$ in the Section 4.5 Notes: 4.50-4.52. The cycle for that graph is in red.

To obtain a cycle for our desired graph (in purple), we:

1) Draw VAs through the midpoints (i.e., inflection points) of the cos graph.

   Warning: In this Example, there is a vertical shift, so do **not** draw VAs through the *x*-intercepts of the cos graph.

2) Draw the "up-U" and "down-U" as described in Notes 4.65. This gives one complete cycle of our sec graph. However, because we are dealing with a sec graph, one of the "U"s will be broken into halves at the edges of the frame. (This is not true for a csc graph, if you use our usual method.)

You may want to lengthen the *y*-axis to make the picture look nicer.