CHAPTER 5: ANALYTIC TRIG

SECTION 5.1: FUNDAMENTAL TRIG IDENTITIES

PART A: WHAT IS AN IDENTITY?

An identity is an equation that is true for all real values of the variable(s) for which all expressions contained within the identity are defined.

For example, $\frac{1}{\left(x^2\right)^3} = \frac{1}{x^6}$ is an identity, because it holds true for all real values of x for which both sides of the equation are defined (i.e., for all real **nonzero** values of x).

If you are given the expression $\frac{1}{\left(x^2\right)^3}$, it may be simplified to form the equivalent expression $\frac{1}{x^6}$.

PART B: LISTS OF FUNDAMENTAL TRIG IDENTITIES

Memorize these in both "directions" (i.e., from left-to-right and from right-to-left).

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sin x = \frac{1}{\csc x}$$

$$\sec x = \frac{1}{\cos x} \qquad \cos x = \frac{1}{\sec x}$$

$$\cot x = \frac{1}{\tan x} \qquad \tan x = \frac{1}{\cot x}$$

Warning: Remember that the reciprocal of $\sin x$ is $\csc x$, not $\sec x$.

Note: We typically treat "0" and "undefined" as reciprocals when we are dealing with trig functions. Your algebra teacher will not want to hear this, though!

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \quad \text{and} \quad \cot x = \frac{\cos x}{\sin x}$$

Technical Note: See Notes 4.10 on why this is consistent with SOH-CAH-TOA.

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \cot^2 x = \csc^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$

See Notes 4.30; see how to derive the last two from the first.

Tip: We know that $\csc x$ and $\sec x$, which have the "Up-U, Down-U" graphs, can never be 0 in value for real x. Their squares are all alone on the right sides of the last two identities. We know that those squares can never be 0 in value; can you see why the same is true of the left sides of the last two identities?

Cofunction Identities

If x is measured in radians, then:

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

We have analogous relationships for tan and cot, and for sec and csc; remember that they are sometimes undefined.

Think: Cofunctions of complementary angles are equal.
See Notes 4.15.

Even/Odd (or Negative Angle) Identities

Among the six basic trig functions, cos (and its reciprocal, sec) are even:

$$\cos(-x) = \cos x$$
$$\sec(-x) = \sec x, \text{ when both sides are defined}$$

However, the other four (sin and csc, tan and cot) are odd:

$$\sin(-x) = -\sin x$$
$$\csc(-x) = -\csc x, \text{ when both sides are defined}$$

$$\tan(-x) = -\tan x, \text{ when both sides are defined}$$
$$\cot(-x) = -\cot x, \text{ when both sides are defined}$$

See Notes 4.29.

Note: When an identity is given, it is typically assumed that it holds in all cases for which all expressions contained within it are defined. However, the Reciprocal Identities, for example, can be helpful even when an expression is undefined.

PART C: SIMPLIFYING TRIG EXPRESSIONS

"Simplifying" can mean different things in different settings. For example, is $\csc x$ "simpler" than $\frac{1}{\sin x}$? Usually, it is, since $\csc x$ is a more "compact" expression, although the latter can be more useful in other settings.

Each step in a simplification process should be a basic arithmetic or algebraic trick, or it should be an application of a Fundamental Trig Identity (for now). As we learn new identities, our arsenal will grow.

PART D: BREAKING THINGS DOWN INTO sin AND cos

If you are dealing with an expression that contains csc, sec, tan, and/or cot, it may be helpful to write some or all of those irritating expressions in terms of sin and cos. The Reciprocal and Quotient Identities can be especially useful for this purpose.

Example

Simplify $\sin^4 x \cot^4 x$.

Solution

By a Quotient Identity, $\cot x = \frac{\cos x}{\sin x}$.

Therefore, $\cot^4 x = (\cot x)^4 = \left(\frac{\cos x}{\sin x}\right)^4 = \frac{(\cos x)^4}{(\sin x)^4} = \frac{\cos^4 x}{\sin^4 x}$.

We usually ignore the middle expressions; they are "automatic" to us.

$$\begin{aligned}\sin^4 x \cot^4 x &= \cancel{\sin^4 x}\left(\frac{\cos^4 x}{\cancel{\sin^4 x}}\right)\\ &= \mathbf{\cos^4 x}\end{aligned}$$

Warning: The given expression simplifies to $\cos^4 x$ **if** x is in the domain of the expression. Don't just assume that the natural (or implied) domain of $\cos^4 x$ is the domain of $\sin^4 x \cot^4 x$. Unless otherwise specified, when we simplify trig expressions, we usually don't state domain restrictions (here, on $\cos^4 x$), although you could argue that that would be more "mathematically proper."

PART E: FACTORING

The old algebraic methodologies still apply.

Example

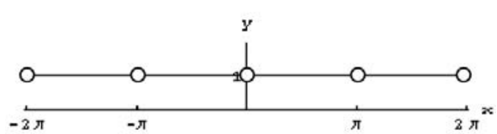
Simplify $\sin^2 x + \sin^2 x \cot^2 x$.

Solution

Factor out the GCF, $\sin^2 x$.

$$\begin{aligned}
\sin^2 x + \sin^2 x \cot^2 x &= \left(\sin^2 x\right)\left(1 + \cot^2 x\right) \\
&= \left(\sin^2 x\right)\left(\csc^2 x\right) && \left(\text{by a Pythagorean Identity}\right) \\
&= \left(\cancel{\sin^2 x}\right)\left(\frac{1}{\cancel{\sin^2 x}}\right) && \left(\text{by a Reciprocal Identity}\right) \\
&= \mathbf{1}
\end{aligned}$$

Warning: The given expression simplifies to 1 **if** x is in the domain of the expression. The graph of $y = \sin^2 x + \sin^2 x \cot^2 x$ would resemble the graph of $y = 1$, except that there would be holes (technically, "removable discontinuities") at the values of x that are not in the domain:



Example

Simplify $1-2\cos^2 x+\cos^4 x$.

Solution

We can either factor directly, or we can let $u=\cos x$, rewrite the expression, and then factor.

$$\begin{aligned}
1-2\cos^2 x+\cos^4 x &= 1-2u^2+u^4 \\
&\qquad \big(\text{This is in Quadratic Form; in fact, it is a PST.}\big) \\
&= \left(1-u^2\right)^2 \\
&\qquad \big(\text{Now, go back to } x.\big) \\
&= \left(1-\cos^2 x\right)^2 \\
&\qquad \big(\text{Now, use the basic Pythagorean Identity.}\big) \\
&= \left(\sin^2 x\right)^2 \\
&= \mathbf{\sin^4 x}
\end{aligned}$$

PART F: ADDING AND SUBTRACTING FRACTIONS

Example (#62 in Larson)

Simplify $\dfrac{1}{\sec x + 1} - \dfrac{1}{\sec x - 1}$.

Solution

The LCD here is the product of the denominators, $(\sec x + 1)(\sec x - 1)$.

Build up both of the given fractions to obtain this common denominator.

$$\frac{1}{\sec x + 1} - \frac{1}{\sec x - 1} = \frac{1}{(\sec x + 1)} \cdot \frac{(\sec x - 1)}{(\sec x - 1)} - \frac{1}{(\sec x - 1)} \cdot \frac{(\sec x + 1)}{(\sec x + 1)}$$

Grouping symbols help make things easier to read.
Unite the right side into a single fraction.

$$= \frac{(\sec x - 1) - (\sec x + 1)}{(\sec x + 1)(\sec x - 1)}$$

The ( ) around $\sec x + 1$ are essential.

$$= \frac{\cancel{\sec x} - 1 \cancel{- \sec x} - 1}{(\sec x + 1)(\sec x - 1)}$$

$$= \frac{-2}{\sec^2 x - 1}$$

We simplified the denominator by using the following multiplication rule from algebra: $(A + B)(A - B) = A^2 - B^2$.

The $\sec^2 x$ and the -1 (or $+1$) term should alert you to the possible application of a Pythagorean Identity. In fact,

$$\begin{aligned} 1+\tan^2 x &= \sec^2 x \\ \tan^2 x &= \sec^2 x - 1 \end{aligned}$$

We now have:

$$\begin{aligned} &= \frac{-2}{\tan^2 x} \\ &= -2\left(\frac{1}{\tan^2 x}\right) \\ &= \mathbf{-2\cot^2 x} \end{aligned}$$

You can think of the $\tan^2 x$ as "jumping up" as $\cot^2 x$.

PART G: TRIG SUBSTITUTIONS

In Calculus: This is a key technique of integration. You will see this in Calculus II: Math 151 at Mesa.

Example

Use the trig substitution $x = 4\sin\theta$ to write the algebraic expression $\sqrt{16 - x^2}$ as a trig function of θ, where θ is acute.

Solution

$$\begin{aligned}\sqrt{16 - x^2} &= \sqrt{16 - (4\sin\theta)^2} \\ &= \sqrt{16 - 16\sin^2\theta} \\ &= \sqrt{16(1 - \sin^2\theta)} \\ &= 4\sqrt{(1 - \sin^2\theta)}\end{aligned}$$

Remember the basic Pythagorean Identity:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \sin^2\theta$$

$$= 4\sqrt{\cos^2\theta}$$

Remember that $\sqrt{(blah)^2} = |blah|$.

$$= 4|\cos\theta|$$

Since θ is acute, $\cos\theta > 0$.

$$= \mathbf{4\cos\boldsymbol{\theta}}$$

SECTION 5.2: VERIFYING TRIG IDENTITIES

PART A: EXAMPLE; STRATEGIES AND "SHOWING WORK"

One Example; Three Solutions

Verify the identity: $\dfrac{\csc\theta + \cot\theta}{\tan\theta + \sin\theta} = \cot\theta\csc\theta$.

Strategies and "Showing Work"

To verify an identity like this one, use the Fundamental Identities and algebraic techniques to simplify the side with the more complicated expression step-by-step until we end up with the expression on the other side. You may think of this as a simplification problem where the "answer" is given to you. The "answer" may be thought of as the top of a jigsaw puzzle box, the **TARGET** that you are aiming for. This is a strategy to keep in mind as you perform your manipulations.

Warning: Instructors generally want their students to "show work." In the simplification or verification process, you should probably write a new expression every time you apply a Fundamental Identity and every time you execute a "major" algebraic step (this may be a matter of judgment). If you are applying Fundamental Identities to different pieces of an expression, you may be able to apply them simultaneously in one step, provided that it is clear how and where they are being applied.

In this class, you will typically not be required to write the names of the various identity types you are using, but they will often be written in solutions for your reference.

The left-hand side (LHS) seems more complicated in this example, so we will operate on it until we obtain the right-hand side (RHS). In principle, you could begin with the RHS, or you could even work on both sides simultaneously until you "meet" somewhere in the middle. Some instructors may object to the latter method, however, perhaps because it may seem "sloppy." Even then, it could still inspire a more linear approach.

There are often different "good" approaches to problems such as these. You don't necessarily have to agree with your book's solutions manual!

Solution (Method 1)

(This may be the least efficient approach, though.)

Remember, we want to verify: $\dfrac{\csc\theta + \cot\theta}{\tan\theta + \sin\theta} = \cot\theta\csc\theta$

$$\frac{\csc\theta + \cot\theta}{\tan\theta + \sin\theta} = \frac{\dfrac{1}{\sin\theta} + \dfrac{\cos\theta}{\sin\theta}}{\dfrac{\sin\theta}{\cos\theta} + \sin\theta} \quad \big(\text{Reciprocal and Quotient Identities}\big)$$

We are breaking things down into expressions involving $\sin\theta$ and $\cos\theta$. They are like common currencies.

We can begin by simplifying the numerator ("N") and the denominator ("D") individually.

Tip: It may help to express $\sin\theta$ as $\dfrac{\sin\theta}{1}$.

$$= \frac{\dfrac{1+\cos\theta}{\sin\theta}}{\dfrac{\sin\theta}{\cos\theta} + \dfrac{\sin\theta}{1}}$$

← We already had a common denominator.

← $\cos\theta$ will be our common denominator.

$$= \frac{\dfrac{1+\cos\theta}{\sin\theta}}{\dfrac{\sin\theta}{\cos\theta} + \dfrac{\sin\theta}{1} \cdot \dfrac{\cos\theta}{\cos\theta}}$$

← We "build up" a fraction so that we have a common denominator.

$$= \frac{\dfrac{1+\cos\theta}{\sin\theta}}{\dfrac{\sin\theta}{\cos\theta} + \dfrac{\sin\theta\cos\theta}{\cos\theta}}$$

← We now have a common denominator. Add the fractions.

$$= \frac{\dfrac{1+\cos\theta}{\sin\theta}}{\dfrac{\sin\theta+\sin\theta\cos\theta}{\cos\theta}}$$

When we divide by a fraction, we are really multiplying by its reciprocal.

$$= \frac{1+\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta+\sin\theta\cos\theta}$$

We can factor the denominator of the second fraction, and we can perform a cancellation.

Tip: It often helps to consider easier factoring problems from Algebra I. If you have difficulty factoring $\sin\theta+\sin\theta\cos\theta$, try factoring $x+xy$. If you can see that $x+xy=(x)(1+y)$, then you should be able to see that $\sin\theta+\sin\theta\cos\theta=(\sin\theta)(1+\cos\theta)$

Tip: Grouping symbols can be very helpful when used appropriately, even when books don't use them as often!

$$= \frac{\cancelto{1}{1+\cos\theta}}{\sin\theta}\cdot\frac{\cos\theta}{(\sin\theta)\,\cancelto{1}{(1+\cos\theta)}}$$

$$= \frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}$$

Keep the TARGET in mind. We are very close!
We will apply the Reciprocal and Quotient Identities to condense our expressions. (At the beginning, we used them to expand.)

$$= \csc\theta\cot\theta$$

Don't forget that multiplication of real quantities is commutative.
This strategy is sometimes overlooked by students!

Warning: Your final expression must look **exactly** like the TARGET.

$$= \quad \cot\theta\csc\theta$$

Solution (Method 2)

Remember, we want to verify: $\dfrac{\csc\theta+\cot\theta}{\tan\theta+\sin\theta}=\cot\theta\csc\theta$

$$\frac{\csc\theta+\cot\theta}{\tan\theta+\sin\theta} = \frac{\dfrac{1}{\sin\theta}+\dfrac{\cos\theta}{\sin\theta}}{\dfrac{\sin\theta}{\cos\theta}+\sin\theta} \quad \text{(Reciprocal and Quotient Identities)}$$

We will multiply the N and the D by the least common denominator (LCD) of the overall fraction. The LCD is $\sin\theta\cos\theta$.

Warning: People often fail to properly apply the Distributive Property, so grouping symbols may help here! Also, it may help to express $\sin\theta$ as $\dfrac{\sin\theta}{1}$.

$$= \frac{\left(\dfrac{1}{\sin\theta}+\dfrac{\cos\theta}{\sin\theta}\right)}{\left(\dfrac{\sin\theta}{\cos\theta}+\dfrac{\sin\theta}{1}\right)}\cdot\frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}$$

Warning: Instead of crossing things out (which is very risky if you have not yet applied the Distributive Property), you may want to cover up other expressions as you multiply things together. When in doubt, carefully write the step where you apply the Distributive Property, and then perform cancellations:

$$= \frac{\left(\frac{1}{\cancel{\sin\theta}} \cdot \cancel{\sin\theta}\cos\theta\right) + \left(\frac{\cos\theta}{\cancel{\sin\theta}} \cdot \cancel{\sin\theta}\cos\theta\right)}{\left(\frac{\sin\theta}{\cancel{\cos\theta}} \cdot \sin\theta\cancel{\cos\theta}\right) + \left(\frac{\sin\theta}{1} \cdot \sin\theta\cos\theta\right)}$$

Warning: "Wishful thinking" tends to creep into these problems involving cancellations in compound fractions. Remember that $\cos\theta$ multiplied by itself is $\cos^2\theta$. Many people incorrectly attempt to cancel and write "1."

$$= \frac{\cos\theta + \cos^2\theta}{\sin^2\theta + \sin^2\theta\cos\theta}$$

Factor the N and the D, and cancel common factors.

$$= \frac{(\cos\theta)\,\cancel{(1+\cos\theta)}^{1}}{(\sin^2\theta)\,\cancel{(1+\cos\theta)}^{1}} \qquad \frac{\leftarrow \text{Think: } x + x^2 = (x)(1+x)}{\leftarrow \text{Think: } y^2 + y^2x = (y^2)(1+x)}$$

Keep the TARGET in mind. We may employ a "peeling" strategy. Remember that $\sin^2\theta = (\sin\theta)(\sin\theta)$, just as $y^2 = (y)(y)$.

$$= \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}$$

Finally, apply the Reciprocal and Quotient Identities to condense.

$$= \cot\theta\csc\theta$$

Solution (Method 3)

Remember, we want to verify: $\dfrac{\csc\theta+\cot\theta}{\tan\theta+\sin\theta}=\cot\theta\csc\theta$

$$\frac{\csc\theta+\cot\theta}{\tan\theta+\sin\theta} = \frac{\dfrac{1}{\sin\theta}+\dfrac{1}{\tan\theta}}{\tan\theta+\sin\theta} \quad \left(\text{Reciprocal Identities only}\right)$$

We could multiply the N and the D by the LCD, $\sin\theta\tan\theta$.

It turns out to be easier to first express the N as a "simple" fraction. Our LCD in the N is, again, $\sin\theta\tan\theta$.

$$= \frac{\dfrac{1}{\sin\theta}\cdot\dfrac{\tan\theta}{\tan\theta}+\dfrac{1}{\tan\theta}\cdot\dfrac{\sin\theta}{\sin\theta}}{\tan\theta+\sin\theta} \quad \leftarrow \text{Build up both fractions in the N.}$$

$$= \frac{\dfrac{\tan\theta+\sin\theta}{\sin\theta\tan\theta}}{\tan\theta+\sin\theta}$$

We may cancel the D and the "N of the N."

$$= \frac{\dfrac{\cancel{\tan\theta+\sin\theta}^{\,1}}{\sin\theta\tan\theta}}{{}_1\cancel{\tan\theta+\sin\theta}}$$

$$= \frac{1}{\sin\theta\tan\theta}$$

Let's "peel apart" (actually, factor) the one fraction as a product of two fractions.

$$= \quad \frac{1}{\sin\theta} \cdot \frac{1}{\tan\theta}$$

Now, apply the Reciprocal Identities to condense.

$$= \quad \csc\theta \cot\theta$$

$$= \quad \cot\theta \csc\theta$$

PART B: "TRIG CONJUGATES"

When we rationalize the D in $\frac{1}{\sqrt{3}+\sqrt{5}}$, we multiply the N and the D by the conjugate of the D, $\sqrt{3}-\sqrt{5}$. This led to squarings that eliminated radicals in the D.

Similarly, when we rationalize (Think "real"-ize) the D in $\frac{1}{3+2i}$, we multiply the N and the D by the complex conjugate of the D, $3-2i$. This led to squarings that eliminated i in the D.

Similarly, we can use "trig conjugates" (not a standard term) to help us simplify, and verify identities involving, fractional expressions, particularly when the resulting squarings lead to expressions that we can attack using the Pythagorean Identities.

Example

Verify the identity: $\dfrac{1}{\sec\alpha - \tan\alpha} = \sec\alpha + \tan\alpha$.

Solution

Begin with the LHS, and multiply the N and the D of the LHS by the trig conjugate of the D, $\sec\alpha + \tan\alpha$.

Warning: Write the LHS (exactly) as your first expression, even if your first manipulation seems straightforward.

$$\frac{1}{\sec\alpha - \tan\alpha} = \frac{1}{\left(\sec\alpha - \tan\alpha\right)} \cdot \frac{\left(\sec\alpha + \tan\alpha\right)}{\left(\sec\alpha + \tan\alpha\right)}$$

For the new D, we will use the algebra rule: $\left(A - B\right)\left(A + B\right) = A^2 - B^2$

$$= \frac{\sec\alpha + \tan\alpha}{\sec^2\alpha - \tan^2\alpha}$$

The Pythagorean Identities may or may not help us now. It turns out that they do. Observe that the Pythagorean Identity $\tan^2\alpha + 1 = \sec^2\alpha$ may be rewritten as: $1 = \sec^2\alpha - \tan^2\alpha$.

$$= \frac{\sec\alpha + \tan\alpha}{1}$$

$$= \sec\alpha + \tan\alpha$$

Controversial Solution

Remember, we want to verify: $\dfrac{1}{\sec\alpha - \tan\alpha} = \sec\alpha + \tan\alpha$

We will treat the proposed identity as an equation. We will write a sequence of equivalent equations until we obtain an identity that we know to be true.

$$\frac{1}{\sec\alpha - \tan\alpha} = \sec\alpha + \tan\alpha$$

We will multiply both sides by $\sec\alpha - \tan\alpha$. For the purposes of verifying the identity, we may assume that both $\sec\alpha$ and $\tan\alpha$ are defined.
We may also assume that $\sec\alpha - \tan\alpha \neq 0$; otherwise, the LHS would be undefined.

$$\begin{aligned} 1 &= (\sec\alpha + \tan\alpha)(\sec\alpha - \tan\alpha) \\ 1 &= \sec^2\alpha - \tan^2\alpha \\ \tan^2\alpha + 1 &= \sec^2\alpha \end{aligned}$$

The last equation is a known Pythagorean Identity.

Although the author is not particularly bothered by this method, it does bother many other instructors, and it will be discouraged. Always follow your instructor's cue, and ask him/her about "good form and procedure" if you are unsure.

Although we tend to disregard domain issues when doing these kinds of problems, we must be very careful about potentially multiplying or dividing both sides of an equation by a quantity that is 0 or undefined. This may be a key reason for the controversy surrounding this method. Addition and subtraction tend to be less controversial operations, as are multiplication and division by nonzero constants. When in doubt, keep domain issues in mind!

PART C: A SUMMARY OF STRATEGIES

This list is not intended to be comprehensive, but it is a nice toolbox!

1) Longer $\rightarrow$ Shorter

We usually want to start with the "longer" (i.e., the more complicated) side and try to get to the "shorter" side. You could tinker with the "shorter" side as necessary as you strategize, or you could re-express it outright.

2) TARGET

Keep the TARGET (the expression you're aiming for) in mind.
This can influence strategies.

3) Fundamental Identities

Keep all the Fundamental Identities in mind.

4) LCDs

Use LCDs for adding and subtracting fractions and for simplifying compound fractions.

5) Trig Conjugates

Consider using trig conjugates in conjunction with Pythagorean Identities, especially when pairs of trig functions found in Pythagorean Identities (sin and cos, tan and sec, cot and csc), 1, and/or -1 are involved.

"DECOMPOSITION" STRATEGIES

6) Go to sin and cos

Consider using the Reciprocal and Quotient Identities to break everything down into expressions involving sin and cos.

7) Factoring

Cancellations may result. Pythagorean Identities may be useful.

8) Splitting a Fraction (Multiplication and Division): "Peeling"

This is like a basic form of factoring. For example, see Notes 5.14:

$$\frac{\cos\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}$$

Keep the TARGET in mind.

9) Splitting a Fraction (Addition and Subtraction): Splitting a Fraction through the N (Numerator)

For example, you may use the template: $\dfrac{x+y}{z} = \dfrac{x}{z} + \dfrac{y}{z}$

Warning: Remember that we cannot split through the D (Denominator) in a similar fashion.

OTHER STRATEGIES

10) Looking at a similar problem in Algebra I, which we did in 9), may help you. Also, remember how to manipulate fractions back in Arithmetic. For example, when you divide by a fraction, you are really multiplying by its reciprocal.

11) In general, be neat, and show work, especially when you are applying the Fundamental Identities and algebraic strategies.

SECTION 5.3: SOLVING TRIG EQUATIONS

PART A: BASIC EQUATIONS IN sin, cos, csc, OR sec (LINEAR FORMS)

Example

Solve: $5\cos x-\sqrt{2}=3\cos x$

(It is assumed that you are to give **all** real solutions and to give them in **exact** form – no approximations – unless otherwise specified.)

Conditional Equations

This is an example of a conditional equation. It is true (i.e., the left side equals the right side) for some real values of x but not for others. In other words, the truth of the equation is conditional, depending on the particular real value that x takes on. You should be used to solving conditional equations in your Algebra courses.

This is different from an identity, which holds true for **all** real values of x (for instance) for which all expressions involved are defined as real quantities. An identity may be thought of as an equation that has as its solution set the intersection (overlap) of the domains of the expressions involved.

Solution

First, solve for $\cos x$. This process is no different from solving the linear equation $5u-\sqrt{2}=3u$ for u. In fact, you could employ the substitution $u=\cos x$ and do exactly that.

$$\begin{aligned} 5\cos x-\sqrt{2}&=3\cos x\\ 2\cos x&=\sqrt{2}\\ \cos x&=\frac{\sqrt{2}}{2}\end{aligned}$$

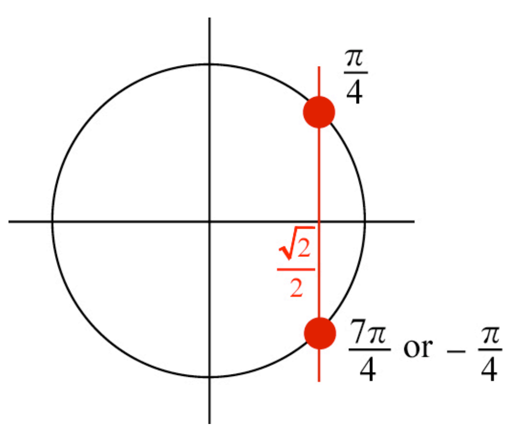
We want to find **all** angles whose cos value is $\frac{\sqrt{2}}{2}$. We will use radian measure, which corresponds to "real number" solutions for x.

Second, because $\cos x$ has period 2π, we will first find solutions in the interval $\left[0, 2\pi\right)$. Later, we will find all of their coterminal "twin" angles. If you are more comfortable with "slightly negative" Quadrant IV angles such as $-\frac{\pi}{4}$ than angles such as $\frac{7\pi}{4}$, then you may want to look in the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$, instead.

Is there an "easy" angle x whose cos value is $\frac{\sqrt{2}}{2}$?

Yes, namely $\frac{\pi}{4}$, which is $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$.

Look at the Unit Circle. Look at the point corresponding to the $\frac{\pi}{4}$ angle. It turns out that there is another point on the Unit Circle that has the same horizontal (or what we used to call "x") coordinate, $\frac{\sqrt{2}}{2}$, so we must look for another angle with that same cos value of $\frac{\sqrt{2}}{2}$. We know that this point lies in Quadrant IV, because, aside from Quadrant I, it is the only other Quadrant in which cos is positive in value.



If you are considering the interval $\left[0, 2\pi\right)$, then this other point corresponds to the $\dfrac{7\pi}{4}$ angle. For the interval $\left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$, it corresponds to $-\dfrac{\pi}{4}$.

> Note: You may have realized that $-\dfrac{\pi}{4}$ was another solution, because we know $\dfrac{\pi}{4}$ is a solution, and the $\cos x$ function is even.

Third, we find all angles coterminal with the two solutions we have already found. There may be different "good" ways of writing the solution set (the general solution) for the equation:

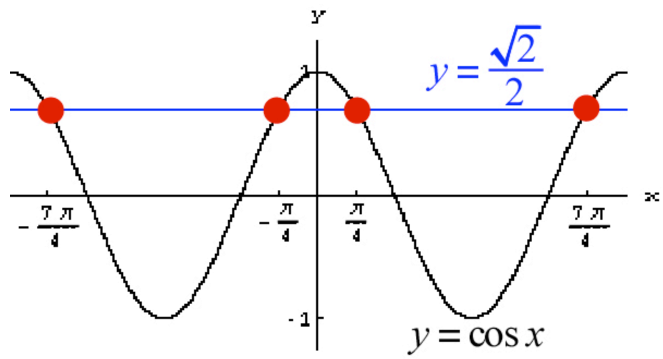
One form: $\left\{ x \,\middle|\, x = \dfrac{\pi}{4} + 2\pi n, \;\text{ or }\; x = \dfrac{7\pi}{4} + 2\pi n \;\; \left(n \text{ integer}\right) \right\}$

Another form: $\left\{ x \,\middle|\, x = \pm\dfrac{\pi}{4} + 2\pi n \;\; \left(n \text{ integer}\right) \right\}$

> Note: The $\pm$ symbol in this context indicates that we are bringing together the values from the "+" case and the values from the "$-$" case. We do not pick one sign over the other; the $\pm$ symbol does not indicate a "choice" or incomplete knowledge.

Fourth, we check to see if there are any nice symmetries or periodicities we can exploit so that we may "simplify" our solution set. At this stage, people may decide to switch from the first form of the solution set (above) to the second.

Note: Graphically, the solutions are the x-coordinates of the red points below where the graph of $y = \cos x$ and the graph of $y = \frac{\sqrt{2}}{2}$ (the blue horizontal line) intersect. Observe that the $\cos x$ function is even.



Note: If you had been asked to only find solutions in the interval $\left[0, 2\pi\right)$, then you would have given $\left\{\frac{\pi}{4}, \frac{7\pi}{4}\right\}$ as your solution set.

Follow-Up Example

Solve: $\sec x - \sqrt{2} = 0$

Solution Sketch

$$\sec x - \sqrt{2} = 0$$

Isolate $\sec x$ on one side.

$$\sec x = \sqrt{2}$$

We can "take the reciprocal of both sides." Remember that we informally treat 0 and "undefined" as reciprocals in trig.

$$\cos x = \frac{1}{\sqrt{2}} \quad \left(\text{or } \frac{\sqrt{2}}{2} \right)$$

We then proceed as in the previous Example….

Example

Solve: $\sin x = 2$

Solution

This equation has no solution, because 2 is outside the range of the $\sin x$ function, $[-1, 1]$. There is no angle with a sin value of 2.

The solution set is the empty set, or null set, denoted $\varnothing$.

Technical Note: In a Complex Variables class, you may see that $\sin x = 2$ actually does have solutions in **C**.

PART B: BASIC EQUATIONS IN tan OR cot (LINEAR FORMS)

Remember that the $\tan x$ and $\cot x$ functions differ from the other four basic trig functions in that they have a period of π, not 2π.

Example

Solve: $\tan x = -\sqrt{3}$

Solution

First, observe that $\tan x$ has already been isolated on one side.

Second, because $\tan x$ has period π, we will first find solutions in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This interval has the advantage of being the range of the arctan $\left(\text{i.e., } \tan^{-1}\right)$ function, which will help us in "calculator problems." However, instead of Quadrants I and IV, some people focus on Quadrants I and II.

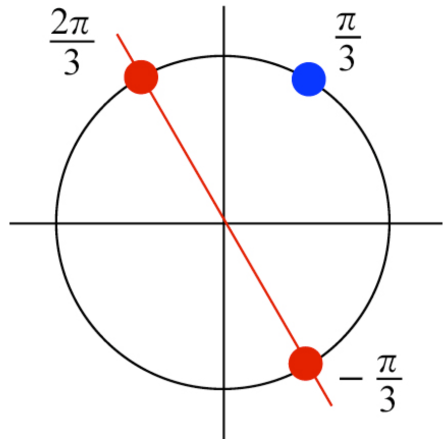
In any case, it will help to find the **reference angle** for our solutions. Here, it is $\frac{\pi}{3}$, because it is acute and $\tan\frac{\pi}{3} = \sqrt{3}$. (Think: High slope.) Reference angles are always acute, and they have only positive basic trig values. (Quadrantal angles are a different story.)

We actually want a **brother angle** whose tan value is $-\sqrt{3}$.

Remember that $\tan x$ is negative in value in Quadrants II and IV.

The desired brother in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is $-\frac{\pi}{3}$, an angle in Quadrant IV. (Remember that the $\tan x$ function is odd.)

If, instead, you want a brother in Quadrant II, then you could use $\frac{2\pi}{3}$.



Third, we find all coterminal angles, and
Fourth, we check to see if there are any nice symmetries or periodicities.

The figure above implies the following form for the solution set:

$$\left\{ x \,\middle|\, x = -\frac{\pi}{3} + 2\pi n, \ \text{ or } \ x = \frac{2\pi}{3} + 2\pi n \quad \left(n \text{ integer}\right) \right\}$$

However, instructors may object to this form as "unsimplified," because we can still easily exploit the periodicity of the solutions. Exploiting other symmetries (particularly about the vertical axis in the Unit Circle picture) is typically considered to be not as critical.

For another form, you may begin with either the $-\frac{\pi}{3}$ or the $\frac{2\pi}{3}$ angle (or any of their coterminal "twin" angles, for that matter), and use the fact that the period of the $\tan x$ function is π. (Think: "Half revolutions" about the Unit Circle.)

One form: $\left\{ x \,\middle|\, x = -\frac{\pi}{3} + \pi n \quad \left(n \text{ integer}\right) \right\}$

Another form: $\left\{ x \,\middle|\, x = \frac{2\pi}{3} + \pi n \quad \left(n \text{ integer}\right) \right\}$

PART C: THE SQUARE ROOT METHOD (FOR QUADRATIC FORMS)

We often grab solutions from all four Quadrants when we apply this method, whichever of the six basic trig functions is primarily involved.

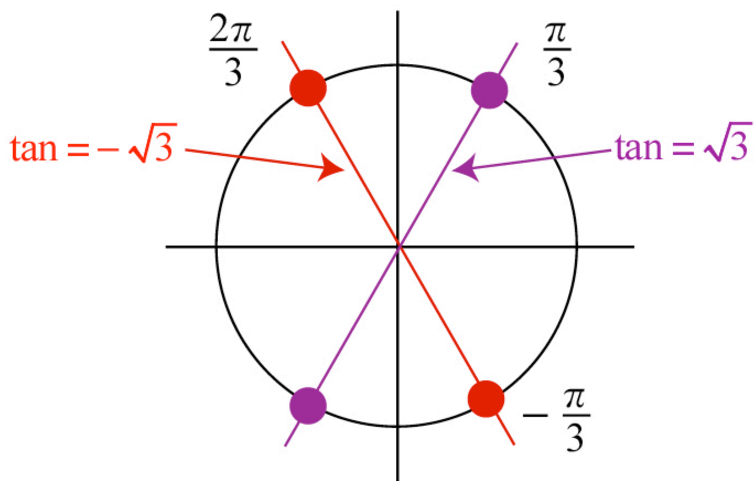
Follow-Up Example

Solve: $\tan^2 x - 3 = 0$

Solution

$$\begin{aligned} \tan^2 x - 3 &= 0 \\ \tan^2 x &= 3 && \text{(The "square" is now isolated.)} \\ \tan x &= \pm\sqrt{3} && \text{(by the Square Root Method)} \end{aligned}$$

Here, we want $\frac{\pi}{3}$ and **all** of its brothers.



Forms for the solution set:

One form: $\left\{ x \;\middle|\; \underbrace{x = \frac{\pi}{3} + \pi n}_{\text{Think: Purple points}}, \text{ or } \underbrace{x = \frac{2\pi}{3} + \pi n}_{\text{Think: Red points}} \quad (n \text{ integer}) \right\}$

Another form: $\left\{ x \;\middle|\; x = \pm\frac{\pi}{3} + \pi n \quad (n \text{ integer}) \right\}$

PART D: FACTORING

Example

Solve: $2\sin^3 x + \sin x = 3\sin^2 x$

Solution

$$2\sin^3 x + \sin x = 3\sin^2 x$$

Warning: Do **not** divide both sides by $\sin x$, because it is "illegal" to divide both sides of an equation by 0, and $\sin x$ could be 0 in value. (We are more careful about these kinds of issues than in the simplification and verification problems of previous Sections.) Instead, we should use Factoring. In this Example, if were to divide both sides by $\sin x$, we would lose solutions x for which $\sin x = 0$. This could, however, be remedied by consideration of the $\sin x = 0$ case as a "Special Case." This technique is often employed in Differential Equations.

The substitution $u = \sin x$ may be helpful here.

$$2u^3 + u = 3u^2$$

Rewrite this polynomial equation in Standard Form
(i.e., with descending powers on one side and 0 isolated on the other).

$$2u^3 - 3u^2 + u = 0$$

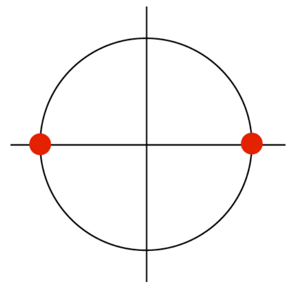
Now, factor. Begin by factoring out the GCF on the left.

$$u\left(2u^2 - 3u + 1\right) = 0$$
$$u\left(2u - 1\right)\left(u - 1\right) = 0$$

Use the Zero Factor Property (ZFP).
Set each factor on the left equal to 0 and solve for u.
Replace u with $\sin x$ and solve each resulting equation.

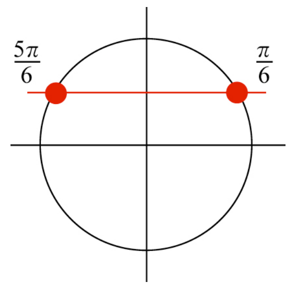
First factor

$$\begin{aligned} u &= 0 \\ \sin x &= 0 \\ x &= \pi n \quad (n \text{ integer}) \end{aligned}$$



Second factor

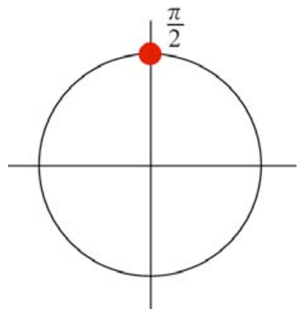
$$\begin{aligned} 2u - 1 &= 0 \\ u &= \frac{1}{2} \\ \sin x &= \frac{1}{2} \\ x &= \frac{\pi}{6} + 2\pi n, \text{ or } x = \frac{5\pi}{6} + 2\pi n \quad (n \text{ integer}) \end{aligned}$$



Note: The more efficient form $\frac{\pi}{2} \pm \frac{\pi}{3} + 2\pi n$ $(n \text{ integer})$ may be overkill!

Third factor

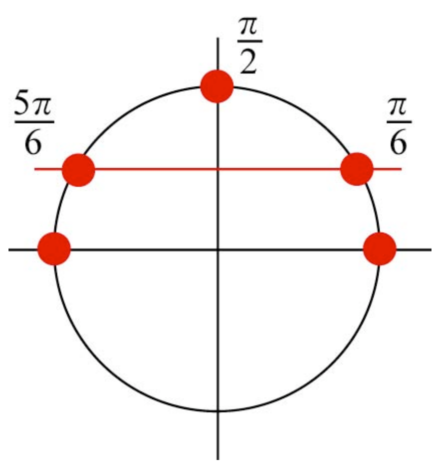
$$\begin{aligned} u-1&=0 \\ u&=1 \\ \sin x&=1 \\ x&=\frac{\pi}{2}+2\pi n \quad (n \text{ integer}) \end{aligned}$$



Solution set:

$$\left\{ x \,\middle|\, x=\pi n,\ \ x=\frac{\pi}{6}+2\pi n,\ \ x=\frac{5\pi}{6}+2\pi n,\ \text{ or }\ x=\frac{\pi}{2}+2\pi n \quad (n \text{ integer}) \right\}$$

When gathering groups of solutions, you should check to see if there are any more nice symmetries or periodicities you could exploit. No easy ones are apparent here:



PART E: PYTHAGOREAN IDENTITIES

Follow-Up Example

Solve: $2\sin^3 x + \sin x + 3\cos^2 x = 3$

Solution

The $\cos^2 x$ seems like the odd man out, but we can make it look more like the powers of $\sin x$ in the equation. We often prefer conformity.

$$
\begin{aligned}
2\sin^3 x + \sin x + 3\cos^2 x &= 3 \\
2\sin^3 x + \sin x + 3\left(1 - \sin^2 x\right) &= 3 \quad \text{(by a Pythagorean Identity)} \\
2\sin^3 x + \sin x \cancel{+3} - 3\sin^2 x &= \cancel{3}^{\,0} \\
2\sin^3 x + \sin x - 3\sin^2 x &= 0 \\
2u^3 + u - 3u^2 &= 0 \quad \left(\text{Let } u = \sin x.\right) \\
2u^3 - 3u^2 + u &= 0
\end{aligned}
$$

We then proceed as in the previous Example.

PART F: EQUATIONS WITH "MULTIPLE ANGLES"

Example

Solve: $2\sin(4x) = -\sqrt{3}$

Solution

$$2\sin(4x) = -\sqrt{3}$$

Isolate the sin expression.

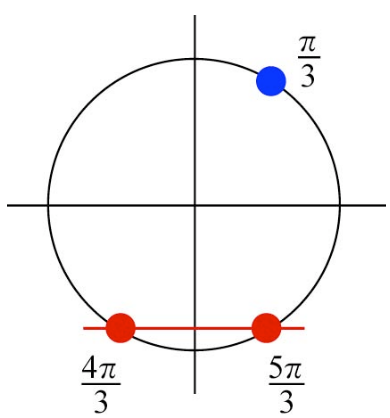
$$\sin\underbrace{(4x)}_{=\theta} = -\frac{\sqrt{3}}{2}$$

Substitution: Let $\theta = 4x$.

$$\sin\theta = -\frac{\sqrt{3}}{2}$$

We will now solve this equation for θ.

Observe that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, so $\frac{\pi}{3}$ will be the reference angle for our solutions for θ. Since $-\frac{\sqrt{3}}{2}$ is a negative sin value, we want brothers of $\frac{\pi}{3}$ in Quadrants III and IV. For multiple angle problems, we may prefer positive brothers, as we will see in our Follow-Up Example.



Our solutions for θ are:

$$\theta = \frac{4\pi}{3} + 2\pi n, \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2\pi n \quad (n \text{ integer})$$

Note: Our solution set will contain the conjunction "or" as an inclusive (not exclusive or limiting) device to gather solutions together. Although the conjunction "and" may have seemed more appropriate in the above phrasing, we will stick with "or" throughout.

From this point on, it is a matter of Algebra.

To find our solutions for x, replace θ with $4x$, and solve for x.

$$4x = \frac{4\pi}{3} + 2\pi n, \qquad \text{or} \qquad 4x = \frac{5\pi}{3} + 2\pi n \qquad (n \text{ integer})$$

$$x = \frac{\frac{4\pi}{3}}{4} + \frac{2\pi}{4} n, \quad \text{or} \qquad x = \frac{\frac{5\pi}{3}}{4} + \frac{2\pi}{4} n \quad (n \text{ integer})$$

$$x = \frac{\pi}{3} + \frac{\pi}{2} n, \qquad \text{or} \qquad x = \frac{5\pi}{12} + \frac{\pi}{2} n \qquad (n \text{ integer})$$

Solution set: $\left\{ x \,\middle|\, x = \frac{\pi}{3} + \frac{\pi}{2} n, \quad \text{or} \quad x = \frac{5\pi}{12} + \frac{\pi}{2} n \quad (n \text{ integer}) \right\}$

Note: The picture for x is considerably more complicated than the picture for θ on the last page. To avoid confusion, you might not want to even think about the picture for x. If you're curious, though, look at the picture at the end of our Follow-Up Example ….

Follow-Up Example

Find all solutions of the equation $2\sin(4x) = -\sqrt{3}$ in the interval $[0, 2\pi)$.
(Surprisingly, this tends to be the much more involved problem!)

Solution

We found the general solution (consisting of two "groups" of solutions) in the previous Example:

$$\left\{ x \,\middle|\, \underbrace{x = \frac{\pi}{3} + \frac{\pi}{2}n}_{\text{Group 1}}, \quad \text{or} \quad \underbrace{x = \frac{5\pi}{12} + \frac{\pi}{2}n}_{\text{Group 2}} \quad (n \text{ integer}) \right\}$$

We could plug in integer values for n and evaluate to obtain particular solutions.

Instead, let's try a more efficient approach using the "Increment" idea we used for trig graphs in Chapter 4. Observe that, every time we increase n by 1, that has the effect of adding $\frac{\pi}{2}$ in both groups of solutions.
(Think: Distributive Property of Multiplication over Addition.)
We can think of $\frac{\pi}{2}$ as our "Increment" in both groups.

<u>Group 1</u>: $x = \frac{\pi}{3} + \frac{\pi}{2}n \quad (n \text{ integer})$

Observe that 6 is the LCD.

Start with: $\frac{\pi}{3}$, which equals $\frac{2\pi}{6}$

Increment: $\frac{\pi}{2}$, which equals $\frac{3\pi}{6}$

The simplified solutions are in bold below.

(Optional)	Particular Solutions
$n = 0$	$\boldsymbol{\frac{\pi}{3}} = \frac{2\pi}{6}$
$n = 1$	$\boldsymbol{\frac{5\pi}{6}}$
$n = 2$	$\frac{8\pi}{6} = \boldsymbol{\frac{4\pi}{3}}$
$n = 3$	$\boldsymbol{\frac{11\pi}{6}}$

Observe that, if the integer $n < 0$, the resulting value is negative, so it cannot be included in the solution set. Remember that we are only looking for solutions in the interval $[0, 2\pi)$. The fact that the $n = 0$ solution "works" here is a result of our preference for the smallest positive brothers that "worked" back in Notes 5.34.

Observe that, if the integer $n > 3$, the resulting value is at least 2π, so it cannot be included, either.

This all works smoothly because $\frac{\pi}{3}$ is the smallest positive solution (angle) in this group. If you had started with another angle in the group, make sure that you add and/or subtract the Increment in such a way that you "sweep through" the interval $[0, 2\pi)$ and pick up all real solutions in there.

Group 2: $x = \frac{5\pi}{12} + \frac{\pi}{2}n \quad (n \text{ integer})$

Observe that 12 is the LCD.

Start with: $\frac{5\pi}{12}$

Increment: $\frac{\pi}{2}$, which equals $\frac{6\pi}{12}$

The simplified solutions are in bold below.

(Optional)	Particular Solutions
$n = 0$	$\mathbf{\frac{5\pi}{12}}$
$n = 1$	$\mathbf{\frac{11\pi}{12}}$
$n = 2$	$\mathbf{\frac{17\pi}{12}}$
$n = 3$	$\mathbf{\frac{23\pi}{12}}$

Observe that, if the integer $n < 0$, the resulting value is negative, so it cannot be included in the solution set.

Observe that, if the integer $n > 3$, the resulting value is at least 2π, so it cannot be included, either.

Solution set:

(Instructors typically do not require numbers in solution sets to be written in increasing order. Sets are typically assumed to be unordered, anyway.)

There are 8 solutions in the interval $\left[0, 2\pi\right)$:

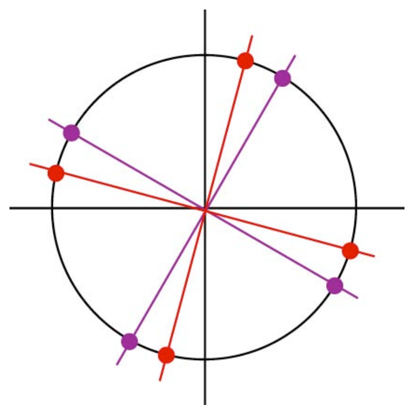
$$\left\{ \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12} \right\}$$

Idea:

The two "spider eggs" at the red points of interest for the θ figure in Notes 5.34 each break open and produce four spiders placed evenly (periodically) along the Unit Circle.

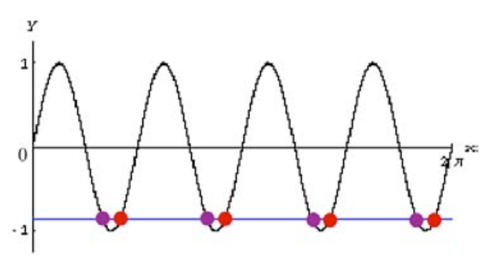
Basically, we take the $\frac{4\pi}{3}$ angle and three of its coterminal "twins" and the $\frac{5\pi}{3}$ angle and three of its coterminal "twins" and divide each of them by 4.

The first group corresponds to the purple points below.
The second group corresponds to the red points below.



These points correspond to the intersection points of the graphs of $y=\sin(4x)$ (in black) and $y=-\frac{\sqrt{3}}{2}$ (in blue) below.

Their x-coordinates are solutions to $\sin(4x)=-\frac{\sqrt{3}}{2}$.



PART G: USING INVERSE FUNCTIONS

Example

Solve: $3\sin x - 1 = 0$. (Find all real solutions.)

Solution

$$3\sin x - 1 = 0$$

Isolate the sin expression.

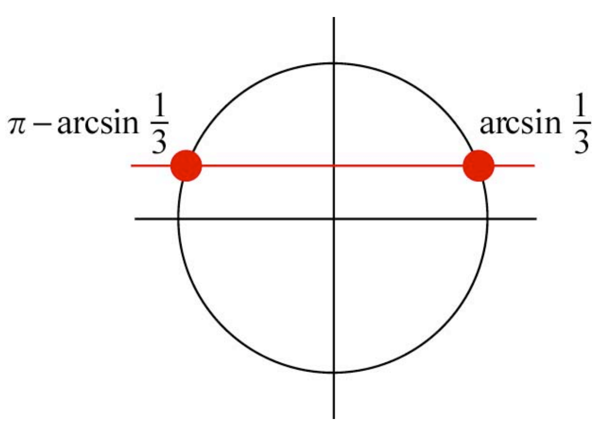
$$\sin x = \frac{1}{3}$$

$\frac{1}{3}$ is not a special sin value, although it does lie in the range of the $\sin x$ function, $\left[-1, 1\right]$. This equation has real solutions, but we need to use "arcsin" or "$\sin^{-1}$" notation to express them exactly.

Since the sin value $\frac{1}{3} > 0$, $\arcsin\left(\frac{1}{3}\right)$ represents an acute angle in Quadrant I, and we also want a brother in Quadrant II.

Remember that radians are the assumed measure for angles.

Note: $\arcsin\left(\frac{1}{3}\right)$ is about 19.5°. Its brother below is about 160.5°.



The two particular solutions in $\left[0, 2\pi\right)$ are $\arcsin\left(\frac{1}{3}\right)$ and $\pi - \arcsin\left(\frac{1}{3}\right)$.

Considering all coterminal "twin" angles, the general solution set is:

$$\left\{ x \;\middle|\; x = \arcsin\left(\frac{1}{3}\right) + 2\pi n, \quad \text{or} \quad x = \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi n \quad \left(n \text{ integer}\right) \right\}$$

… or, using $\sin^{-1}$ notation, …

$$\left\{ x \;\middle|\; x = \sin^{-1}\left(\frac{1}{3}\right) + 2\pi n, \quad \text{or} \quad x = \pi - \sin^{-1}\left(\frac{1}{3}\right) + 2\pi n \quad \left(n \text{ integer}\right) \right\}$$