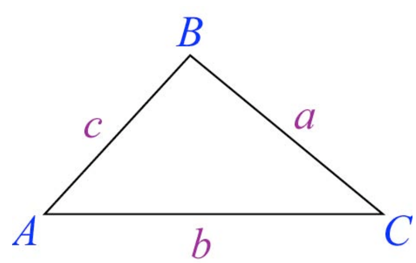
CHAPTER 6: ADDITIONAL TOPICS IN TRIG

SECTION 6.1: THE LAW OF SINES

PART A: THE SETUP AND THE LAW

The Law of Sines and the Law of Cosines will allow us to analyze and solve oblique (i.e., non-right) triangles, as well as the right triangles we have been used to dealing with.

Here is an example of a conventional setup for a triangle:



There are 6 parts: 3 angles and 3 sides.

Observe that Side a "faces" Angle A, b faces B, and c faces C.
(In a right triangle, C was typically the right angle.)

When we refer to a, we may be referring to the line segment $\overline{BC}$ or its length.

The Law of Sines

For such a triangle:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Equivalently:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

PART B: WHAT MUST BE TRUE OF ALL TRIANGLES?

We assume that *A*, *B*, and *C* are angles whose degree measures are strictly between 0° and 180°.

The 180° Rule

The sum of the interior angles of a triangle must be 180°.
That is, $A + B + C = 180^\circ$.

The Triangle Inequality

The sum of any two sides (i.e., side lengths) of a triangle must exceed the third.
That is, $a + b > c$, $b + c > a$, and $a + c > b$.

Think: Detours. Any detour in the plane from point *A* to point *B*, for example, must be longer than the straight route from *A* to *B*.

Example: There can be no triangle with side lengths 3 cm, 4 cm, and 10 cm, because $3 + 4 \not> 10$. If you had three "pick-up" sticks with those lengths, you could not form a triangle with them if you were only allowed to connect them at their endpoints.

The "Eating" Rule

For a given triangle, larger angles face (or "eat") longer sides.

You can use this to check to see if your answers are sensible.

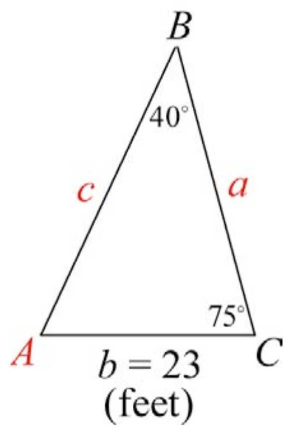
PART C: EXAMPLE

Example

Given: $B = 40^\circ$, $C = 75^\circ$, $b = 23$ ft.
Solve the triangle. In your final answers, round off lengths to the nearest foot.

Solution

Sketch a model triangle.
(Information yet to be determined is in red.)



Find Angle A:

Use the 180° Rule.

$$A = 180^\circ - 40^\circ - 75^\circ$$
$$\boldsymbol{A = 65^\circ}$$

Use the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 65^\circ} = \frac{23}{\sin 40^\circ} = \frac{c}{\sin 75^\circ}$$

Observe that the middle ratio is "known," so we should use it when we solve for both a and c.

Find a:

Solve $\dfrac{a}{\sin 65^\circ} = \dfrac{23}{\sin 40^\circ}$ for a.

$$a = \frac{23\sin 65^\circ}{\sin 40^\circ}$$

$$\boldsymbol{a \approx 32 \textbf{ feet}}$$

Warning: Make sure your calculator is in DEGREE mode.

Warning: Avoid approximations for trig values in intermediate steps. Excessive rounding can render your final approximations inaccurate. Memory buttons may help. If feasible, you should keep exact expressions such as $\sin 65^\circ$ throughout your solution until the end.

Warning: Don't forget units where they are appropriate!

Find c:

Solve $\dfrac{23}{\sin 40^\circ} = \dfrac{c}{\sin 75^\circ}$ for c.

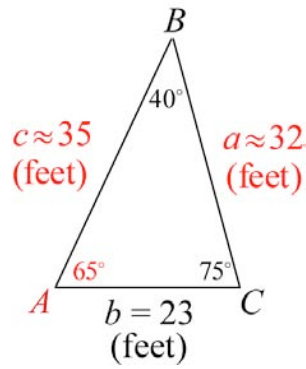
$$\frac{23\sin 75^\circ}{\sin 40^\circ} = c$$

$$\boldsymbol{c \approx 35 \textbf{ feet}}$$

Warning: Although you could, in principle, use the $\dfrac{a}{\sin 65^\circ}$ ratio instead of the $\dfrac{23}{\sin 40^\circ}$ ratio, it is ill-advised to use your rough approximation for a as a foundation for your new calculations.

Use the "Eating" Rule to Check:

Make sure that larger angles "eat" longer sides.



PART D: CASES

If you are given 3 parts of a triangle (including one side), you can solve the triangle by finding the other 3 parts or by discovering that there is no triangle that supports the given configuration (in which case there is "no solution"). Two triangles are considered to be the same if they have the same values for A, B, C, a, b, and c.

• If you are only given the 3 angles (the AAA case), then you have an entire family of similar triangles of varying sizes that have those angles.

• On the other hand, it is possible that **no** triangle can support the given configuration. For example, maybe the Triangle Inequality is being violated.

• In the "Ambiguous SSA" case, which is discussed in Part E, two different triangles may work.

The Law of Sines is applied in cases where you know two angles and one side.

For example, it is applied in:

• The AAS case (such as in the previous Example, in which we are given Angle B, Angle C, and Side b, a "nonincluded" side), and

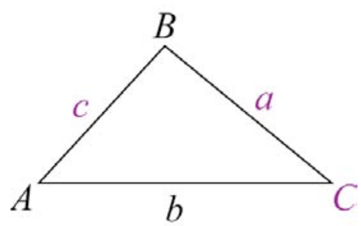
• The ASA case (in which, for example, we are given Angle A, Side b, and Angle C; here, Side b is "included" between the two given angles).

Can you see how these cases can yield either no triangle or exactly one?

The Law of Sines is also applied in the "Ambiguous SSA" case, described next.

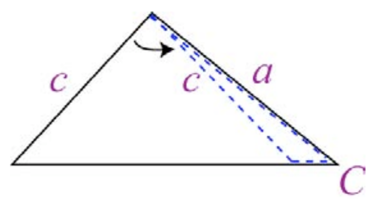
PART E: THE AMBIGUOUS SSA CASE

The SSA case is when you are given two sides and a nonincluded angle.
For example, you could be given Side c, Side a, and Angle C (in purple below).



The SSA case is called the "ambiguous case," because two triangles (that is, two triangles that are not congruent) may arise from the given information. Bear in mind that the possibility of "no triangles" potentially plagues all cases.

In the figure above, imagine the side labeled c being rotated about Point B.
We can obtain a second triangle (in blue dashed lines below) for which the given information still holds!



How is this issue reflected in the Law of Sines?

Look at the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

In order to solve the triangle, we find Angle A using the Law of Sines. We could then use the 180° Rule to find the remaining angle, Angle B.

The problem is that we must first find $\sin A$, and an acute angle and its supplementary obtuse angle may share the same sin value. (Look at the Unit Circle!) These two possibilities may yield two different triangles, provided that the 180° Rule does not fall apart (the obtuse angle may "eat up" too many degrees). Also, it's "game over" if $\sin A$ was not in $\left(0, 1\right]$ to begin with.

PART F: THE AREA OF A TRIANGLE

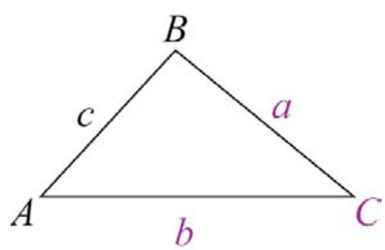
In the SAS Case (described below), you can quickly compute the area of a triangle.

Area of a Triangle (SAS Case)

Let a and b be two sides (i.e., two side lengths) of a triangle, and let C be the included angle between them. Then, the area of the triangle is given by:

$$\text{Area} = \frac{1}{2}ab\sin C$$

Think: Half the product of two sides and the sine of the included angle between them (represented by a different letter).



Warning: Avoid writing A for Area, since we often use A to name a vertex on the triangle.

What happens if C is a right angle?

Variations

The following also hold:

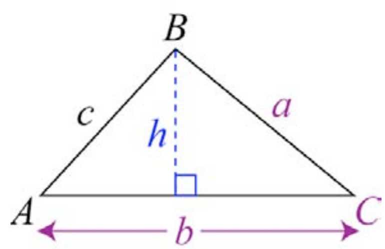
$$\text{Area} = \frac{1}{2}bc\sin A$$

$$\text{Area} = \frac{1}{2}ac\sin B$$

If you know one of the area formulas above, you can figure out the other two. If you are given a word problem or an unlabeled triangle, you could assign labels in a manner best suited for the formula you are most familiar with.

Proof

"Without loss of generality," let's say we are given a, b, and C in the figure below. The h represents the height of the triangle, provided that b is taken as the base.



Since $\sin C = \dfrac{h}{a}$, the height h of the triangle is given by: $h = a \sin C$.
The area is then given by:

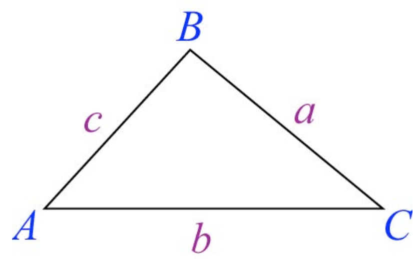
$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}bh = \frac{1}{2}b(a\sin C) = \frac{1}{2}ab\sin C$$

This proof is similar to the proof for the Law of Sines on p.468 of Larson, in which expressions for the height of a triangle involving the sines of the three angles are equated.

SECTION 6.2: THE LAW OF COSINES

PART A: THE SETUP AND THE LAW

Remember our example of a conventional setup for a triangle:



Observe that Side a "faces" Angle A, b faces B, and c faces C.

The Law of Cosines

For such a triangle:

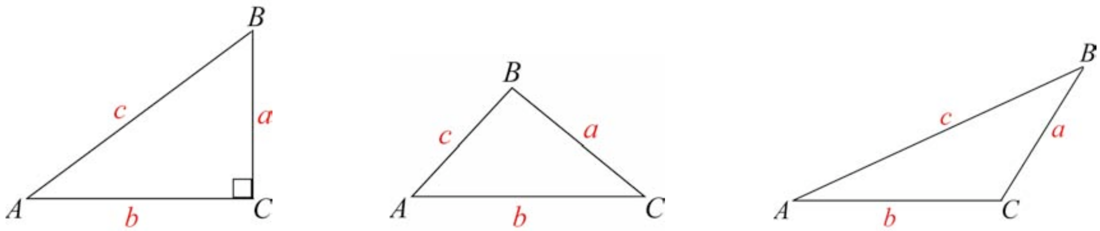
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Think / For Memorization Purposes: This looks like the Pythagorean Theorem, except that there is a third term. "The square of one side equals the sum of the squares of the other two sides, **minus** twice their product times the cosine of the angle included between them." Notice that the formula is symmetric in a and b; we have $2ab$ in the formula as opposed to $2bc$ or $2ac$. Angle C is the one we take the cosine of, because it is the "special" angle that faces the side indicated on the left.

The (very involved) proof is on p.469 of Larson. There is a nicer "Proof Without Words" available on my website.

PART B: A THOUGHT EXPERIMENT

How does the formula gibe with our geometric intuition?



Let's say we fix lengths a and b, but we allow the other parts to vary.
Imagine rotating the side labeled a about the point C so that Angle C changes.

If C is a right angle (left figure above), then we obtain the Pythagorean Theorem as a special case:

$$c^2 = a^2 + b^2 - \underbrace{2ab\underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0}}_{=0}$$

$$c^2 = a^2 + b^2$$

If C is acute (middle figure above), then:

$$c^2 = a^2 + b^2 \underbrace{- 2ab\underbrace{\cos C}_{>0}}_{<0}$$

$$c^2 < a^2 + b^2$$

This reflects the fact that c in this case is smaller than c in the right angle case.

If C is obtuse (right figure above), then:

$$c^2 = a^2 + b^2 \underbrace{- 2ab\underbrace{\cos C}_{<0}}_{>0}$$

$$c^2 > a^2 + b^2$$

This reflects the fact that c in this case is larger than c in the right angle case.

PART C: VARIATIONS OF THE LAW

The form given in Part A is the only one you need to memorize, but you should be aware of variations.

There is nothing "special" about side c and Angle C. "Role-switching" yields analogous formulas for the other side-angle pairs.

Variations of the Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$

Without loss of generality, the proof of one variation yields the others, as well.

The previous formulas may also be solved for the cos expressions.

More Variations of the Law of Cosines

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

In radians, A, B, and C must be in the interval $(0, \pi)$, which is a subset of the range of the $\cos^{-1}$ function, $[0, \pi]$. Therefore, we can conclude that:

$C = \cos^{-1}\left(\dfrac{a^2 + b^2 - c^2}{2ab}\right)$, and so forth. However, books tend not to give these as variations of the Law of Cosines, because, if we are working in degrees, they may be deemed inappropriate. (Inverse trig values are not supposed to be directly given in degrees.) As a practical matter, though, it helps to be aware of them.

PART D: WHAT MUST BE TRUE OF ALL TRIANGLES?

We assume that A, B, and C are angles whose degree measures are strictly between 0° and 180°.

The 180° Rule, the Triangle Inequality, and the "Eating" Rule from Notes 6.02 still apply.

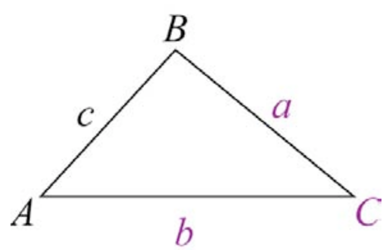
PART E: CASES

Remember that the Law of Sines is applied in cases where you know two angles and one side (ASA and AAS) and in the ambiguous SSA case.

The Law of Cosines is applied in:

- The SSS case, in which all three sides are given, and

- The SAS case, in which two sides and their included angle are given.
  For example, we could be given side a, Angle C, and side b (see the figure below). This is actually the case for which the area of the triangle can be readily computed using the formula from Section 6.1, whose proof was related to the proof for the Law of Sines.



The Law of Sines cannot handle these cases.

Can you see how these cases can yield either no triangle or exactly one?

Variations such as $c^2 = a^2 + b^2 - 2ab\cos C$ are directly suited for the SAS case.

Variations such as $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$ are directly suited for the SSS case.

PART F: APPLYING THE LAW OF COSINES

(See the Flowchart for Sections 6.1-6.2 on my website. It is better suited for practical use than for memorization.)

In the SSS case, you can quickly check to see if the Triangle Inequality is violated. Then, there are no triangles as solutions.

Strategies get a little complicated if you are going to use the Law of Sines in conjunction with the Law of Cosines ….

Technical Note: If you mix the two laws, you should find the largest angle (i.e., the angle "eating" the longest side) first in the SSS case. Then, you can eliminate any further candidates for obtuse angles (since a triangle can have at most one, and you have found the largest angle), and no SSA-style ambiguities arise.

In the SAS case, you should find the smaller angle of the two unknown angles (i.e., the one "eating" the shorter of the two given sides), because you know that that angle is acute; the remaining angle can then be obtained by subtraction via the 180° Rule.

You could, instead, use the Law of Cosines throughout, in which case there are no such ambiguities. (The computations may be a bit more involved, though, which is why some books prefer to mix in the Law of Sines.) As far as triangles go, although a sin value in the interval $(0, 1)$ can yield two possible angles (an acute angle and its supplement), a cos value in the interval $(-1, 1)$ can yield only one possible angle.

It is "game over" for possible solutions if you ever find that the cosine of an angle has to be outside the interval $(-1, 1)$.

If you are given a word problem, or if you are given a choice as to how to name the parts of the triangle at hand, you may want to label parts in a manner consistent with the first formula given for the Law of Cosines in Notes 6.09.

PART G: EXAMPLES

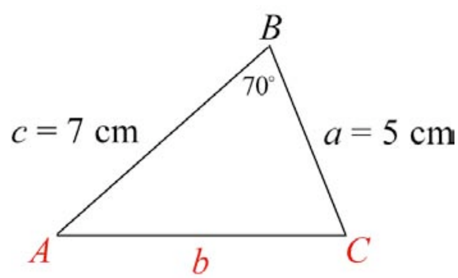
Example: SAS Case

Given: $a = 5$ cm, $c = 7$ cm, and $B = 70^\circ$.

Solve the triangle. In your final answers, round off lengths to the nearest hundredth of a centimeter and angles to the nearest tenth of a degree.

Solution

Sketch a model triangle.
(Information yet to be determined is in red.)



Find Side b:

Use a variation of the Law of Cosines.

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$b = \sqrt{(5)^2 + (7)^2 - 2(5)(7)\cos 70^\circ} \quad \text{(Take the "+" root.)}$$

$$\boldsymbol{b \approx 7.08\ \text{cm}}$$

Calculator Comments: You may want to work out the term with $\cos 70^\circ$ first. Remember to incorporate the "$-$." Remember to take the square root at the end.

We will continue to use variations of the Law of Cosines instead of mixing in the Law of Sines, because the latter requires more strategizing about order. However, using the Law of Cosines may require more concentration on your part with respect to memorization, algebraic manipulations, and calculator computations.

<u>Find Angle A</u>:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

You can derive the above formula by starting with $a^2 = b^2 + c^2 - 2bc\cos A$ and solving for $\cos A$.

$$\approx \frac{(7.08)^2 + (7)^2 - (5)^2}{2(7.08)(7)}$$

<u>Calculator comments</u>: If possible, try to keep your calculator's result for b in memory so that you may recall it. Also, process the numerator before dividing by the three factors in the denominator.

$$\approx 0.748$$

<u>Calculator comments</u>: Keep your calculator's result (not your rounded version) when you press the $\cos^{-1}$ button. Make sure your calculator is in DEGREE mode, so that it automatically converts from radians (which really correspond to inverse trig values) to degrees.

<u>Note</u>: If we had obtained a cos value outside of the interval $(-1, 1)$, then no solution triangle would have existed.

$$\boldsymbol{A \approx 41.6^\circ}$$

<u>Find Angle C</u>:

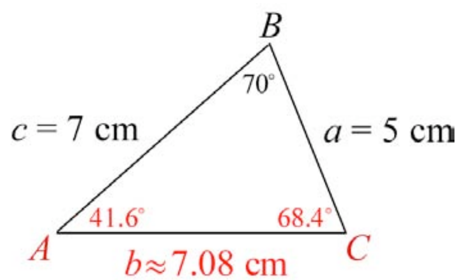
Use the 180° Rule. (It's a lot easier than the Law of Cosines.)

$$C \approx 180^\circ - 70^\circ - 41.6^\circ$$

$$\boldsymbol{C \approx 68.4^\circ}$$

Use the "Eating" Rule to Check:

Make sure that larger angles "eat" longer sides.
The relative side lengths seem appropriate, also.



Example: SSS Case

Given: $a = 4$ in., $b = 5$ in., $c = 7$ in.

Solve the triangle. In your final answers, round off angles to the nearest tenth of a degree.

Solution

A model triangle may not be as helpful here.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{(4)^2 + (5)^2 - (7)^2}{2(4)(5)} \quad \Rightarrow \quad \boldsymbol{C \approx 101.5^{\circ}}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(5)^2 + (7)^2 - (4)^2}{2(5)(7)} \quad \Rightarrow \quad \boldsymbol{A \approx 34.0^{\circ}}$$

$B \approx 180^{\circ} - 101.5^{\circ} - 34.0^{\circ}$

$\boldsymbol{B \approx 44.5^{\circ}}$

Observe that larger angles "eat" longer sides.
(C "eats" c, the longest side, and so forth.)

PART H: THE AREA OF A TRIANGLE: HERON'S FORMULA (SSS CASE)

If you know the three sides of a triangle (if one exists), Heron's Formula can be used to quickly find its area.

Remember that, if you have SAS information, you can use the formulas in Notes 6.07.

Area of a Triangle: Heron's Formula (SSS Case)

Let a, b, and c be the three sides (i.e., side lengths) of a triangle.

Let the semiperimeter $s = \dfrac{a+b+c}{2}$. (This is half of the perimeter.)

Then, the area of the triangle is given by:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

Observe that the formula is symmetric in a, b, and c (so it does not matter which label goes with which side), and it does not require knowledge of any angles.

Note: If the sum of any two sides of the triangle does **not** exceed the third (i.e., if the Triangle Inequality falls apart), then the formula does **not** yield a positive real number. Let's say $a+b \not> c$. Then, $s-c \le 0$. (It takes a little bit of work to see why.)

The proof on p.470 of Larson employs an area formula from Notes 6.07 and the Law of Cosines.

See p.422 of Larson to see an Example and some info on Heron of Alexandria.

SOLVING A TRIANGLE (T)

AAA → Can't solve:
Family of similar triangles (if angle sum = 180^{o})

Assume that any given angle measure (or the sum of any two given angle measure lies in the interval (0^{o}, 180^{o}). Otherwise, there is no such T.

"Find 3rd angle": Subtract the other two angle measures from 180^{o}.

2 angles, 1 side (SAA, ASA) → Find 3rd angle → Use **Law of Sines** to find 2nd, 3rd sides

SSA (Two sides and a non-included angle) | AMBIGUOUS CASE!! (Possibly 2 triangles) →

Find candidates for 2nd angle using **Law of Sines**

→ If get a sin value > 1.... → **No such T exists!!**

→ $\sin^{-1}$ result → Find 3rd angle → Find 3rd side

→ supplementary angle → Find 3rd angle → Find 3rd side

If angle $\leq 0^{o}$,
KILL THIS CASE!!
There is only
one triangle.

CONTINUED →

SAS (Two sides and their included angle) → Use **Law of Cosines** to find 3rd side →

- Find 2nd angle using **Law of Cosines** → Find 3rd angle

OR

- Find 2nd angle using **Law of Sines**. Find the smaller of the two remaining angles (compare their opposite sides to see which is smaller); this angle must be acute. → Find 3rd angle

SSS →

- If the Triangle Inequality fails... → **No such T exists!!**

Otherwise...

- Find an angle using **Law of Cosines**. → Find a 2nd angle using **Law of Cosines**. → Find 3rd angle
- Find the largest angle (it faces the longest side) if you intend to use the Law of Sines later. The other two angles must be acute. → Find a 2nd angle using **Law of Sines**. → Find 3rd angle

FINDING THE AREA OF A TRIANGLE (T)

SSS →

- If the Triangle Inequality fails... → **No such T exists!!**
- Otherwise... → Use Heron's Formula

SAS → Use (for example) Area = $\frac{1}{2}ab\sin C$

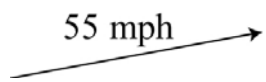
SECTION 6.3: VECTORS IN THE PLANE

Assume *a*, *b*, *c*, and *d* are real numbers.

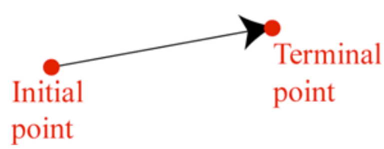
PART A: INTRO

A scalar has magnitude but not direction. We think of real numbers as scalars, even if they are negative. For example, a speed such as 55 mph is a scalar quantity.

A vector has both magnitude and direction. A vector **v** (written as $\vec{v}$ of $\vec{v}$ if you can't write in boldface) has magnitude $\|\mathbf{v}\|$. The length of a vector indicates its magnitude. For example, the directed line segment ("arrow") below is a velocity vector:



An equal vector (together with labeled parts) is shown below. Vectors with the same magnitude and direction (but not necessarily the same position) are equal.

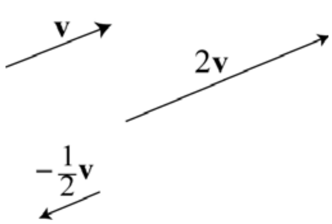


PART B: SCALAR MULTIPLICATION OF VECTORS

A scalar multiple of **v** is given by $c\mathbf{v}$, where *c* is some real scalar.

This new vector, $c\mathbf{v}$, is $|c|$ times as long as **v**.
If $c < 0$, then $c\mathbf{v}$ points in the opposite direction from the direction **v** points in.

Examples:



The vector $-\dfrac{1}{2}\mathbf{v}$ is referred to as "the opposite of $\dfrac{1}{2}\mathbf{v}$."

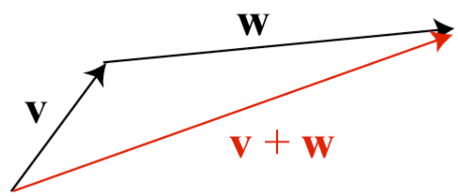
PART C: VECTOR ADDITION

Vector addition can correspond to combined (or net) effects.
For example, if **v** and **w** are force vectors, the resultant vector $\mathbf{v}+\mathbf{w}$ represents net force.

Vector subtraction may be defined as follows: $\mathbf{v}-\mathbf{w}=\mathbf{v}+\left(-\mathbf{w}\right)$.

There are two easy ways we can graphically represent vector addition:

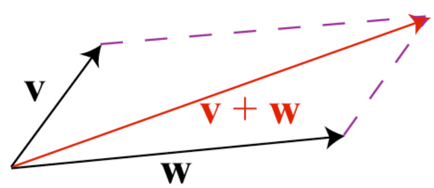
Triangle Law



To draw $\mathbf{v}+\mathbf{w}$, we place the tail of **w** at the head of **v**, and we draw an arrow from the tail of **v** to the head of **w**.

This may be better for representing sequential effects and displacements.

Parallelogram Law



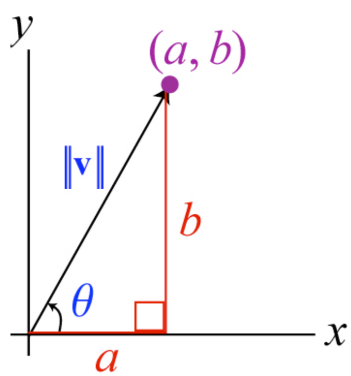
To draw $\mathbf{v}+\mathbf{w}$, we draw **v** and **w** so that they have the same initial point, we construct the parallelogram (if any) that they determine, and we draw an arrow from the common initial point to the opposing corner of the parallelogram.

This may be better for representing simultaneous effects and net force.

PART D: VECTORS IN THE RECTANGULAR (CARTESIAN) PLANE

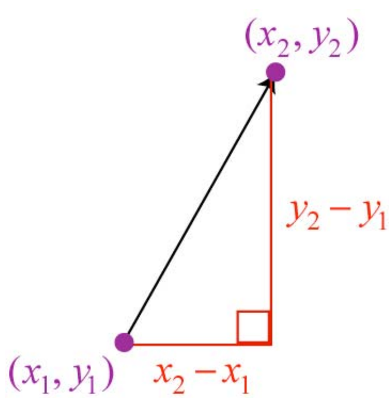
Let $\mathbf{v} = \langle a, b \rangle$. This is called the component form of v. We call a the horizontal component of **v**, b the vertical component, and the $\langle \ \rangle$ symbols angle brackets.

The position vector for **v** is drawn from the origin to the point (a, b).
It is the most convenient representation of **v**.

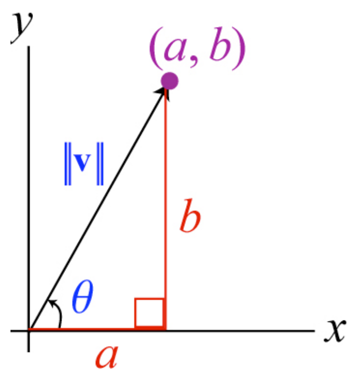


θ is a direction angle for **v**. We treat direction angles as standard angles here.
Remember that $\|\mathbf{v}\|$ is the magnitude, or length, of **v**.

A directed line segment drawn from the point (x_1, y_1) to the point (x_2, y_2) represents the vector $\langle x_2 - x_1,\ y_2 - y_1 \rangle$.



PART E: FORMULAS



If we are given a and b ...

By the Distance Formula (or the Pythagorean Theorem),

$$\boxed{\|\mathbf{v}\| = \sqrt{a^2 + b^2}}$$

How can we relate θ, a, and b?

Choose θ such that:

$$\boxed{\begin{array}{l} \tan\theta = \dfrac{b}{a} \quad (\text{if } a \neq 0), \text{ and} \\ \theta \text{ is in the correct Quadrant} \end{array}}$$

Example

If $\mathbf{v} = \langle -3, 5 \rangle$, find $\| \mathbf{v} \|$ and θ, where $0 \le \theta < 360^\circ$.
Round off θ to the nearest tenth of a degree.

Solution

Find $\|\mathbf{v}\|$:

$$\begin{aligned} \| \mathbf{v} \| &= \sqrt{a^2 + b^2} \\ &= \sqrt{(-3)^2 + (5)^2} \\ &= \sqrt{34} \end{aligned}$$

Find θ:

$$\begin{aligned} \tan\theta &= \frac{b}{a} \\ &= \frac{5}{-3} \\ &= -\frac{5}{3} \end{aligned}$$

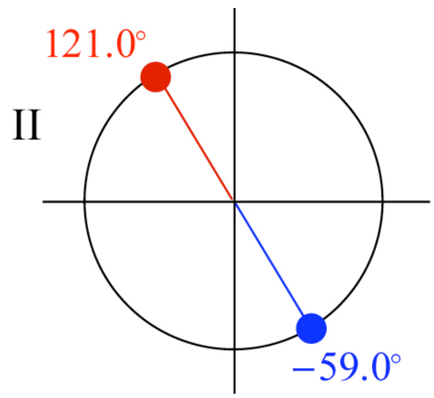
Warning: Make sure your calculator is in DEGREE mode when you press the $\tan^{-1}$ button.

Warning: The result may not be your answer. In fact, for this problem, it is not.

In degrees, $\tan^{-1}\left(-\frac{5}{3}\right) \approx -59.0^\circ$. However, this would be an inappropriate choice for θ, even without the restriction $0 \le \theta < 360^\circ$. This is because -59.0° is a Quadrant IV angle, whereas the point $(-3, 5)$ (and, therefore, the position vector for $\mathbf{v} = \langle -3, 5 \rangle$) is in Quadrant II.

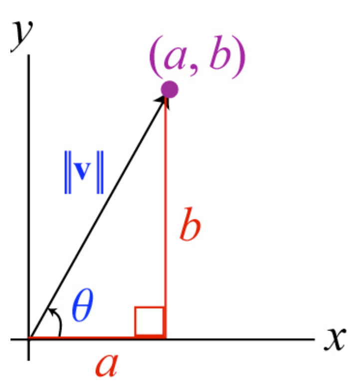
We require θ to be a Quadrant II angle in $\left[0^\circ, 360^\circ\right)$.
There is only one such angle:

$$\theta \approx -59.0^\circ + 180^\circ$$
$$\theta \approx 121.0^\circ$$



Answers: $\boxed{\left\| \mathbf{v} \right\| = \sqrt{34},\ \theta \approx 121.0^\circ}$

<u>If we are given $\left\| \mathbf{v} \right\|$ and θ ...</u>



$$\cos\theta = \frac{a}{\left\| \mathbf{v} \right\|} \qquad \sin\theta = \frac{b}{\left\| \mathbf{v} \right\|}$$

$$a = \left\| \mathbf{v} \right\| \cos\theta \qquad b = \left\| \mathbf{v} \right\| \sin\theta$$

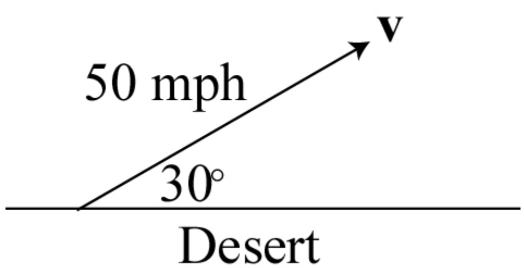
Therefore, $\boxed{\begin{aligned} \mathbf{v} &= \left\langle a, b \right\rangle \\ &= \left\langle \left\| \mathbf{v} \right\| \cos\theta,\ \left\| \mathbf{v} \right\| \sin\theta \right\rangle \end{aligned}}$

These formulas allow us to <u>resolve</u> a vector into its horizontal and vertical components.

Example

Out in the flat desert, a projectile is shot at a speed of 50 mph and an angle of elevation of 30°. Give the component form of the initial velocity vector **v**.

Solution



$\|\mathbf{v}\| = 50$ (mph), and $\theta = 30^\circ$.

$$\begin{aligned}\mathbf{v} &= \left\langle \|\mathbf{v}\|\cos\theta,\ \|\mathbf{v}\|\sin\theta \right\rangle \\ &= \left\langle 50\cos 30^\circ,\ 50\sin 30^\circ \right\rangle \\ &= \left\langle 50\left(\frac{\sqrt{3}}{2}\right),\ 50\left(\frac{1}{2}\right) \right\rangle \\ &== \left\langle 25\sqrt{3},\ 25 \right\rangle\end{aligned}$$

We now know the horizontal and vertical components of the initial velocity vector.

PART F: COMPUTATIONS WITH VECTORS

To add or subtract vectors, we add or subtract (in order) the components of the vectors.

$$\langle a, b\rangle + \langle c, d\rangle = \langle a+c,\ b+d\rangle$$
$$\langle a, b\rangle - \langle c, d\rangle = \langle a-c,\ b-d\rangle$$

To multiply a vector by a scalar, we multiply each component of the vector by the scalar.

$$c\langle a, b\rangle = \langle ca, cb\rangle$$

Example

If $\mathbf{v} = \langle 3, 5\rangle$ and $\mathbf{w} = \langle -1, -2\rangle$, find $4\mathbf{v} - 2\mathbf{w}$.

Solution

$$\begin{aligned} 4\mathbf{v} - 2\mathbf{w} &= 4\langle 3, 5\rangle - 2\langle -1, -2\rangle \\ &= \langle 12, 20\rangle + \langle 2, 4\rangle \qquad \text{(Adding is easier!)} \\ &= \langle 14, 24\rangle \end{aligned}$$

Although "scalar division" is a bit informal, we can define (if $c \neq 0$):

$$\frac{\langle a, b\rangle}{c} = \frac{1}{c}\langle a, b\rangle = \left\langle \frac{a}{c}, \frac{b}{c}\right\rangle$$

PART G: UNIT VECTORS

A unit vector has length (or magnitude) 1. Unit vectors are often denoted by **u**.

Given a vector **v**, the unit vector in the direction of **v** is given by:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \quad \left(\text{or } \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right)$$

It turns out that this normalization process is useful in Multivariable Calculus (Calculus III: Math 252 at Mesa) and Linear Algebra (Math 254 at Mesa).

Example

Find the unit vector in the direction of the vector **v**, if **v** can be represented by a directed line segment from $(1, 2)$ to $(4, 6)$.

Solution

Find **v**: $\mathbf{v} = \langle 4-1,\ 6-2 \rangle = \langle 3, 4 \rangle$

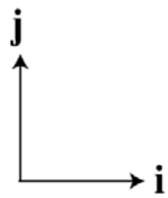
Find its magnitude: $\|\mathbf{v}\| = \|\langle 3, 4 \rangle\| = \sqrt{(3)^2 + (4)^2} = 5$

The desired unit vector is: $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{\langle 3, 4 \rangle}{5} = \boxed{\left\langle \dfrac{3}{5}, \dfrac{4}{5} \right\rangle}$

<u>Standard Unit Vectors</u>

$\mathbf{i} = \langle 1, 0 \rangle$, and

$\mathbf{j} = \langle 0, 1 \rangle$



These are often used in physics.

The vector $\langle a, b \rangle$ can be written as $a\mathbf{i} + b\mathbf{j}$, a <u>linear combination</u> of **i** and **j**.

For example, the answer in the previous Example, $\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, can be written as

$\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$.

Read the Historical Note on p.431 about Hamilton and Maxwell.

SECTION 6.4: VECTORS AND DOT PRODUCTS

Assume v_1, v_2, w_1, and w_2 are real numbers. For now, we will deal with vectors in the plane.

PART A: DOT PRODUCTS

How do we multiply vectors? There are two common types of products of vectors: the dot product (also known as the Euclidean inner product), and the cross product (also known as the vector product).

Dot Product (Algebraic Definition)

If $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$, then the dot product of $\mathbf{v}$ and $\mathbf{w}$ is given by:

$$\mathbf{v} \bullet \mathbf{w} = v_1 w_1 + v_2 w_2$$

In words, you add the products of corresponding components.
Dot products, themselves, are scalars.

Example

$$\begin{aligned} \langle 7, 3 \rangle \bullet \langle 2, -4 \rangle &= (7)(2) + (3)(-4) \\ &= 14 - 12 \\ &= 2 \end{aligned}$$

PART B: PROPERTIES OF THE DOT PRODUCT

See p.440 in Larson. All but Property #4 are shared by the operation of multiplication of real numbers.

Property #1) The dot product is commutative: $\mathbf{v} \bullet \mathbf{w} = \mathbf{w} \bullet \mathbf{v}$

Property #3) The dot product distributes over vector addition: $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

Property #4) Relating dot product and vector length: $\mathbf{v} \bullet \mathbf{v} = \|\mathbf{v}\|^2$

Observe that both sides equal $v_1^2 + v_2^2$.

PART C: THE ANGLE BETWEEN TWO VECTORS

Given two nonzero vectors **v** and **w**, let θ be the angle between them (i.e., between their position vectors) that is in the interval $\left[0^\circ, 180^\circ\right]$.

$$\cos\theta = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$$

Think: The dot product over the product of the lengths.

The proof employs the Law of Cosines.

Note: Again, books avoid the $\cos^{-1}$ notation here, because angles between vectors are often given in degrees. The $\cos^{-1}$ button works nicely, though, because (if you are in DEGREE mode), you are guaranteed to get an angle in $\left[0^\circ, 180^\circ\right]$.

Note: Observe that the formula is symmetric in **v** and **w**, so it doesn't matter which way you name the vectors.

Technical Note: The Cauchy-Schwarz Inequality guarantees that the right-hand side of the formula is a value in $\left[-1, 1\right]$.

Two vectors **v** and **w** are orthogonal $\Leftrightarrow$ $\mathbf{v} \bullet \mathbf{w} = 0$.

This happens $\Leftrightarrow$ Either vector is **0**, the zero vector $\langle 0, 0\rangle$ in the plane, or if $\theta = 90^\circ$. The terms "orthogonal" and "perpendicular" are frequently interchangeable.

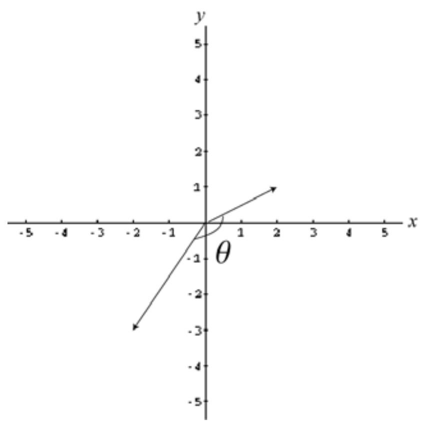
We say that **0** is orthogonal to every vector in the plane.

Example

Find the angle between $\mathbf{v} = \langle 2, 1 \rangle$ and $\mathbf{w} = \langle -2, -3 \rangle$ to the nearest tenth of a degree.

Solution

(Optional sketch:)



$$\begin{aligned}
\cos\theta &= \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \\
&= \frac{\langle 2, 1 \rangle \bullet \langle -2, -3 \rangle}{\|\langle 2, 1 \rangle\| \|\langle -2, -3 \rangle\|} \\
&= \frac{-4 - 3}{\sqrt{(2)^2 + (1)^2}\sqrt{(-2)^2 + (-3)^2}} \\
&= \frac{-7}{\sqrt{5}\sqrt{13}} \\
&= -\frac{7}{\sqrt{65}} \quad (\approx -0.868)
\end{aligned}$$

Make sure you are in DEGREE mode on your calculator.
Press the $\cos^{-1}$ button.

$$\theta = \cos^{-1}\left(-\frac{7}{\sqrt{65}}\right)$$

$$\theta \approx 150.3^\circ$$

PART D: WORK

The work W applied on an object by a constant force $\mathbf{F}$ as the object moves along the displacement vector $\mathbf{d}$ is given by: $W = \mathbf{F} \bullet \mathbf{d}$

Preliminaries: Dot Product (Geometric Definition)

From Notes 6.29, we had: $\cos\theta = \dfrac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$

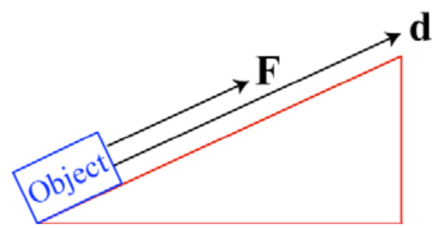
We can solve for $\mathbf{v} \bullet \mathbf{w}$ and obtain a geometric definition for the dot product:

$$\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta$$

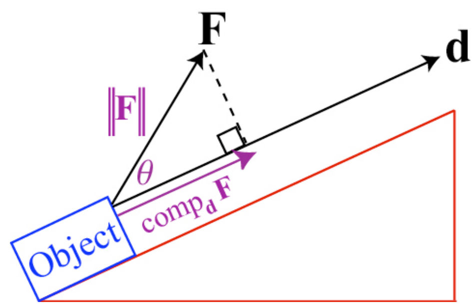
Reasoning Behind the Work Formula

If the force $\mathbf{F}$ being applied acts in the same direction as the direction of motion (i.e., the direction of $\mathbf{d}$), then the work done is given by:

$$\begin{aligned} W &= (\text{magnitude of force})(\text{distance traveled}) \\ &= \|\mathbf{F}\| \|\mathbf{d}\| \end{aligned}$$



In general, if **F** does not necessarily act in the same direction as **d**, then we replace the magnitude of **F**, denoted by $\|\mathbf{F}\|$, with the <u>component</u> of **F** in the direction of **d**, denoted by $\text{comp}_{\mathbf{d}}\mathbf{F}$. The latter represents the "relevant aspect" of the force in the direction of motion.



The work done is given by:

$$\begin{aligned} W &= \left(\text{comp}_{\mathbf{d}}\mathbf{F}\right)\left(\text{distance traveled}\right) \\ &= \left(\|\mathbf{F}\|\cos\theta\right)\left(\|\mathbf{d}\|\right) \\ &= \|\mathbf{F}\|\,\|\mathbf{d}\|\cos\theta \\ &= \mathbf{F}\bullet\mathbf{d} \end{aligned}$$

(by the Geometric Definition of Dot Product)

<u>In Calculus</u>: You will discuss the work applied on an object by a nonconstant (or variable) force field as the object moves along a nonlinear path, or even a path in three-dimensional space. You see this in Multivariable Calculus (Calculus III: Math 252 at Mesa).

<u>Note</u>: The ideas of this section extend naturally to three dimensions and (for the algebraic perspective on dot products) even higher dimensions.

SECTION 6.5: TRIG (AND EULER / EXPONENTIAL) FORMS OF A COMPLEX NUMBER

See the Handout on my website.

PART A: DIFFERENT FORMS OF A COMPLEX NUMBER

Let a, b, and r be real numbers, and let θ be measured in either degrees or radians (in which case it could be treated as a real number.)

We often let z denote a complex number.

Standard or Rectangular Form: $z = a + bi$

We saw this form in Section 2.4.

The complex number $a + bi$ may be graphed in the complex plane as either the point (a, b) or as the position vector $\langle a, b \rangle$, in which case our analyses from Section 6.3 become helpful.

Trig Form: $z = r(\cos\theta + i\sin\theta)$

r takes on the role of $\|\mathbf{v}\|$ from our discussion of vectors. It is the distance of the point representing the complex number from 0.

θ has a role similar to the one it had in our discussion of vectors, namely as a direction angle, but this time in the complex plane.

This is derived from the Standard Form through the relations:

$$\boxed{a = r\cos\theta, \text{ and } b = r\sin\theta}$$

Recall from Section 6.3, with r replacing $\|\mathbf{v}\|$:

$$r = \sqrt{a^2 + b^2}$$

Choose θ such that:

$$\tan\theta = \frac{b}{a} \quad (\text{if } a \neq 0), \text{ and } \theta \text{ is in the correct Quadrant}$$

Euler (Exponential) Form: $z = re^{i\theta}$

This is useful to derive various formulas in the Handout.
In this class, you will not be required to use this form.

PART B: EXAMPLE

Example

Express $z = 6 - 6i\sqrt{3}$ in Trig Form.

Solution

Find r:

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{(6)^2 + \left(-6\sqrt{3}\right)^2} \\ &= \sqrt{36 + (36)(3)} \\ &= \sqrt{144} \\ &= 12 \end{aligned}$$

Find θ:

$$\begin{aligned} \tan\theta &= \frac{b}{a} \\ &= \frac{-6\sqrt{3}}{6} \\ &= -\sqrt{3} \end{aligned}$$

Because $z = 6 - 6i\sqrt{3}$, we know that θ must be in Quadrant IV.
An appropriate choice for θ would be 300°, or $\frac{5\pi}{3}$ radians.

Find the (really, "a") Trig Form:

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 12\left(\cos 300^\circ + i\sin 300^\circ\right) \quad \text{or} \quad z = 12\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

Note: In Euler (Exponential) Form, we have $12e^{\frac{5\pi}{3}i}$. We need a radian measure such as $\frac{5\pi}{3}$ here.

PART C: OPERATIONS ON COMPLEX NUMBERS USING TRIG FORM

Again, see the Handout on my website.

When we **multiply** complex numbers (in Trig Form), we multiply their moduli, but we **add** their arguments; we go one step down in the order of operations when we deal with arguments. You can see this from the Euler (Exponential) Form.

When we **divide** complex numbers, we divide their moduli, but we **subtract** their arguments.

When we **raise** a complex number **to a power**, we raise the modulus to that power, but we **multiply** the argument by that power.

Examples

Let $z_1 = 3\left(\cos 50^\circ + i\sin 50^\circ\right)$, and $z_2 = 4\left(\cos 10^\circ + i\sin 10^\circ\right)$.

Then (in Trig Form):

$$z_1 z_2 = 12\left(\cos 60^\circ + i\sin 60^\circ\right) \quad \leftarrow \text{This is } 6 + 6i\sqrt{3} \text{ in Rectangular Form.}$$

$$\frac{z_1}{z_2} = \frac{3}{4}\left(\cos 40^\circ + i\sin 40^\circ\right)$$

$$\begin{aligned}(z_1)^4 &= (3)^4 \cdot \left(\cos\left(4\cdot 50^\circ\right) + i\sin\left(4\cdot 50^\circ\right)\right)\\ &= 81\left(\cos 200^\circ + i\sin 200^\circ\right)\end{aligned}$$

Think About It: What happens to a point representing a complex number in the complex plane when we multiply the number by i? How does this relate to our discussion on the powers of i back in Section 2.4?

PART D: ROOTS OF A COMPLEX NUMBER

This is the most complicated story we have in this section.

We say: $\sqrt{9} = 3$, because 3 is the principal square root of 9. This is because 3 is the **nonnegative** square root of 9. You could also say that -2 is the principal cube root of -8, because -2 is the only **real** cube root of -8.

In this section, however, we will consider both 3 and -3 as square roots of 9, because the square of both numbers is 9. (In fact, they are the only complex numbers whose square is 9.)

More generally, a nonzero complex number has n complex n^{th} roots, where n is a counting number (i.e., a positive integer) greater than 1.

Example

Find the fourth (complex) roots of $16i$.
In other words, find all complex solutions of $x^4 = 16i$.

Solution

The modulus of $16i$ is: $r = 16$. We will call this "old r."

All four fourth roots will have modulus: new $r = \sqrt[4]{\text{old } r} = \sqrt[4]{16} = 2$

The argument of $16i$ is: $\theta = 90^\circ$. We will call this "old θ."

According to the Handout, the arguments of the roots are given by:
$\dfrac{\theta + 2\pi k}{n}$ $\left(\text{or } \dfrac{\theta}{n} + \dfrac{2\pi}{n}k\right)$, $k = 0, 1, 2, \ldots, n-1$, where θ is any suitable argument in radians. The degree version looks like:
$\dfrac{\theta + \left(360^\circ\right)k}{n}$ $\left(\text{or } \dfrac{\theta}{n} + \dfrac{360^\circ}{n}k\right)$, $k = 0, 1, 2, \ldots, n-1$

The forms in the parentheses present an easier approach:

Take $\dfrac{\theta}{n}$ as the argument of one of the roots. Here, this would be:

$$\frac{90^\circ}{4} = 22.5^\circ, \text{ or } \frac{\pi/2}{4} = \frac{\pi}{8} \text{ in radians.}$$

The roots will be regularly spaced about the circle of radius 2 centered at 0. The "period" for the roots will be:

$$\frac{360^\circ}{n} = \frac{360^\circ}{4} = 90^\circ, \text{ or } \frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ in radians.}$$

Let's deal with degrees in this problem.

Remember, all four fourth roots have modulus 2.

The following arguments are appropriate for the four roots:

$$22.5^\circ \xrightarrow{+90^\circ} 112.5^\circ \xrightarrow{+90^\circ} 202.5^\circ \xrightarrow{+90^\circ} 292.5^\circ$$

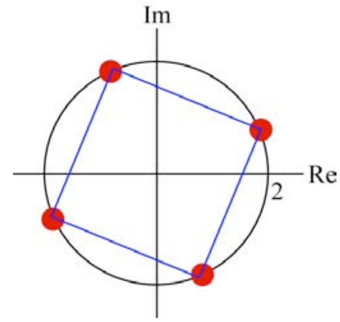
If we were to add 90° to the last argument, we would get an angle coterminal with 22.5°, and we do not take that as another root. Observe that we would get a Trig Form corresponding to the same Standard Form as the first root; we are looking for four **distinct** roots.

Trig Form for the roots:

<u>Note</u>: There are many acceptable possibilities for the arguments.

$$\begin{aligned}
z_0 &= 2\left(\cos 22.5^\circ + i\sin 22.5^\circ\right)\\
z_1 &= 2\left(\cos 112.5^\circ + i\sin 112.5^\circ\right)\\
z_2 &= 2\left(\cos 202.5^\circ + i\sin 202.5^\circ\right)\\
z_3 &= 2\left(\cos 292.5^\circ + i\sin 292.5^\circ\right)
\end{aligned}$$

Here is a graph of the four roots (represented by points in red):



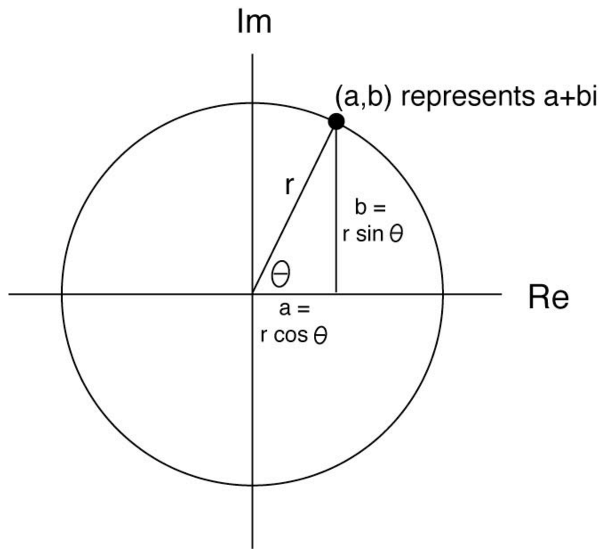
This should remind you of solving trig equations with multiple angles back in Section 5.3.

In general, the n^{th} roots of a nonzero complex number correspond to the vertices of a regular n-gon. They exhibit the nice symmetry around a circle that we found in Section 5.3 with trig equations with multiple angles.

SECTION 6.5: TRIG (and EULER) FORMS OF A COMPLEX NUMBER

A Picture

Let $z = a + bi$. This is plotted as (a, b) in the complex plane.



r (modulus)

The absolute value (or modulus, plural moduli) of z is

$$\begin{aligned} r &= |a+bi| \\ &= \sqrt{a^2+b^2} \end{aligned}$$

If z is a real number, then $b = 0$ and $r = |a|$, which is consistent with the notation for the absolute value of a real number.

θ (argument)

θ is an argument of z in the picture.
(Remember that infinitely many coterminal angles can be the argument.)
θ can be anything real if $z = 0$.

Finding θ:

$\tan\theta = \dfrac{b}{a}$ (Which quadrant of the complex plane does z lie in?)
(Maybe this is undefined.)

Trig (or "Polar") Form of a Complex Number

$z = \underbrace{(r\cos\theta)}_{a} + \underbrace{(r\sin\theta)}_{b}i$ or, more simply,

$z = r(\cos\theta + i\sin\theta)$

<u>Euler Form of a Complex Number</u>

<u>Euler's Formula</u>: $e^{i\theta} = \cos\theta + i\sin\theta$

<u>Famous Case</u>

If $\theta = \pi$, we get

$$e^{i\pi} = \cos\pi + i\underbrace{\sin\pi}_{0}$$

$$e^{\pi i} = -1 \quad \leftarrow \text{We have } e^{\text{something}} = \text{(a negative number)!!!}$$

$$e^{\pi i} + 1 = 0$$

This last formula relates five of the most basic constants in mathematics: $e, \pi, i, 1,$ and 0 !!!

<u>From Trig Form to Euler Form</u>

$$z = r\underbrace{(\cos\theta + i\sin\theta)}_{e^{i\theta}} \quad \leftarrow \text{Trig (Polar) Form}$$

$$z = re^{i\theta} \qquad \leftarrow \text{Euler Form}$$

Euler Form may be convenient when performing operations on complex numbers and when deriving related properties. Let's see…. (The textbook has different approaches.)

<u>Multiplying Complex Numbers in Trig Form</u>

<u>Multiply</u> the moduli, and <u>add</u> the arguments.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ i.e., $z_1 = r_1e^{i\theta_1}$
and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ i.e., $z_2 = r_2e^{i\theta_2}$

Then, $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$ i.e., $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$

Derivation using Euler Form

Why do we add the arguments instead of multiplying them?
When we <u>multiply</u> powers of e, we <u>add</u> the exponents.

$$\begin{aligned} z_1z_2 &= \left(r_1e^{i\theta_1}\right)\left(r_2e^{i\theta_2}\right) \\ &= r_1r_2e^{i\theta_1 + i\theta_2} \\ &= \underbrace{r_1r_2}_{\text{new } r}\, e^{i\overbrace{(\theta_1+\theta_2)}^{\text{new } \theta}} \\ &= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

<u>Dividing Complex Numbers in Trig Form</u>

<u>Divide</u> the moduli, and <u>subtract</u> the arguments.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ i.e., $z_1 = r_1 e^{i\theta_1}$
and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) \neq 0$ i.e., $z_2 = r_2 e^{i\theta_2} \neq 0$

Then, $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$ i.e., $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

Derivation using Euler Form

Why do we subtract the arguments instead of dividing them?
When we <u>divide</u> powers of *e*, we <u>subtract</u> the exponents.

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} \\
&= \frac{r_1}{r_2} e^{i\theta_1 - i\theta_2} \\
&= \underbrace{\frac{r_1}{r_2}}_{\text{new } r} e^{i\overbrace{(\theta_1 - \theta_2)}^{\text{new } \theta}} \\
&= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]
\end{aligned}$$

<u>Taking the *n*th Power of a Complex Number in Trig Form (*n* is a positive integer)</u>

Take the <u>*n*th power</u> of the modulus, and <u>multiply</u> the argument <u>by *n*</u>.

If $z = r(\cos\theta + i\sin\theta)$ i.e., $z = re^{i\theta}$
Then, $\underbrace{z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]}_{\text{DeMoivre's Theorem}}$ i.e., $z^n = r^n e^{i(n\theta)}$

Derivation using Euler Form

When we <u>raise</u> a power of *e* <u>to a power</u>, we <u>multiply</u> the exponents.

$$\begin{aligned}
z &= re^{i\theta} \\
z^n &= \left(re^{i\theta}\right)^n \\
&= r^n e^{i\theta n} \\
&= \underbrace{r^n}_{\text{new } r} e^{i\overbrace{(n\theta)}^{\text{new } \theta}} \\
&= r^n\left[\cos(n\theta) + i\sin(n\theta)\right]
\end{aligned}$$

Finding *n*th Roots of a Complex Number in Trig Form (*n* is a positive integer)

Take the nth root of the modulus, and divide *n* consecutive versions of the argument by *n*.

If $z = r(\cos\theta + i\sin\theta)$ i.e., $z = re^{i\theta}$

Then, the n distinct nth roots of z (provided $z \neq 0$) are given by:

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right] \quad \text{i.e.,} \quad z_k = \sqrt[n]{r}e^{i\left(\frac{\theta+2\pi k}{n}\right)}$$

where $k = 0, 1, 2, ..., n-1$.

The nth power of each of these is z.
(This should remind you of solving trig equations involving multiples of angles.)

Derivation using Euler Form

When we take the nth root of a power of e, we divide the exponent by n.

$$z = re^{i(\theta + 2\pi k)}, \; k = 0, 1, 2, ..., n-1$$

$$z_k = \left[re^{i(\theta+2\pi k)}\right]^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}} e^{\frac{i(\theta+2\pi k)}{n}}$$

$$= \underbrace{\sqrt[n]{r}}_{\substack{\text{new} \\ r}} e^{i\overbrace{\left(\frac{\theta+2\pi k}{n}\right)}^{\text{new }\theta}}$$

$$= \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

Summary

Trig (Polar) Form: $z = r(\cos\theta + i\sin\theta)$

where $r = \sqrt{a^2 + b^2}$, and $\tan\theta = \dfrac{b}{a}$ (and consider the Quadrant that z lies in)

Euler Form: $z = re^{i\theta}$ (good for deriving properties)

	new r	new θ
$z_1 z_2$	$r_1 r_2$	$\theta_1 + \theta_2$
$\dfrac{z_1}{z_2}$	$\dfrac{r_1}{r_2}$	$\theta_1 - \theta_2$
z^n (DeMoivre)	r^n	$n\theta$
Roots	$\sqrt[n]{r}$	$\dfrac{\theta + 2\pi k}{n}$, $k = 0, 1, 2, ..., n-1$