1

Functions and Models

1.1 Four Ways to Represent a Function

Four Ways to Represent a Function

Functions arise whenever one quantity depends on another. Consider the following four situations.

A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a *function* of r.

Four Ways to Represent a Function

B. The human population of the world P depends on the time t. The table gives estimates of the world population $P(t)$ at time t, for certain years. For instance,

$$P(1950) \approx 2{,}560{,}000{,}000$$

But for each value of the time t there is a corresponding value of P, and we say that P is a function of t.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

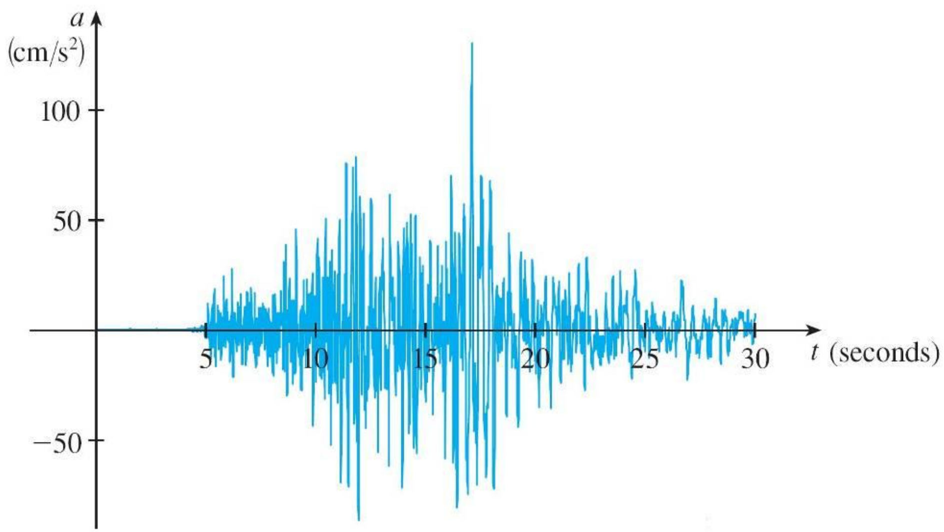
Four Ways to Represent a Function

C. The cost *C* of mailing a large envelope depends on the weight *w* of the envelope. Although there is no simple formula that connects *w* and *C*, the post office has a rule for determining *C* when *w* is known.

D. The vertical acceleration *a* of the ground as measured by a seismograph during an earthquake is a function of the elapsed time *t*.

Four Ways to Represent a Function

Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of *t*, the graph provides a corresponding value of *a*.



Vertical ground acceleration during the Northridge earthquake

Figure 1

Four Ways to Represent a Function

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain** of the function.

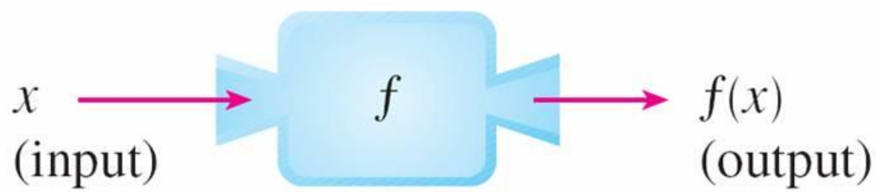
The number $f(x)$ is the **value of f at x** and is read "f of x." The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**.

Four Ways to Represent a Function

A symbol that represents a number in the *range* of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.

It's helpful to think of a function as a **machine** (see Figure 2).



Machine diagram for a function f

Figure 2

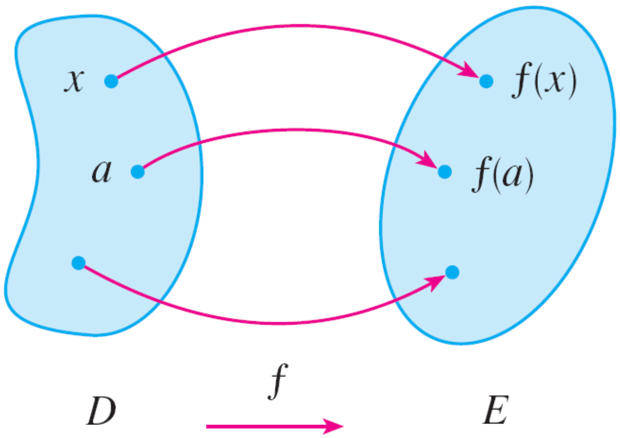
Four Ways to Represent a Function

If x is in the domain of the function f, then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function.

Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

Four Ways to Represent a Function

Another way to picture a function is by an **arrow diagram** as in Figure 3.



Arrow diagram for *f*

Figure 3

Each arrow connects an element of D to an element of E. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

Four Ways to Represent a Function

The most common method for visualizing a function is its graph. If f is a function with domain D, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

The graph of a function f gives us a useful picture of the behavior or “life history” of a function.

Four Ways to Represent a Function

Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 4).

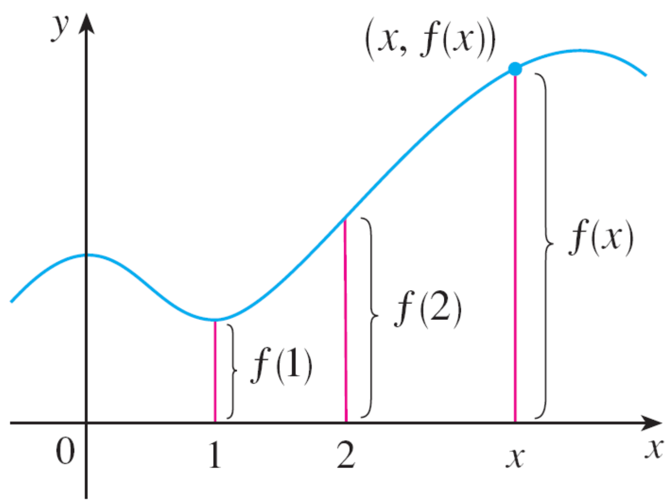


Figure 4

Four Ways to Represent a Function

The graph of *f* also allows us to picture the domain of *f* on the *x*-axis and its range on the *y*-axis as in Figure 5.

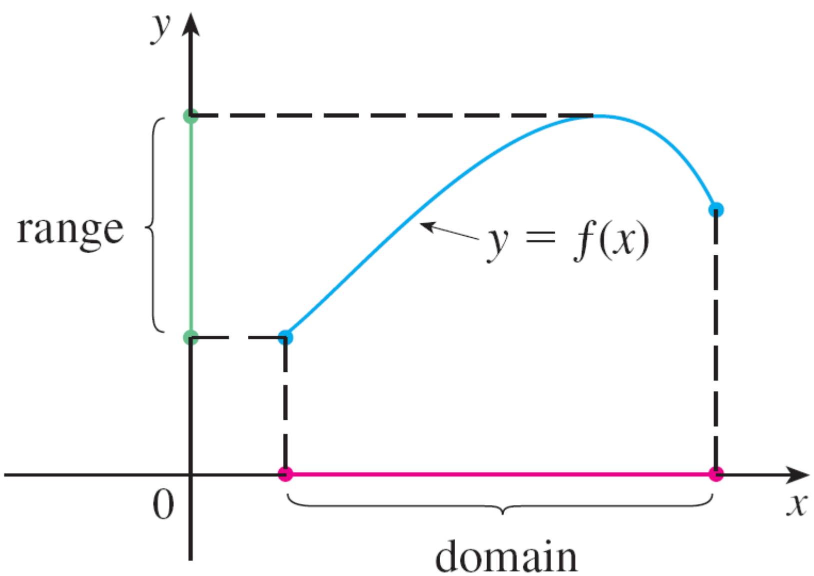


Figure 5

Example 1

The graph of a function f is shown in Figure 6.

(a) Find the values of $f(1)$ and $f(5)$.

(b) What are the domain and range of f ?

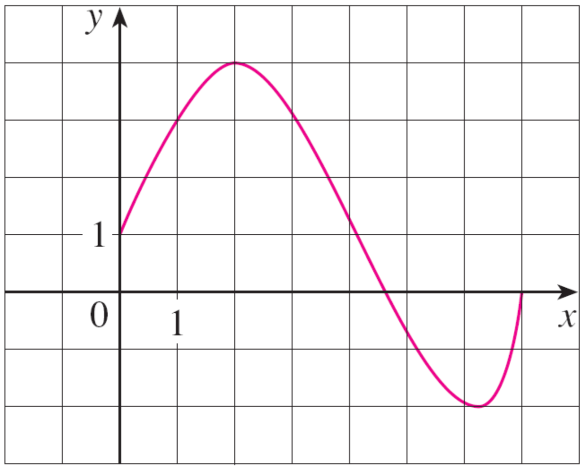


Figure 6

The notation for intervals is given in Appendix A.

Example 1 – *Solution*

(a) We see from Figure 6 that the point (1, 3) lies on the graph of f, so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x-axis.)

When $x = 5$, the graph lies about 0.7 unit below the x-axis, so we estimate that $f(5) \approx -0.7$.

(b) We see that $f(x)$ is defined when $0 \leq x \leq 7$, so the domain of f is the closed interval [0, 7]. Notice that f takes on all values from –2 to 4, so the range of f is

$$\{y \mid -2 \leq y \leq 4\} = [-2, 4]$$

Representations of Functions

Representations of Functions

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

Example 4

When you turn on a hot-water faucet, the temperature *T* of the water depends on how long the water has been running. Draw a rough graph of *T* as a function of the time *t* that has elapsed since the faucet was turned on.

Solution:

The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes.

When the water from the hot-water tank starts flowing from the faucet, *T* increases quickly. In the next phase, *T* is constant at the temperature of the heated water in the tank.

Example 4 – *Solution*

cont'd

When the tank is drained, T decreases to the temperature of the water supply.

This enables us to make the rough sketch of T as a function of t in Figure 11.

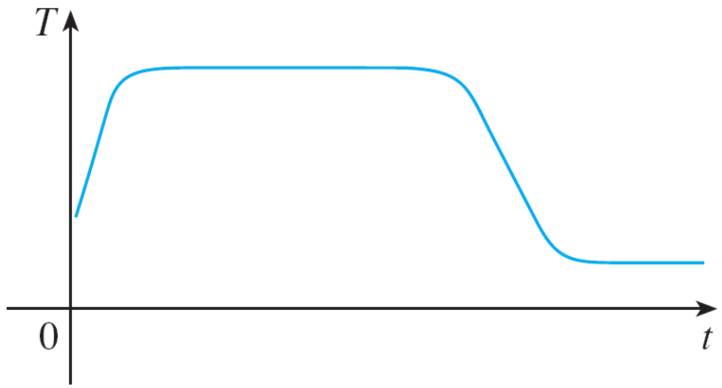


Figure 11

Representations of Functions

The graph of a function is a curve in the *xy*-plane. But the question arises: Which curves in the *xy*-plane are graphs of functions? This is answered by the following test.

The Vertical Line Test A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

The reason for the truth of the Vertical Line Test can be seen in Figure 13.

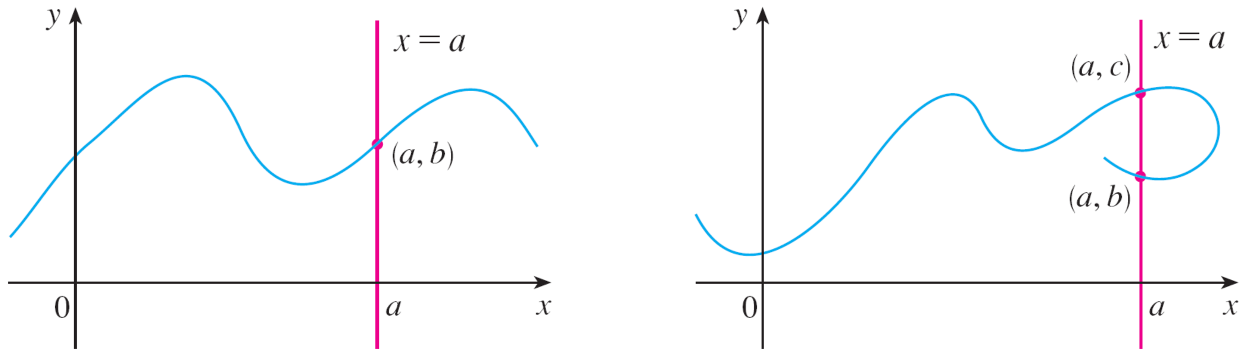


Figure 13

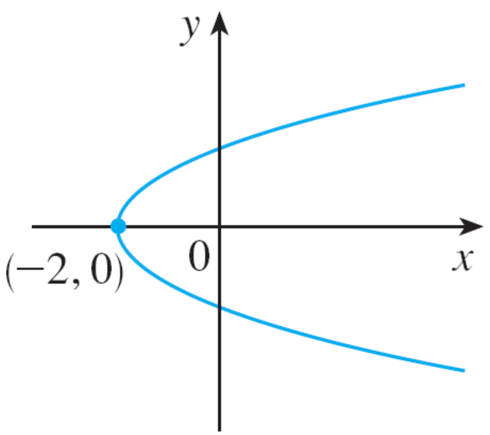
Representations of Functions

If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one functional value is defined by $f(a) = b$.

But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve can’t represent a function because a function can’t assign two different values to a.

Representations of Functions

For example, the parabola $x = y^2 - 2$ shown in Figure 14(a) is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice. The parabola, however, does contain the graphs of *two* functions of x.



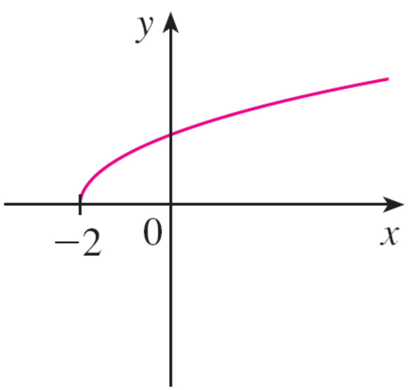
$x = y^2 - 2$

Figure 14(a)

Representations of Functions

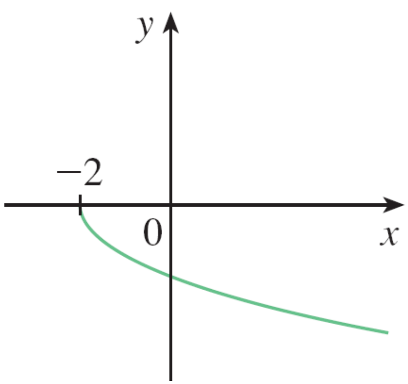
Notice that the equation $x = y^2 - 2$ implies $y^2 = x + 2$, so $y = \pm\sqrt{x+2}$.

Thus the upper and lower halves of the parabola are the graphs of the functions $f(x) = \sqrt{x+2}$ and $g(x) = -\sqrt{x+2}$ [See Figures 14(b) and (c).]



$y = \sqrt{x+2}$

Figure 14(b)



$y = -\sqrt{x+2}$

Figure 14(c)

Representations of Functions

We observe that if we reverse the roles of x and y, then the equation $x = h(y) = y^2 - 2$ *does* define x as a function of y (with y as the independent variable and x as the dependent variable) and the parabola now appears as the graph of the function h.

Piecewise Defined Functions

Example 7

A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

Solution:
Remember that a function is a rule. For this particular function the rule is the following:

First look at the value of the input x. If it happens that $x \leq -1$, then the value of $f(x)$ is $1 - x$.

Example 7 – *Solution*

cont'd

On the other hand, if x > –1, then the value of $f(x)$ is x^2.

Since (–2 ≤ –1, we have $f(-2) = 1 - (-2) = 3$.

Since –1 ≤ –1, we have $f(-1) = 1 - (-1) = 2$.

Since 0 > –1, we have $f(0) = 0^2 = 0$.

How do we draw the graph of f ? We observe that if $x \le -1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of the vertical line $x = -1$ must coincide with the line $y = 1 - x$, which has slope –1 and y-intercept 1.

Example 7 – *Solution*

cont'd

If $x > -1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = -1$ must coincide with the graph of $y = x^2$, which is a parabola. This enables us to sketch the graph in Figure 15.

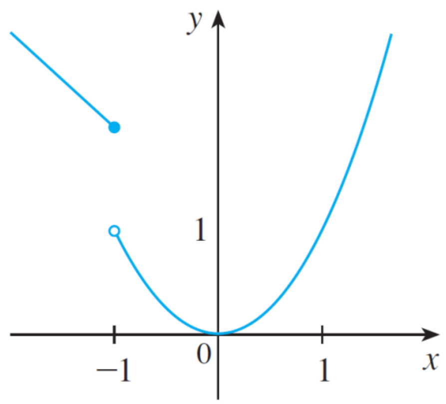


Figure 15

The solid dot indicates that the point $(-1, 2)$ is included on the graph; the open dot indicates that the point $(-1, 1)$ is excluded from the graph.

Piecewise Defined Functions

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geq 0 \qquad \text{for every number } a$$

For example,

$|3| = 3 \quad |-3| = 3 \quad |0| = 0 \quad |\sqrt{2} - 1| = \sqrt{2} - 1$

$|3 - \pi| = \pi - 3$

Piecewise Defined Functions

In general, we have

$$\boxed{\begin{aligned} |a| &= a \qquad \text{if } a \geqslant 0 \\ |a| &= -a \quad \text{if } a < 0 \end{aligned}}$$

(Remember that if a is negative, then $-a$ is positive.)

Example 8

Sketch the graph of the absolute value function $f(x) = |x|$.

Solution:

From the preceding discussion we know that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example 8 – *Solution*

cont'd

Using the same method as in Example 7, we see that the graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the line $y = -x$ to the left of the y-axis (see Figure 16).

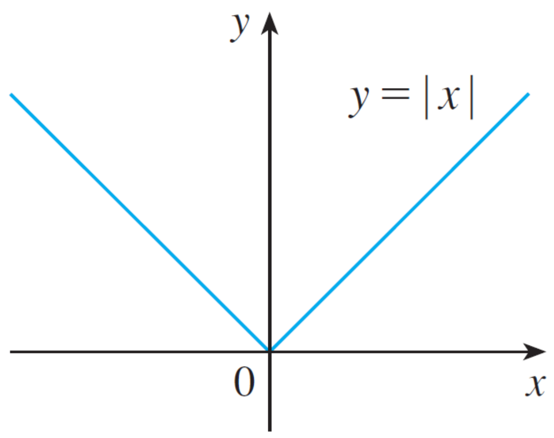


Figure 16

Example 10

In Example C at the beginning of this section we considered the cost $C(w)$ of mailing a large envelope with weight w.

In effect, this is a piecewise defined function because, from the table of values on page 13, we have

$$C(w) = \begin{cases} 0.88 & \text{if } 0 < w \le 1 \\ 1.05 & \text{if } 1 < w \le 2 \\ 1.22 & \text{if } 2 < w \le 3 \\ 1.39 & \text{if } 3 < w \le 4 \\ \vdots & \end{cases}$$

Example 10

cont'd

The graph is shown in Figure 18.

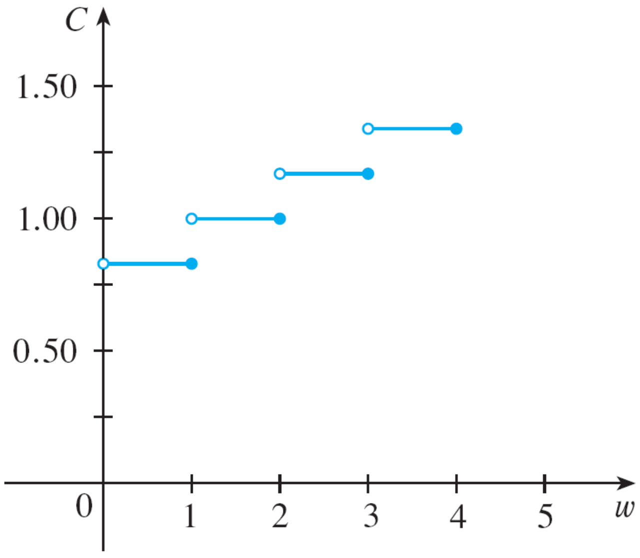


Figure 18

You can see why functions similar to this one are called **step functions**—they jump from one value to the next.

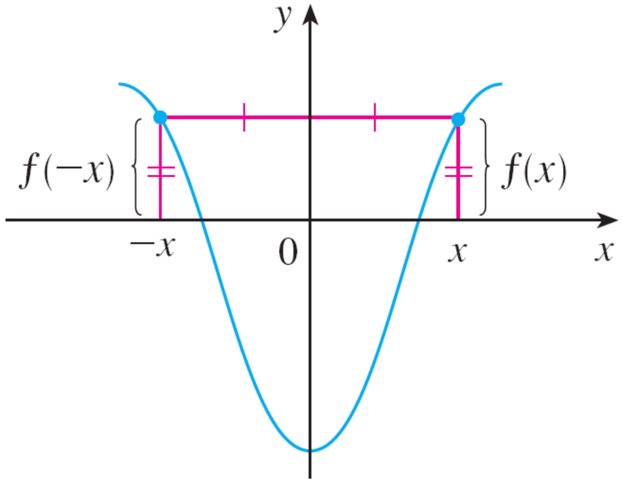
Symmetry

Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect to the y-axis (see Figure 19).



An even function

Figure 19

Symmetry

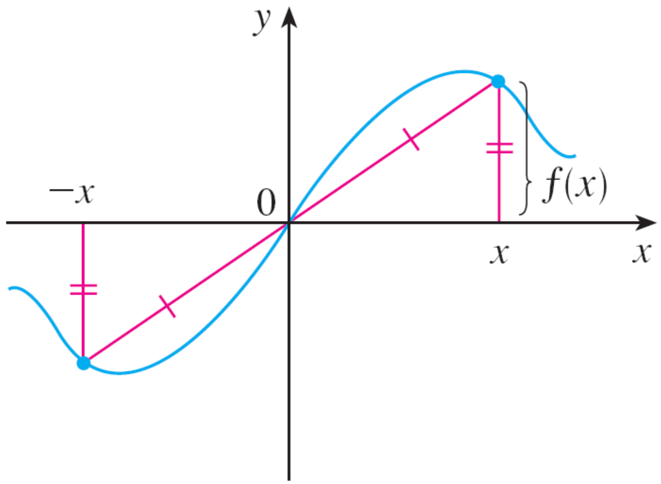
This means that if we have plotted the graph of f for $x \geq 0$, we obtain the entire graph simply by reflecting this portion about the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

Symmetry

The graph of an odd function is symmetric about the origin (see Figure 20).



An odd function

Figure 20

If we already have the graph of *f* for $x \geq 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

Example 11

Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ **(b)** $g(x) = 1 - x^4$ **(c)** $h(x) = 2x - x^2$

Solution:

(a) $f(-x) = (-x)^5 + (-x) = (-1)^5 x^5 + (-x)$

$= -x^5 - x = -(x^5 + x)$

$= -f(x)$

Therefore f is an odd function.

Example 11 – *Solution*

cont'd

(b) $g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$

So g is even.

(c) $h(-x) = 2(-x) - (-x)^2 = -2x - x^2$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd.

Symmetry

The graphs of the functions in Example 11 are shown in Figure 21. Notice that the graph of *h* is symmetric neither about the *y*-axis nor about the origin.

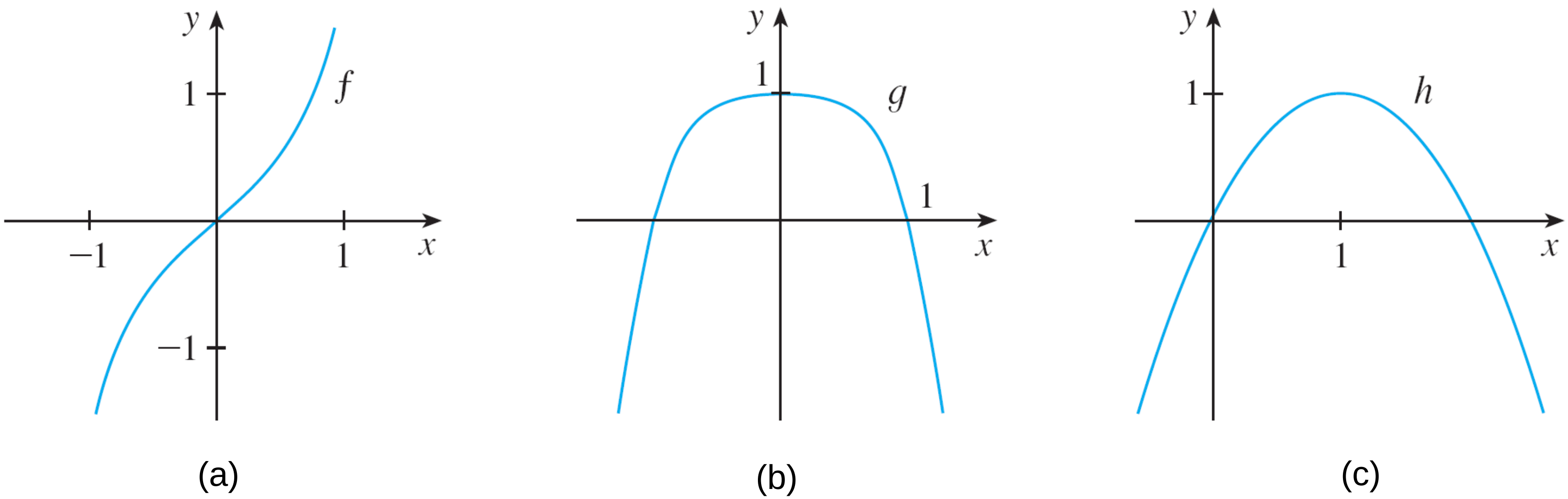


Figure 21

Increasing and Decreasing Functions

Increasing and Decreasing Functions

The graph shown in Figure 22 rises from *A* to *B*, falls from *B* to *C*, and rises again from *C* to *D*. The function *f* is said to be increasing on the interval [*a*, *b*], decreasing on [*b*, *c*], and increasing again on [*c*, *d*].

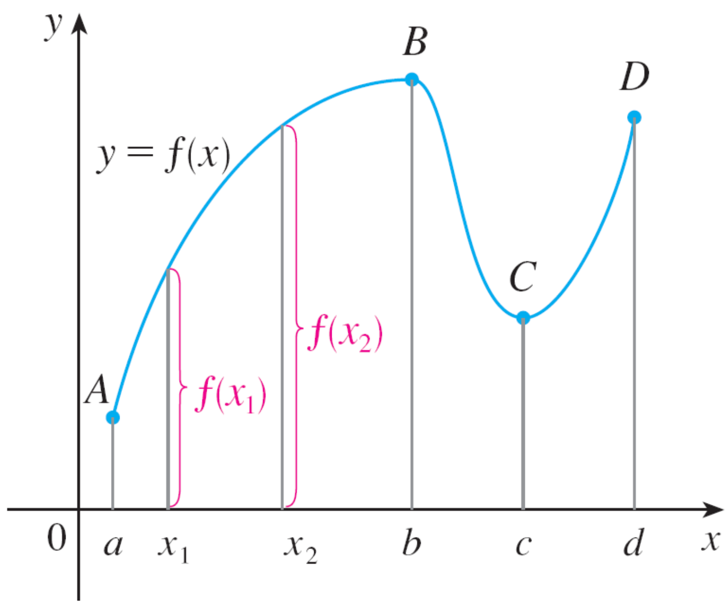


Figure 22

Increasing and Decreasing Functions

Notice that if x_1 and x_2 are any two numbers between a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$.

We use this as the defining property of an increasing function.

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

Increasing and Decreasing Functions

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

You can see from Figure 23 that the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

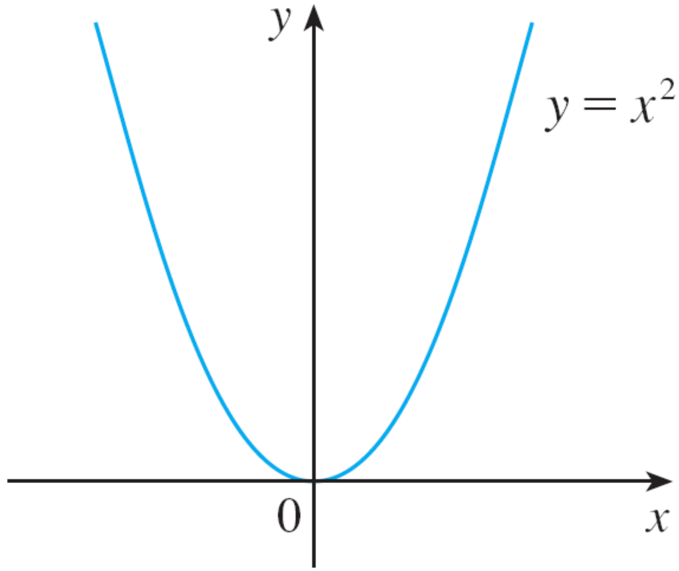


Figure 23

1

Functions and Models

1.2

Mathematical Models: A Catalog of Essential Functions

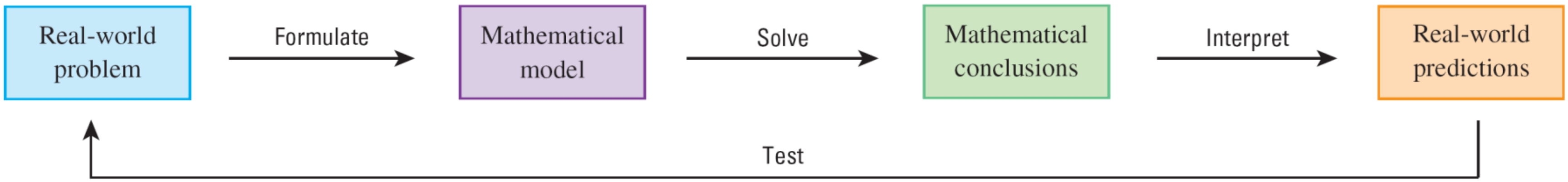
Mathematical Models: A Catalog of Essential Functions

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emission reductions.

The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Mathematical Models: A Catalog of Essential Functions

Figure 1 illustrates the process of mathematical modeling.



The modeling process

Figure 1

Mathematical Models: A Catalog of Essential Functions

A mathematical model is never a completely accurate representation of a physical situation—it is an *idealization.* A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions.

It is important to realize the limitations of the model. In the end, Mother Nature has the final say.

There are many different types of functions that can be used to model relationships observed in the real world. In what follows, we discuss the behavior and graphs of these functions and give examples of situations appropriately modeled by such functions.

Linear Models

Linear Models

When we say that y is a **linear function** of x, we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as

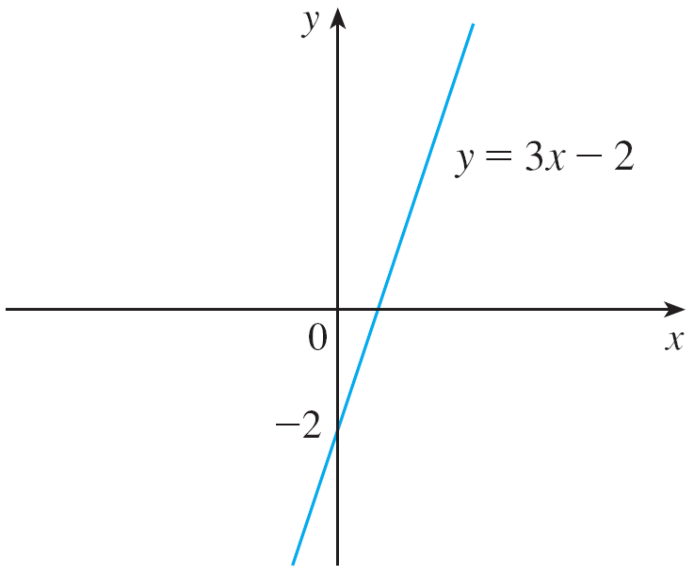
$$y = f(x) = mx + b$$

where m is the slope of the line and b is the y-intercept.

Linear Models

A characteristic feature of linear functions is that they grow at a constant rate.

For instance, Figure 2 shows a graph of the linear function *f*(*x*) = 3*x* – 2 and a table of sample values.



x	$f(x) = 3x - 2$
1.0	1.0
1.1	1.3
1.2	1.6
1.3	1.9
1.4	2.2
1.5	2.5

Figure 2

Linear Models

Notice that whenever x increases by 0.1, the value of $f(x)$ increases by 0.3.

So $f(x)$ increases three times as fast as x. Thus the slope of the graph $y = 3x - 2$, namely 3, can be interpreted as the rate of change of y with respect to x.

Example 1

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature T (in °C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.

(b) Draw the graph of the function in part (a). What does the slope represent?

(c) What is the temperature at a height of 2.5 km?

Example 1(a) – *Solution*

Because we are assuming that T is a linear function of h, we can write

$$T = mh + b$$

We are given that $T = 20$ when $h = 0$, so

$$20 = m \cdot 0 + b = b$$

In other words, the y-intercept is $b = 20$.

We are also given that $T = 10$ when $h = 1$, so

$$10 = m \cdot 1 + 20$$

The slope of the line is therefore $m = 10 – 20 = –10$ and the required linear function is

$$T = –10h + 20$$

Example 1(b) – *Solution*

cont'd

The graph is sketched in Figure 3.

The slope is m = –10°C/km, and this represents the rate of change of temperature with respect to height.

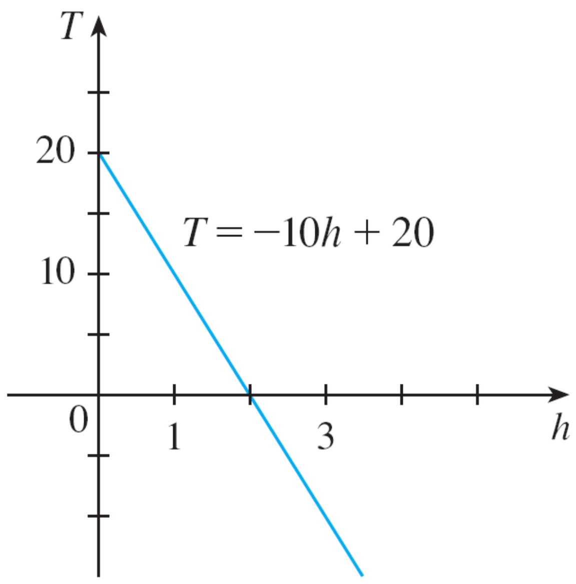


Figure 3

Example 1(c) – *Solution*

cont'd

At a height of $h = 2.5$ km, the temperature is

$$T = -10(2.5) + 20 = \mathbf{-5°C}$$

Linear Models

If there is no physical law or principle to help us formulate a model, we construct an **empirical model**, which is based entirely on collected data.

We seek a curve that “fits” the data in the sense that it captures the basic trend of the data points.

Polynomials

Polynomials

A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial.

The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is n. For example, the function

$$P(x) = 2x^6 - x^4 + \tfrac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

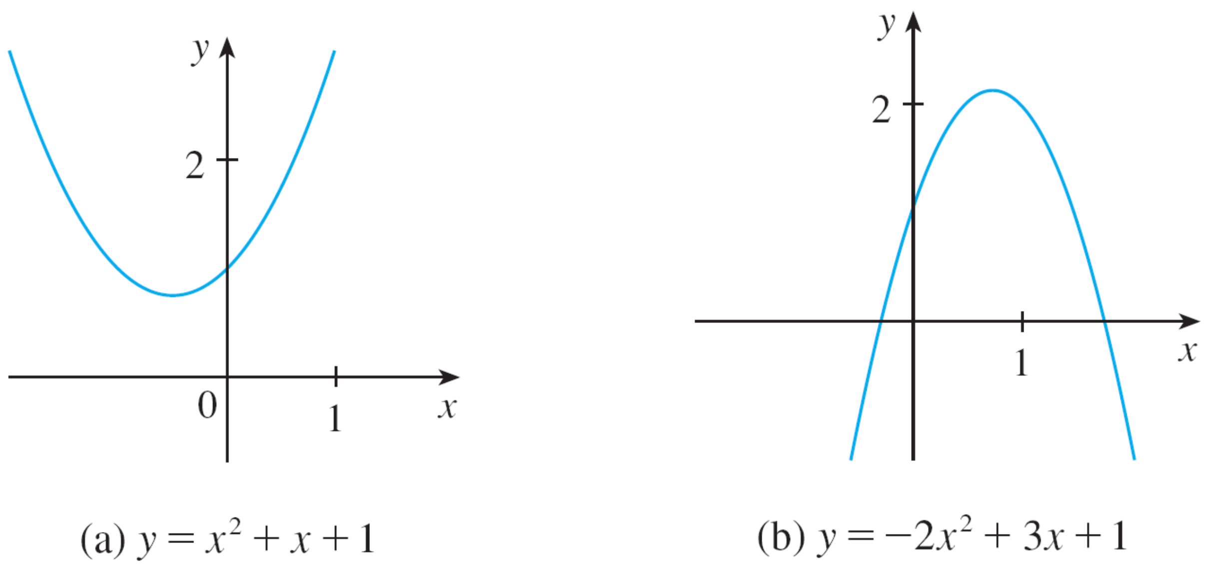
Polynomials

A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function.

A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**.

Polynomials

Its graph is always a parabola obtained by shifting the parabola $y = ax^2$. The parabola opens upward if $a > 0$ and downward if $a < 0$. (See Figure 7.)



The graphs of quadratic functions are parabolas.

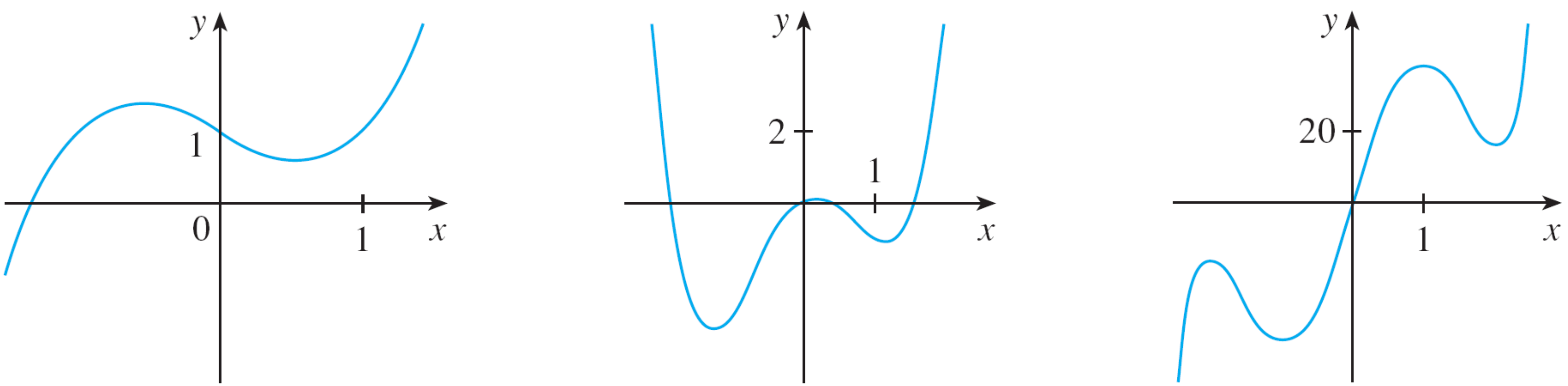
Figure 7

Polynomials

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \qquad a \neq 0$$

and is called a **cubic function**. Figure 8 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c).



(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

Figure 8

Example 4

A ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground, and its height h above the ground is recorded at 1-second intervals in Table 2.

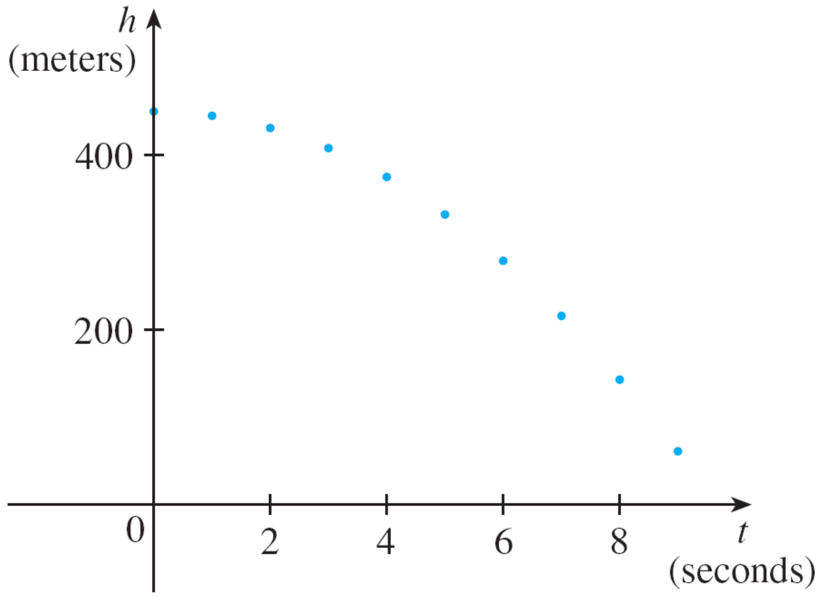
Find a model to fit the data and use the model to predict the time at which the ball hits the ground.

TABLE 2

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

Example 4 – *Solution*

We draw a scatter plot of the data in Figure 9 and observe that a linear model is inappropriate.



Scatter plot for a falling ball

Figure 9

Example 4 – *Solution*

cont'd

But it looks as if the data points might lie on a parabola, so we try a quadratic model instead.

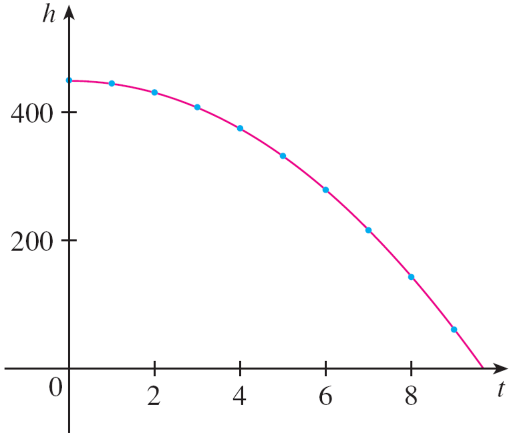
Using a graphing calculator or computer algebra system (which uses the least squares method), we obtain the following quadratic model:

$$\boxed{3} \qquad h = 449.36 + 0.96t - 4.90t^2$$

Example 4 – *Solution*

cont'd

In Figure 10 we plot the graph of Equation 3 together with the data points and see that the quadratic model gives a very good fit.



Quadratic model for a falling ball

Figure 10

The ball hits the ground when $h = 0$, so we solve the quadratic equation

$$-4.90t^2 + 0.96t + 449.36 = 0$$

Example 4 – *Solution*

cont'd

The quadratic formula gives

$$t = \frac{-0.96 \pm \sqrt{(0.96)^2 - 4(-4.90)(449.36)}}{2(-4.90)}$$

The positive root is $t \approx 9.67$, so we predict that the ball will hit the ground after about 9.7 seconds.

Power Functions

Power Functions

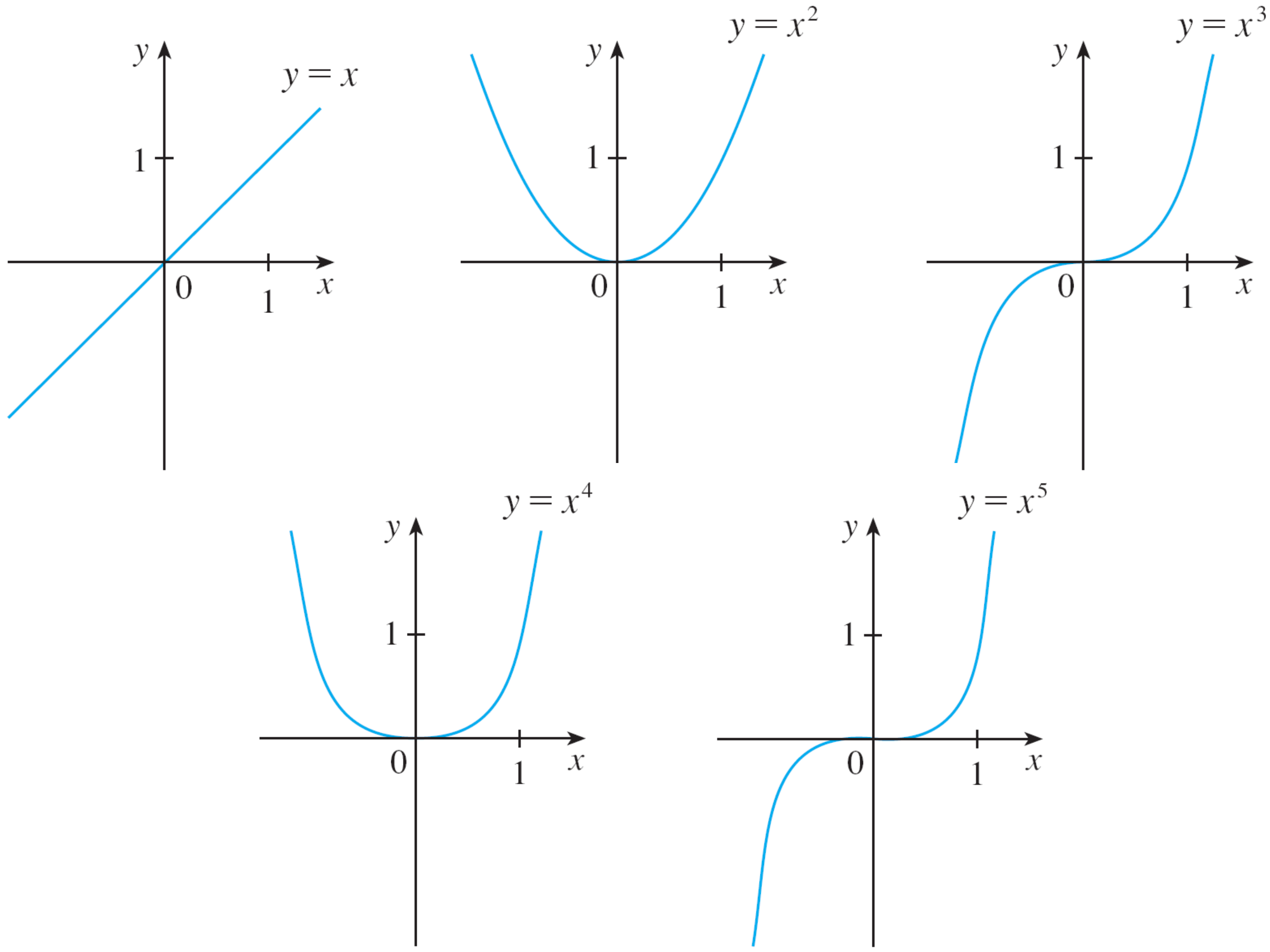
A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider several cases.

(i) ***a* = *n*, where *n* is a positive integer**

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4,$ and 5 are shown in Figure 11. (These are polynomials with only one term.)

We already know the shape of the graphs of $y = x$ (a line through the origin with slope 1) and $y = x^2$ (a parabola).

Power Functions



Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

Figure 11

Power Functions

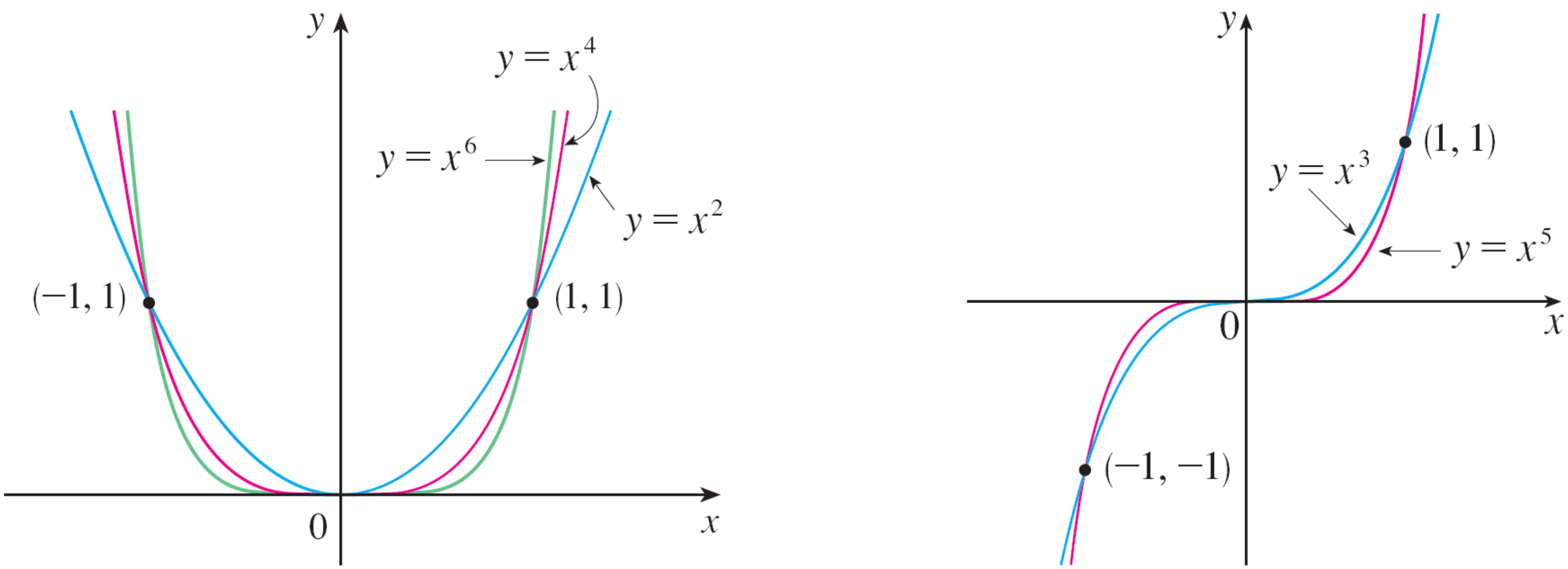
The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.

If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$.

If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$.

Power Functions

Notice from Figure 12, however, that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geq 1$. (If x is small, then x^2 is smaller, x^3 is even smaller, x^4 is smaller still, and so on.)



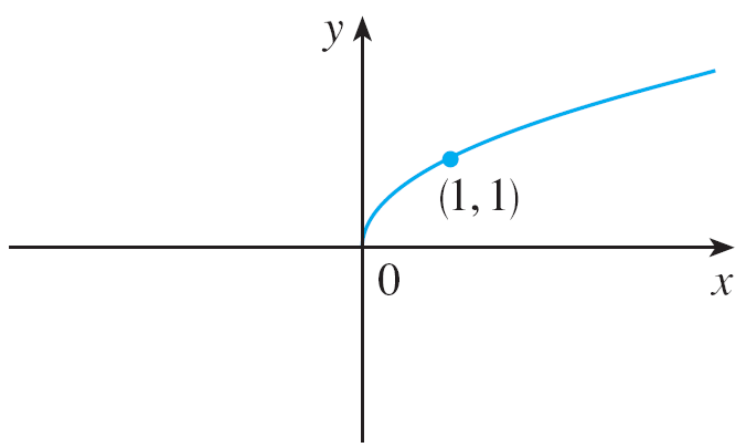
Families of power functions

Figure 12

Power Functions

(ii) ***a* = 1/*n*, where *n* is a positive integer**

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For *n* = 2 it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$. [See Figure 13(a).]



$f(x) = \sqrt{x}$

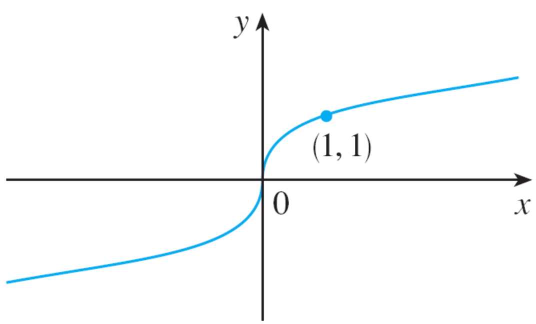
Graph of root function

Figure 13(a)

Power Functions

For other even values of *n*, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$.

For *n* = 3 we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$ (recall that every real number has a cube root) and whose graph is shown in Figure 13(b). The graph of $y = \sqrt[n]{x}$ for *n* odd (*n* > 3) is similar to that of $y = \sqrt[3]{x}$.



$f(x) = \sqrt[3]{x}$

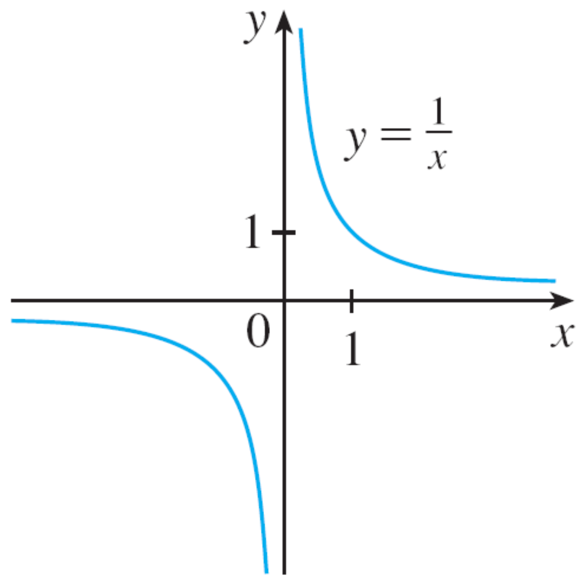
Graph of root function

Figure 13(b)

Power Functions

(iii) ***a* = –1**

The graph of the **reciprocal function** $f(x) = x^{-1} = 1/x$ is shown in Figure 14. Its graph has the equation $y = 1/x$, or $xy = 1$, and is a hyperbola with the coordinate axes as its asymptotes.



The reciprocal function

Figure 14

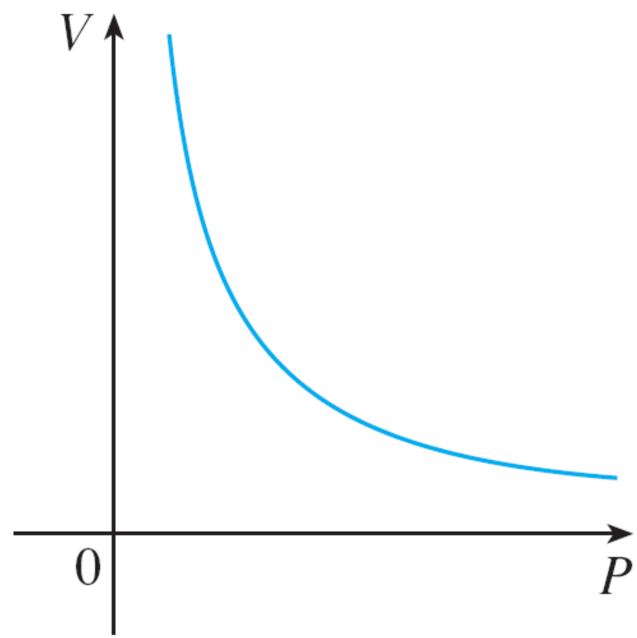
Power Functions

This function arises in physics and chemistry in connection with Boyle's Law, which says that, when the temperature is constant, the volume V of a gas is inversely proportional to the pressure P:

$$V = \frac{C}{P}$$

where C is a constant.

Thus the graph of V as a function of P (see Figure 15) has the same general shape as the right half of Figure 14.



Volume as a function of pressure at constant temperature

Figure 15

Rational Functions

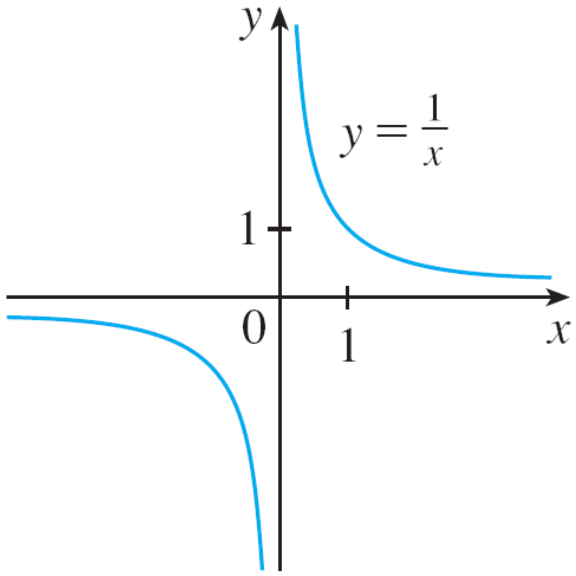
Rational Functions

A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain consists of all values of x such that $Q(x) \neq 0$.

A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in Figure 14.



The reciprocal function

Figure 14

Rational Functions

The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 16.

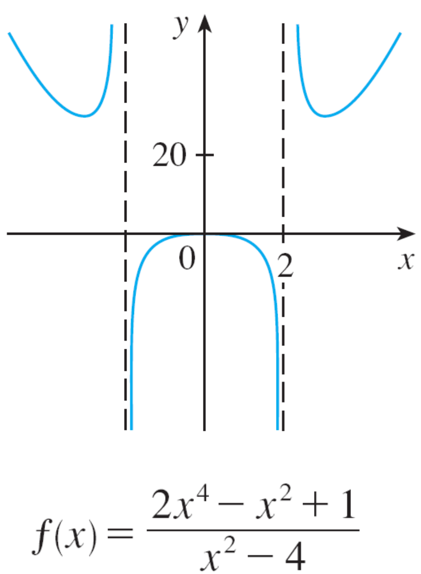


Figure 16

Algebraic Functions

Algebraic Functions

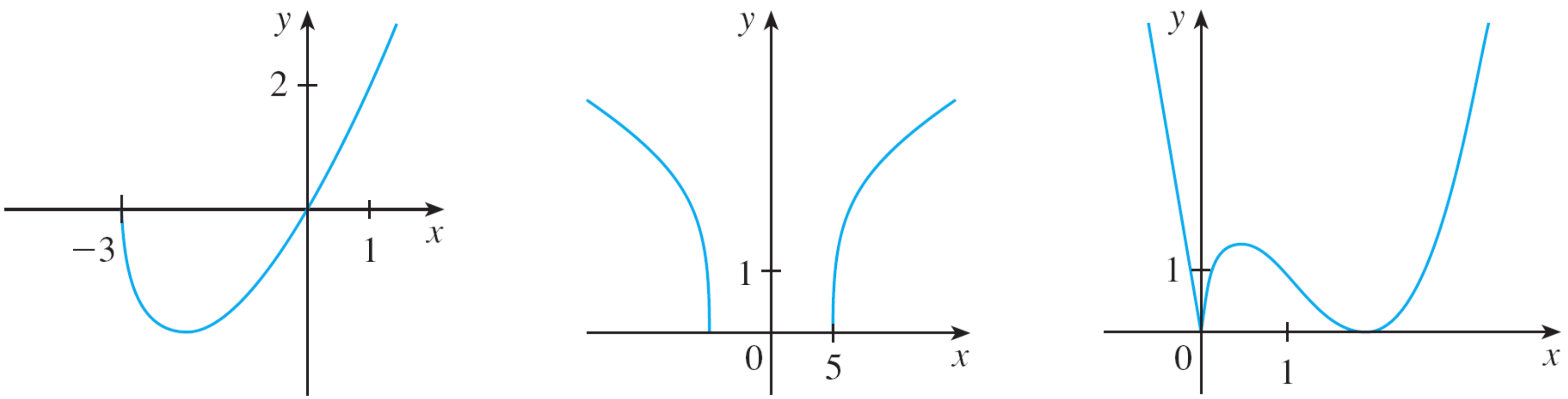
A function *f* is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function.

Here are two more examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

Algebraic Functions

The graphs of algebraic functions can assume a variety of shapes. Figure 17 illustrates some of the possibilities.



(a) $f(x) = x\sqrt{x+3}$ (b) $g(x) = \sqrt[4]{x^2 - 25}$ (c) $h(x) = x^{2/3}(x-2)^2$

Figure 17

Algebraic Functions

An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity *v* is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and
$c = 3.0 \times 10^5$ km/s is the speed of light in a vacuum.

Trigonometric Functions

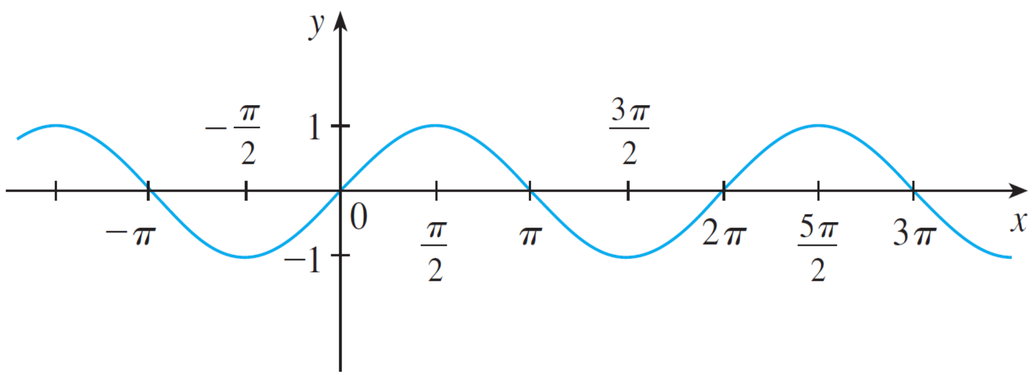
Trigonometric Functions

In calculus the convention is that radian measure is always used (except when otherwise indicated).

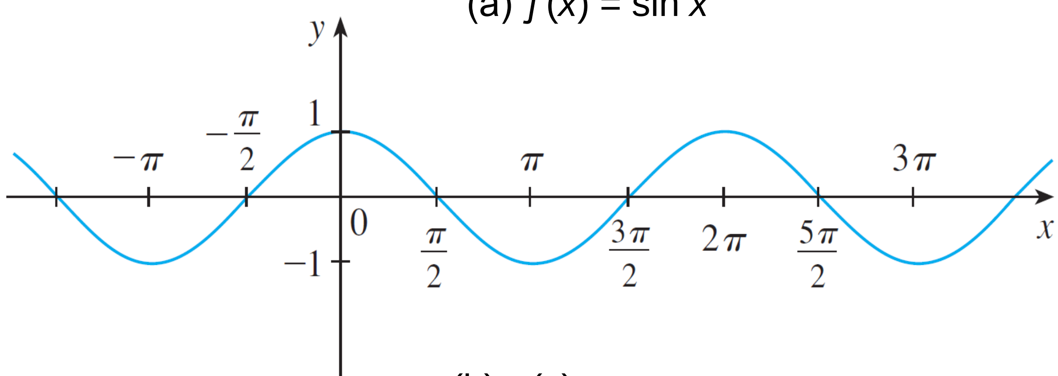
For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x.

Trigonometric Functions

Thus the graphs of the sine and cosine functions are as shown in Figure 18.



(a) $f(x) = \sin x$



(b) $g(x) = \cos x$

Figure 18

Trigonometric Functions

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval [–1, 1].

Thus, for all values of *x*, we have

$$\boxed{-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1}$$

or, in terms of absolute values,

$$|\sin x| \leq 1 \qquad |\cos x| \leq 1$$

Trigonometric Functions

Also, the zeros of the sine function occur at the integer multiples of π; that is,

$$\sin x = 0 \qquad \text{when} \qquad x = n\pi \qquad n \text{ an integer}$$

An important property of the sine and cosine functions is that they are periodic functions and have period 2π.

This means that, for all values of x,

$$\boxed{\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x}$$

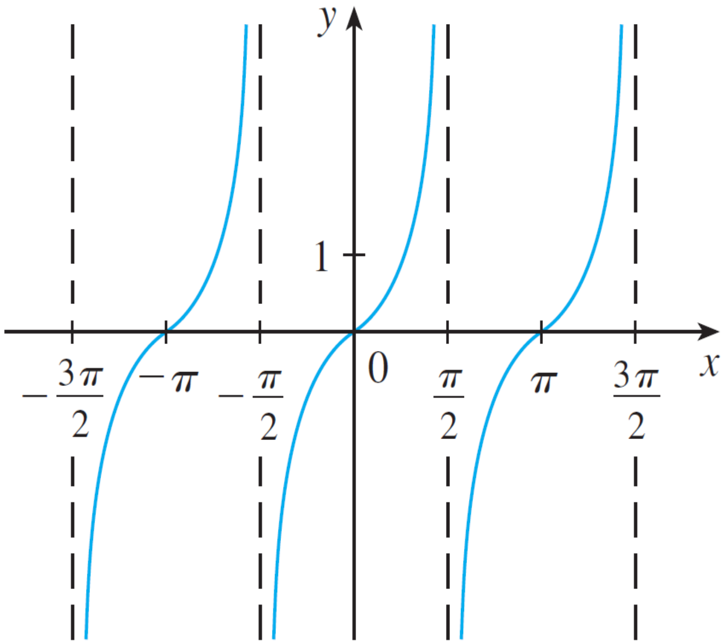
Trigonometric Functions

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

and its graph is shown in Figure 19. It is undefined whenever cos x = 0, that is, when $x = \pm\pi/2, \pm 3\pi/2, \ldots$.

Its range is $(-\infty, \infty)$.



$y = \tan x$

Figure 19

Trigonometric Functions

Notice that the tangent function has period π:

$$\tan(x + \pi) = \tan x \qquad \text{for all } x$$

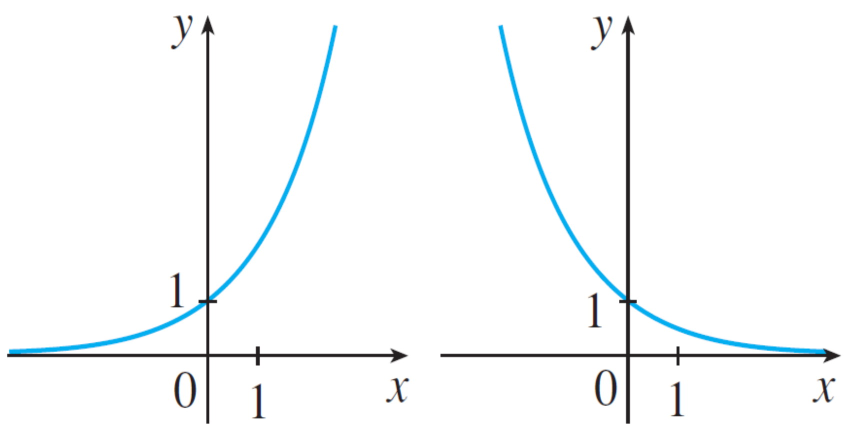
The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions.

Exponential Functions

Exponential Functions

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant.

The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure 20. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.



(a) $y = 2^x$ (b) $y = (0.5)^x$

Figure 20

Exponential Functions

Exponential functions are useful for modeling many natural phenomena, such as population growth (if $a > 1$) and radioactive decay (if $a < 1$).

Logarithmic Functions

Logarithmic Functions

The **logarithmic functions** $f(x) = \log_a x$, where the base a is a positive constant, are the inverse functions of the exponential functions. Figure 21 shows the graphs of four logarithmic functions with various bases.

In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

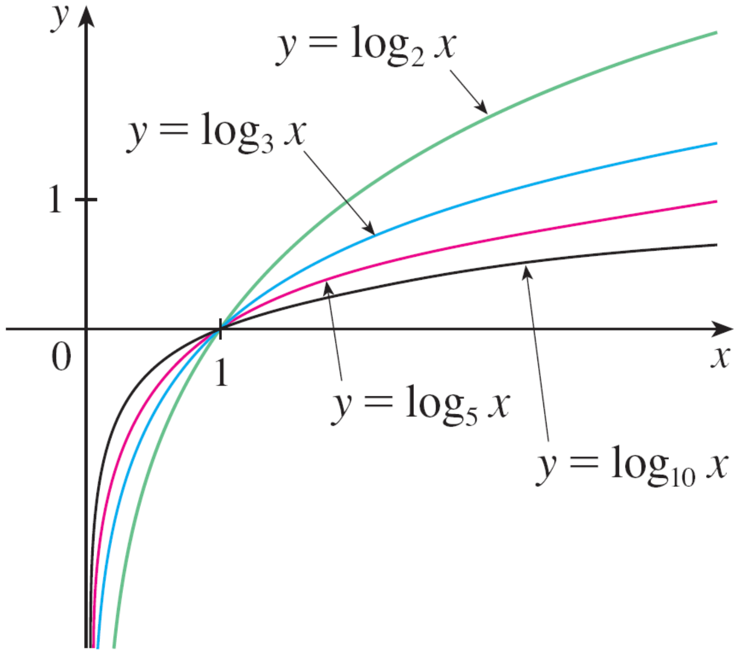


Figure 21

Example 5

Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

Example 5 – *Solution*

(a) $f(x) = 5^x$ is an exponential function.
(The x is the exponent.)

(b) $g(x) = x^5$ is a power function. (The x is the base.)
We could also consider it to be a polynomial of degree 5.

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$ is an algebraic function.

(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4.

1 Functions and Models

1.3 New Functions from Old Functions

Transformations of Functions

Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of certain related functions.

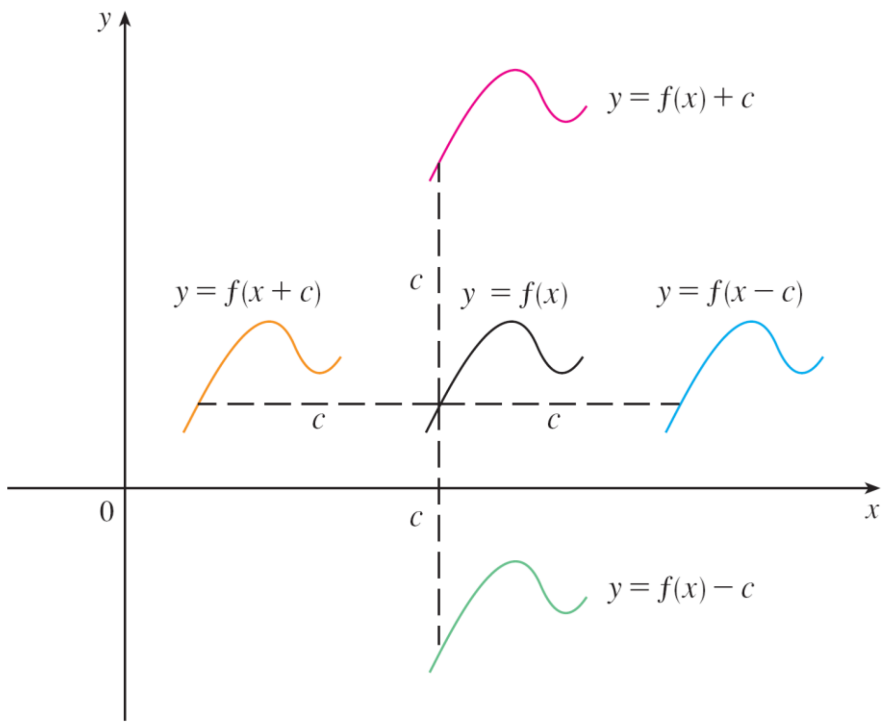
This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.

Let's first consider **translations**. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y-coordinate is increased by the same number c).

Transformations of Functions

Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x).

Therefore the graph of $y = f(x - c)$, is just the graph of $y = f(x)$ shifted c units to the right (see Figure 1).



Translating the graph of f

Figure 1

Transformations of Functions

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

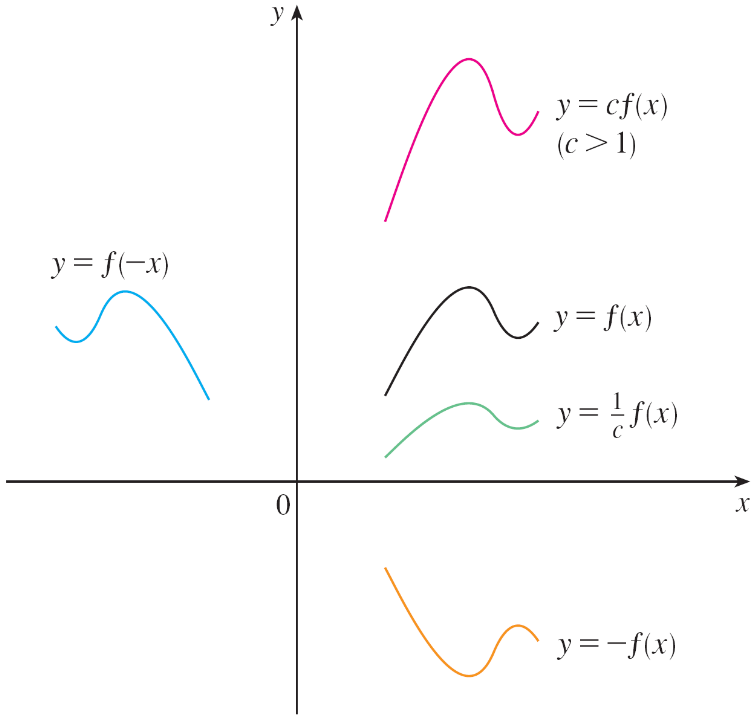
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Now let's consider the **stretching** and **reflecting** transformations. If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y-coordinate is multiplied by the same number c).

Transformations of Functions

The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis because the point (x, y) is replaced by the point $(x, -y)$.

(See Figure 2 and the following chart, where the results of other stretching, shrinking, and reflecting transformations are also given.)



Stretching and reflecting the graph of *f*

Figure 2

Transformations of Functions

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c

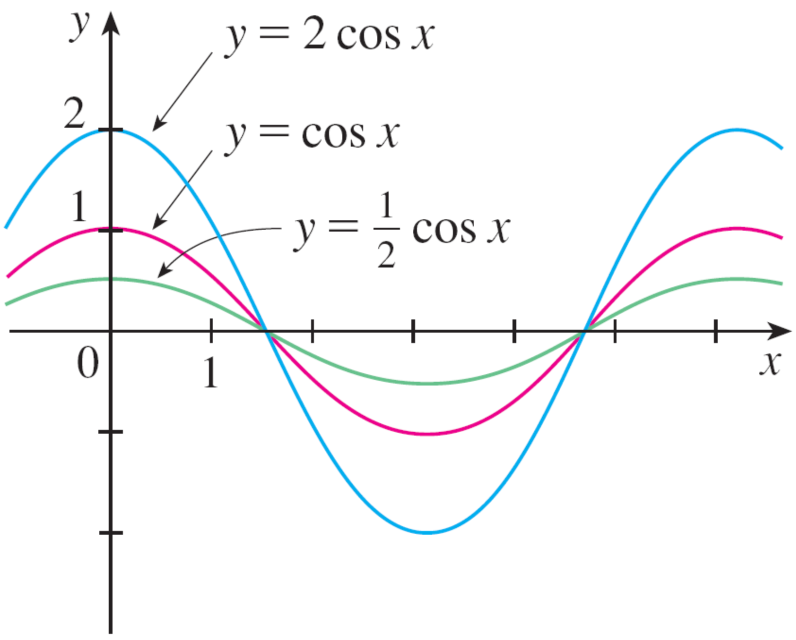
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Transformations of Functions

Figure 3 illustrates these stretching transformations when applied to the cosine function with c = 2.



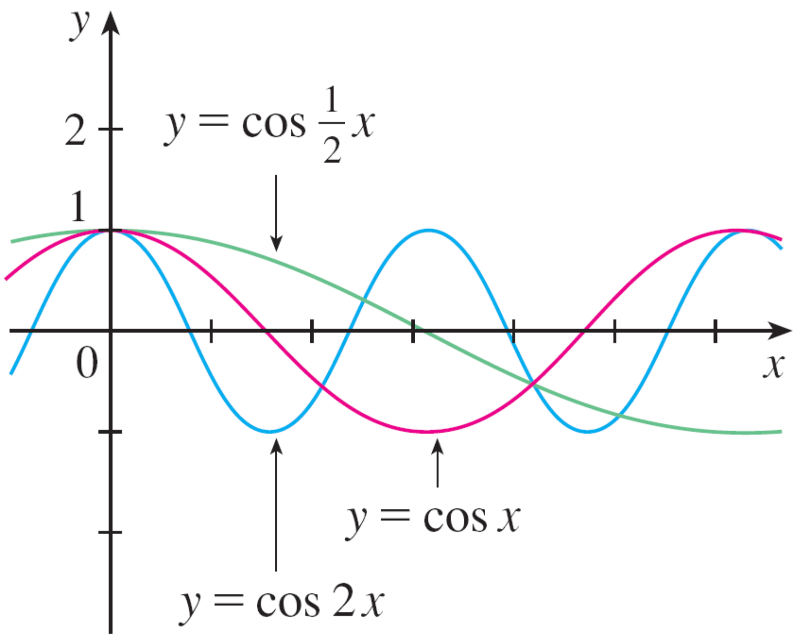


Figure 3

Transformations of Functions

For instance, in order to get the graph of $y = 2 \cos x$ we multiply the y-coordinate of each point on the graph of $y = \cos x$ by 2.

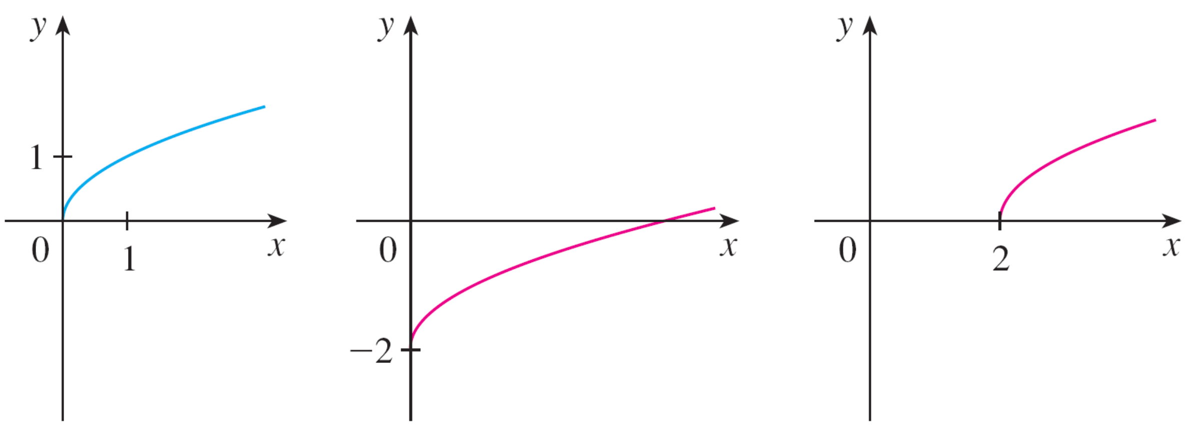
This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

Example 1

Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

Solution:

The graph of the square root function $y = \sqrt{x}$, is shown in Figure 4(a).



(a) $y = \sqrt{x}$ (b) $y = \sqrt{x} - 2$ (c) $y = \sqrt{x - 2}$

Figure 4

Example 1 – *Solution*

cont'd

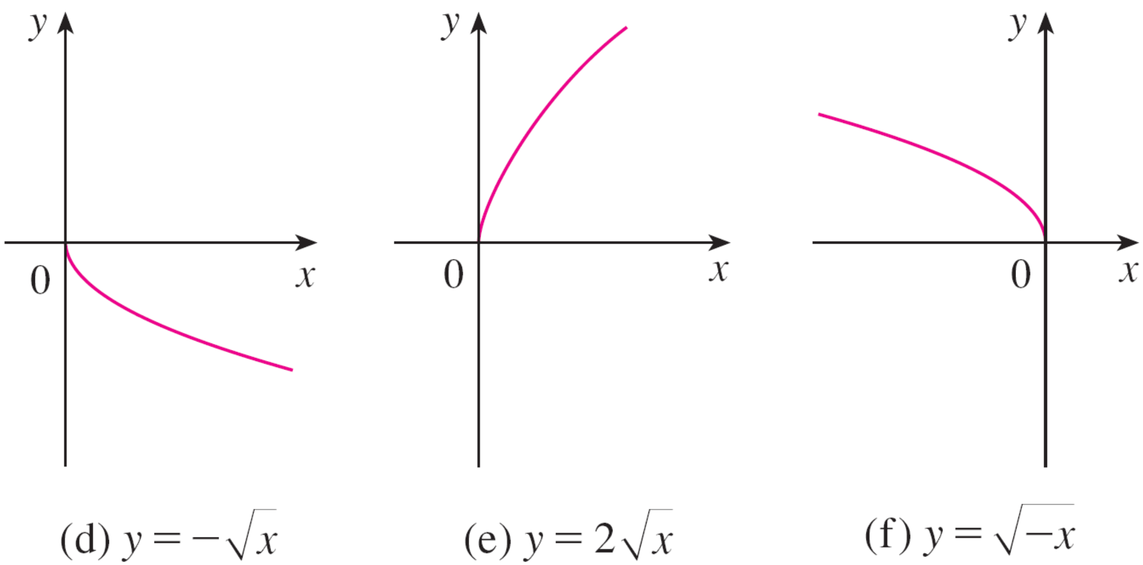


Figure 4

In the other parts of the figure we sketch $y = \sqrt{x} - 2$ by shifting 2 units downward, $y = \sqrt{x - 2}$ by shifting 2 units to the right, $y = -\sqrt{x}$ by reflecting about the *x*-axis, $y = 2\sqrt{x}$ by stretching vertically by a factor of 2, and $y = \sqrt{-x}$ by reflecting about the *y*-axis.

Transformations of Functions

Another transformation of some interest is taking the *absolute value* of a function. If $y = |f(x)|$, then according to the definition of absolute value, $y = f(x)$ when $f(x) \geq 0$ and $y = -f(x)$ when $f(x) < 0$.

This tells us how to get the graph of $y = |f(x)|$ from the graph of $y = f(x)$: The part of the graph that lies above the x-axis remains the same; the part that lies below the x-axis is reflected about the x-axis.

Combinations of Functions

Combinations of Functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. The sum and difference functions are defined by

$$(f + g)(x) = f(x) + g(x) \qquad\qquad (f - g)(x) = f(x) - g(x)$$

If the domain of f is A and the domain of g is B, then the domain of $f + g$ is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined.

For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}$ is $A \cap B = [0, 2]$.

Combinations of Functions

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of fg is $A \cap B$, but we can't divide by 0 and so the domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$.

For instance, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $(f/g)(x) = x^2/(x - 1)$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

Combinations of Functions

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$.

Since y is a function of u and u is, in turn, a function of x, it follows that y is ultimately a function of x. We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g.

Combinations of Functions

In general, given any two functions *f* and *g*, we start with a number *x* in the domain of *g* and find its image *g*(*x*). If this number *g*(*x*) is in the domain of *f*, then we can calculate the value of *f*(*g*(*x*)).

The result is a new function *h*(*x*) = *f*(*g*(*x*)) obtained by substituting *g* into *f*. It is called the *composition* (or *composite*) of *f* and *g and* is denoted by $f \circ g$ ("*f* circle *g*").

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

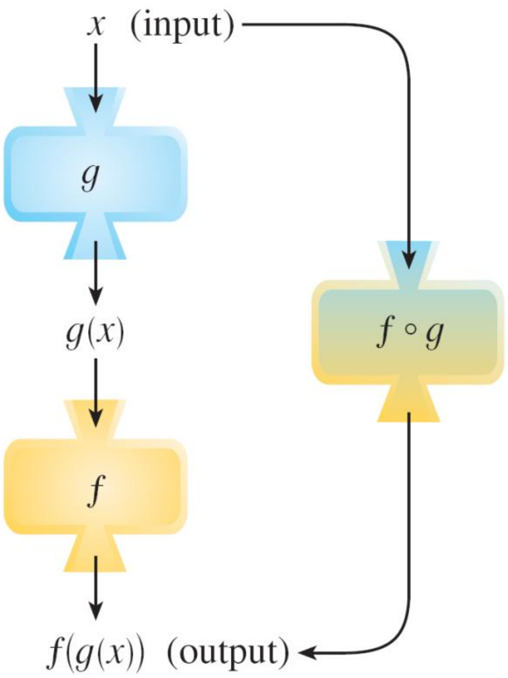
$$(f \circ g)(x) = f(g(x))$$

Combinations of Functions

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

Figure 11 shows how to picture $f \circ g$ in terms of machines.



The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

Figure 11

Example 6

If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

Solution:

We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

Combinations of Functions

Remember, the notation $f \circ g$ means that the function g is applied first and then f is applied second. In Example 6, $f \circ g$ is the function that *first* subtracts 3 and *then* squares; $g \circ f$ is the function that *first* squares and *then* subtracts 3.

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

1 Functions and Models

1.4 Graphing Calculators and Computers

Graphing Calculators and Computers

Graphing calculators and computers can give very accurate graphs of functions.

A graphing calculator or computer displays a rectangular portion of the graph of a function in a **display window** or **viewing screen**, which we refer to as a **viewing rectangle**.

The default screen often gives an incomplete or misleading picture, so it is important to choose the viewing rectangle with care.

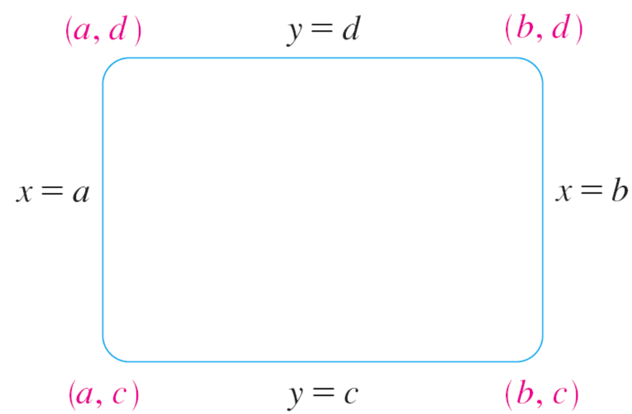
Graphing Calculators and Computers

If we choose the *x*-values to range from a minimum value of $X\,min = a$ to a maximum value of $X\,max = b$ and the *y*-values to range from a minimum of $Y\,min = c$ to a maximum of $Y\,max = d$, then the visible portion of the graph lies in the rectangle

$$[a, b] \times [c, d] = \{(x, y) \mid a \le x \le b,\ c \le y \le d\}$$

shown in Figure 1.

We refer to this rectangle as the [*a*, *b*] *by* [*c*, *d*] *viewing rectangle.*



The viewing rectangle [*a*, *b*] by [*c*, *d*]

Figure 1

Graphing Calculators and Computers

The machine draws the graph of a function *f* much as you would.

It plots points of the form $(x, f(x))$ for a certain number of equally spaced values of x between a and b.

If an x-value is not in the domain of *f*, or if $f(x)$ lies outside the viewing rectangle, it moves on to the next x-value.

The machine connects each point to the preceding plotted point to form a representation of the graph of *f*.

Example 1

Draw the graph of the function $f(x) = x^2 + 3$ in each of the following viewing rectangles.

(a) [–2, 2] by [–2, 2]

(b) [–4, 4] by [–4, 4]

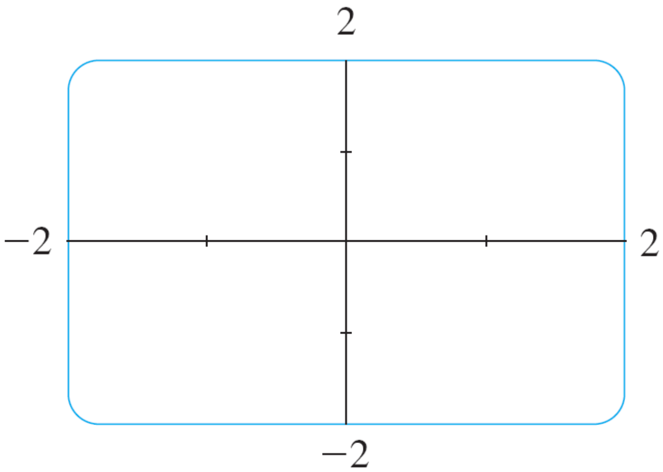
(c) [–10, 10] by [–5, 30]

(d) [–50, 50] by [–100, 1000]

Example 1 – *Solution*

For part (a) we select the range by setting $X min = -2$, $X max = 2$, $Y min = -2$, and $Y max = 2$.

The resulting graph is shown in Figure 2(a). The display window is blank!



[–2, 2] by [–2, 2]

Graph of $f(x) = x^2 + 3$

Figure 2(a)

Example 1 – *Solution*

cont'd

A moment's thought provides the explanation: Notice that $x^2 \geq 0$ for all x, so $x^2 + 3 \geq 3$ for all x.

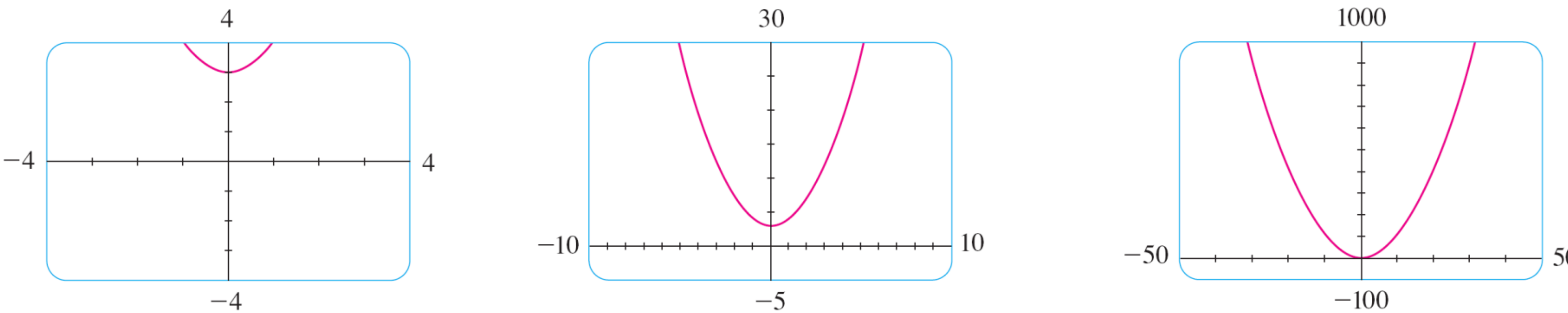
Thus the range of the function $f(x) = x^2 + 3$ is $[3, \infty)$.

This means that the graph of f lies entirely outside the viewing rectangle $[-2, 2]$ by $[-2, 2]$.

Example 1 – *Solution*

cont'd

The graphs for the viewing rectangles in parts (b), (c), and (d) are also shown in Figure 2.



(b) [–4, 4] by [–4, 4]

(c) [–10, 10] by [–5, 30]

(d) [–50, 50] by [–100, 1000]

Graphs of $f(x) = x^2 + 3$

Figure 2

Observe that we get a more complete picture in parts (c) and (d), but in part (d) it is not clear that the y-intercept is 3.

Graphing Calculators and Computers

To understand how the expression for a function relates to its graph, it's helpful to graph a **family of functions**, that is, a collection of functions whose equations are related.

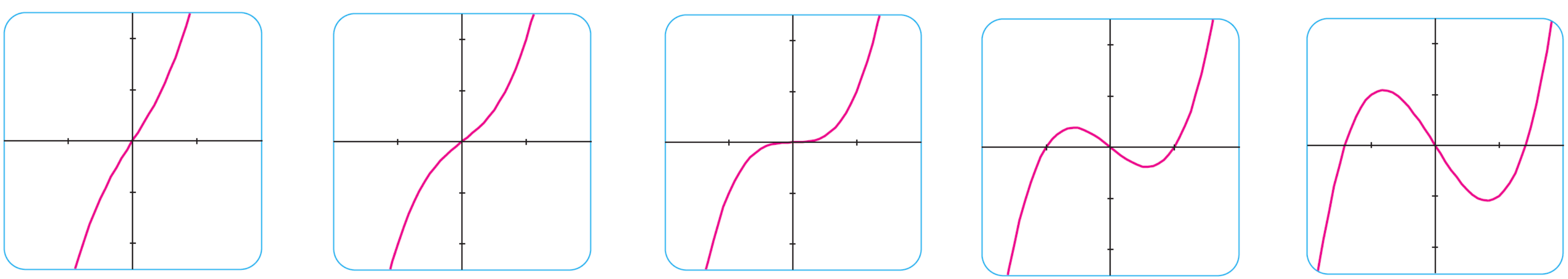
In the next example we graph members of a family of cubic polynomials.

Example 8

Graph the function $y = x^3 + cx$ for various values of the number c. How does the graph change when c is changed?

Solution:

Figure 13 shows the graphs of $y = x^3 + cx$ for $c = 2, 1, 0, -1$, and -2.



(a) $y = x^3 + 2x$ (b) $y = x^3 + x$ (c) $y = x^3$ (d) $y = x^3 - x$ (e) $y = x^3 - 2x$

Several members of the family of functions $y = x^3 + cx$, all graphed in the viewing rectangle $[-2, 2]$ by $[-2.5, 2.5]$

Figure 13

Example 8 – *Solution*

cont'd

We see that, for positive values of c, the graph increases from left to right with no maximum or minimum points (peaks or valleys).

When $c = 0$, the curve is flat at the origin.

When c is negative, the curve has a maximum point and a minimum point.

As c decreases, the maximum point becomes higher and the minimum point lower.

1 Functions and Models

1.5 Exponential Functions

Exponential Functions

The function $f(x) = 2^x$ is called an *exponential function* because the variable, x, is the exponent. It should not be confused with the power function $g(x) = x^2$, in which the variable is the base.

In general, an **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant. Let's recall what this means. If $x = n$, a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

Exponential Functions

If $x = 0$, then $a^0 = 1$, and if $x = -n$, where n is a positive integer, then

$$a^{-n} = \frac{1}{a^n}$$

If x is a rational number, $x = p/q$, where p and q are integers and $q > 0$, then

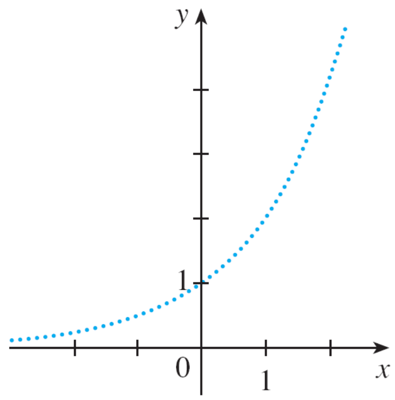
$$a^x = a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

But what is the meaning of a^x if x is an irrational number? For instance, what is meant by $2^{\sqrt{3}}$ or 5^{π}?

Exponential Functions

To help us answer this question we first look at the graph of the function $y = 2^x$, where x is rational. A representation of this graph is shown in Figure 1.

We want to enlarge the domain of $y = 2^x$ to include both rational and irrational numbers.



Representation of $y = 2^x$, x rational

Figure 1

There are holes in the graph in Figure 1 corresponding to irrational values of x.

We want to fill in the holes by defining $f(x) = 2^x$, where $x \in \mathbb{R}$, so that f is an increasing function.

Exponential Functions

In particular, since the irrational number $\sqrt{3}$ satisfies

$$1.7 < \sqrt{3} < 1.8$$

we must have

$$2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$$

and we know what $2^{1.7}$ and $2^{1.8}$ mean because 1.7 and 1.8 are rational numbers.

Exponential Functions

Similarly, if we use better approximations for $\sqrt{3}$, we obtain better approximations for $2^{\sqrt{3}}$.

$$
\begin{array}{rcl}
1.73 < \sqrt{3} < 1.74 & \Rightarrow & 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74} \\
1.732 < \sqrt{3} < 1.733 & \Rightarrow & 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733} \\
1.7320 < \sqrt{3} < 1.7321 & \Rightarrow & 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321} \\
1.73205 < \sqrt{3} < 1.73206 & \Rightarrow & 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206} \\
\vdots \quad\quad \vdots & & \vdots \quad\quad \vdots
\end{array}
$$

Exponential Functions

It can be shown that there is exactly one number that is greater than all of the numbers

$$2^{1.7}, \quad 2^{1.73}, \quad 2^{1.732}, \quad 2^{1.7320}, \quad 2^{1.73205}, \quad \ldots$$

and less than all of the numbers

$$2^{1.8}, \quad 2^{1.74}, \quad 2^{1.733}, \quad 2^{1.7321}, \quad 2^{1.73206}, \quad \ldots$$

We define $2^{\sqrt{3}}$ to be this number. Using the preceding approximation process we can compute it correct to six decimal places:

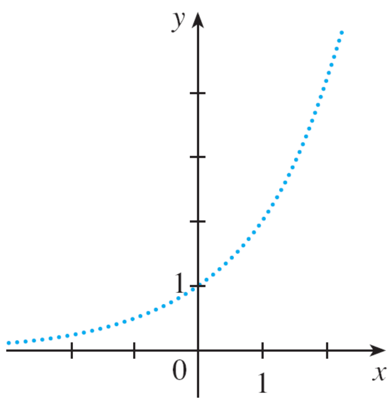
$$2^{\sqrt{3}} \approx 3.321997$$

Exponential Functions

Similarly, we can define 2^x (or a^x, if $a > 0$) where x is any irrational number.

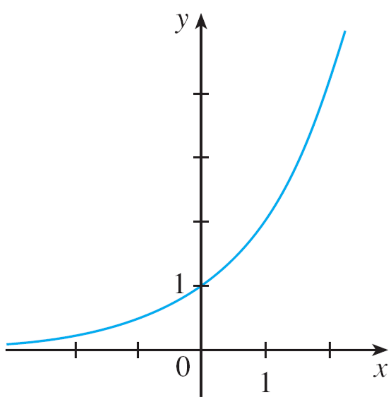
Figure 2 shows how all the holes in Figure 1 have been filled to complete the graph of the function

$$f(x) = 2^x,\ x \in \mathbb{R}.$$



Representation of $y = 2^x$, x rational

Figure 1



$y = 2^x$, x real

Figure 2

Exponential Functions

The graphs of members of the family of functions $y = a^x$ are shown in Figure 3 for various values of the base a.

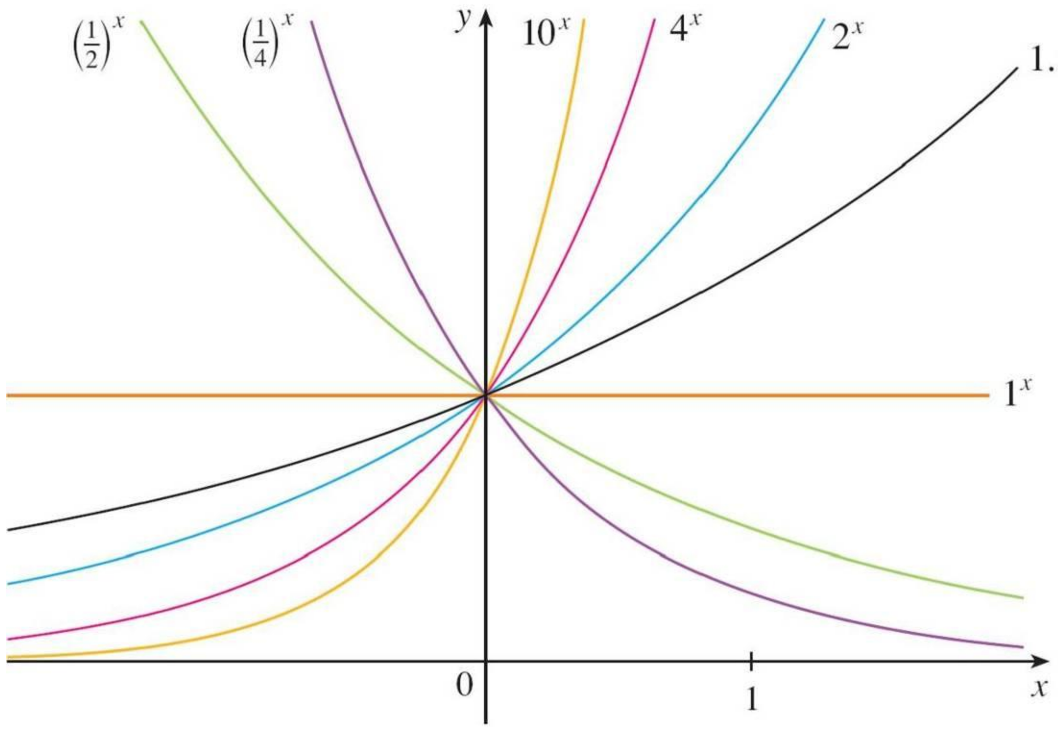


Figure 3

Exponential Functions

Notice that all of these graphs pass through the same point (0, 1) because $a^0 = 1$ for $a \neq 0$.

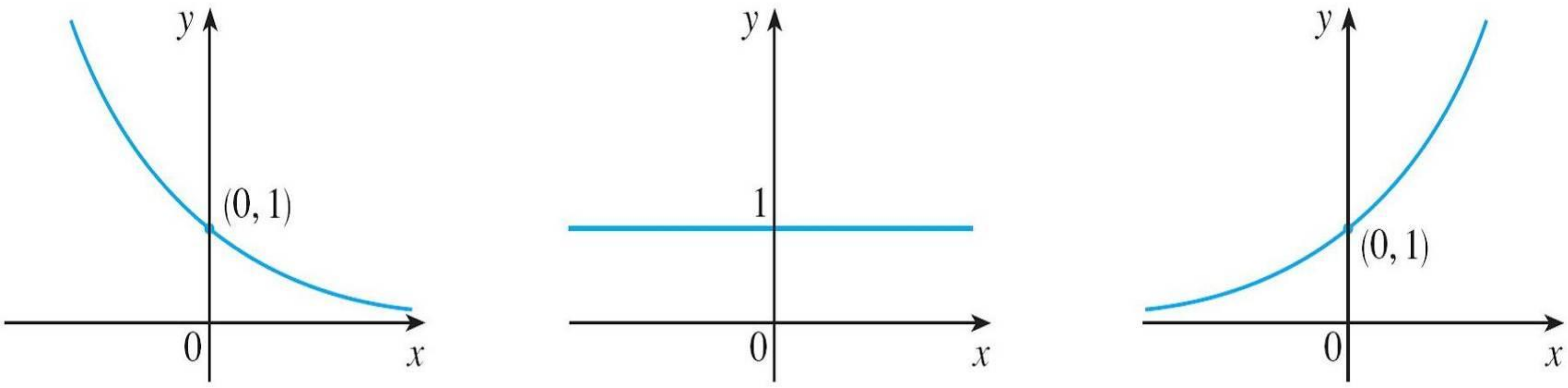
Notice also that as the base a gets larger, the exponential function grows more rapidly (for $x > 0$).

You can see from Figure 3 that there are basically three kinds of exponential functions $y = a^x$.

If $0 < a < 1$, the exponential function decreases; if $a = 1$, it is a constant; and if $a > 1$, it increases.

Exponential Functions

These three cases are illustrated in Figure 4.



(a) $y = a^x$, $0 < a < 1$ **(b)** $y = 1^x$ **(c)** $y = a^x$, $a > 1$

Figure 4

Exponential Functions

Observe that if $a \neq 1$, then the exponential function $y = a^x$ has domain $\mathbb{R}$ and range $(0, \infty)$.

Notice also that, since $(1/a)^x = 1/a^x = a^{-x}$, the graph of $y = (1/a)^x$ is just the reflection of the graph of $y = a^x$ about the y-axis.

Exponential Functions

One reason for the importance of the exponential function lies in the following properties.

If *x* and *y* are rational numbers, then these laws are well known from elementary algebra. It can be proved that they remain true for arbitrary real numbers *x* and *y*.

Laws of Exponents If a and b are positive numbers and x and y are any real numbers, then

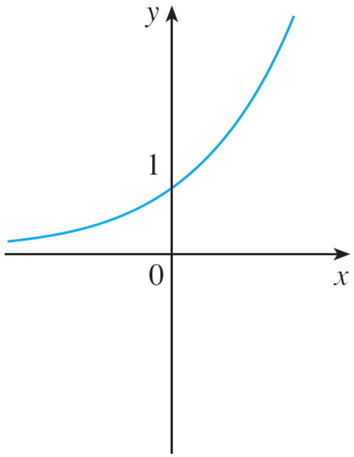
1. $a^{x+y} = a^x a^y$ **2.** $a^{x-y} = \dfrac{a^x}{a^y}$ **3.** $(a^x)^y = a^{xy}$ **4.** $(ab)^x = a^x b^x$

Example 1

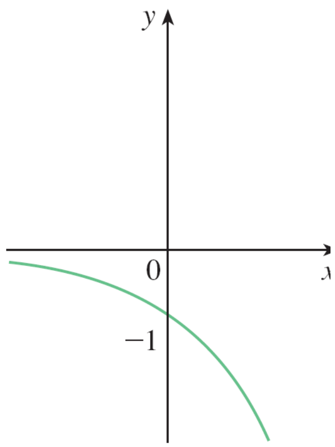
Sketch the graph of the function $y = 3 - 2^x$ and determine its domain and range.

Solution:

First we reflect the graph of $y = 2^x$ [shown in Figure 2 and Figure 5(a)] about the x-axis to get the graph of $y = -2^x$ in Figure 5(b).



(b) $y = 2^x$



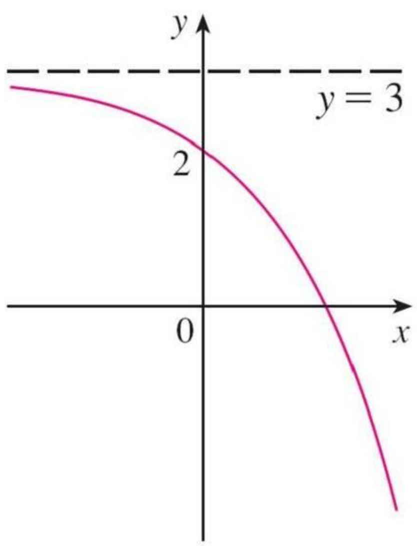
(b) $y = -2^x$

Figure 5

Example 1 – *Solution*

cont'd

Then we shift the graph of $y = –2^x$ upward 3 units to obtain the graph of $y = 3 – 2^x$ in Figure 5(c).



(c) $y = 3 – 2^x$

Figure 5

The domain is $\mathbb{R}$ and the range is $(-\infty, 3)$.

Applications of Exponential Functions

Applications of Exponential Functions

The exponential function occurs very frequently in mathematical models of nature and society. Here we indicate briefly how it arises in the description of population growth.

First we consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour.

Applications of Exponential Functions

If the number of bacteria at time t is $p(t)$, where t is measured in hours, and the initial population is $p(0) = 1000$, then we have

$$p(1) = 2p(0) = 2 \times 1000$$

$$p(2) = 2p(1) = 2^2 \times 1000$$

$$p(3) = 2p(2) = 2^3 \times 1000$$

It seems from this pattern that, in general,

$$p(t) = 2^t \times 1000 = (1000)2^t$$

Applications of Exponential Functions

This population function is a constant multiple of the exponential function $y = 2^t$, so it exhibits the rapid growth.

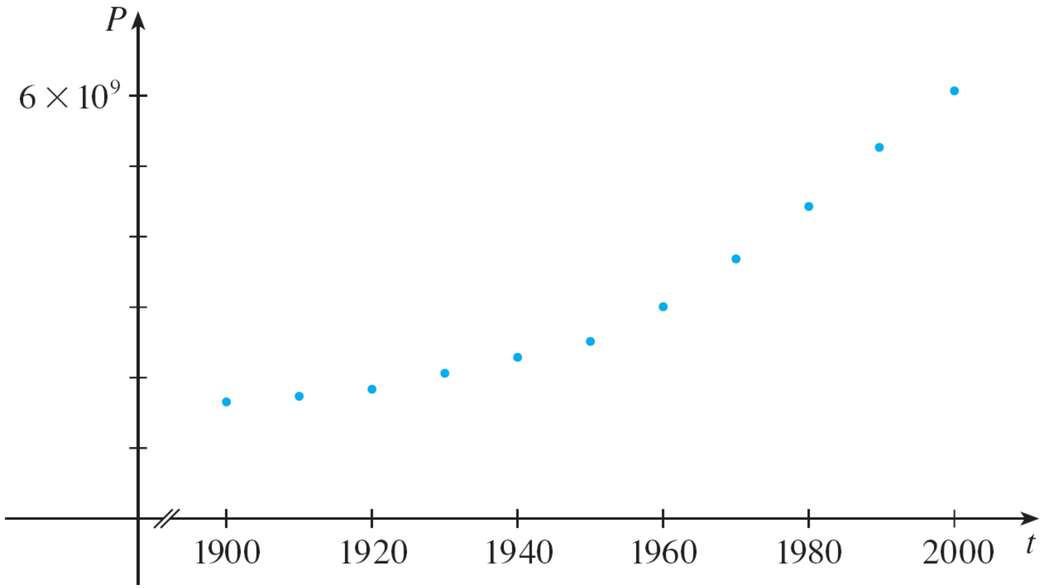
Under ideal conditions (unlimited space and nutrition and absence of disease) this exponential growth is typical of what actually occurs in nature.

Applications of Exponential Functions

What about the human population? Table 1 shows data for the population of the world in the 20th century and Figure 8 shows the corresponding scatter plot.

TABLE 1

t	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870



Scatter plot for world population growth

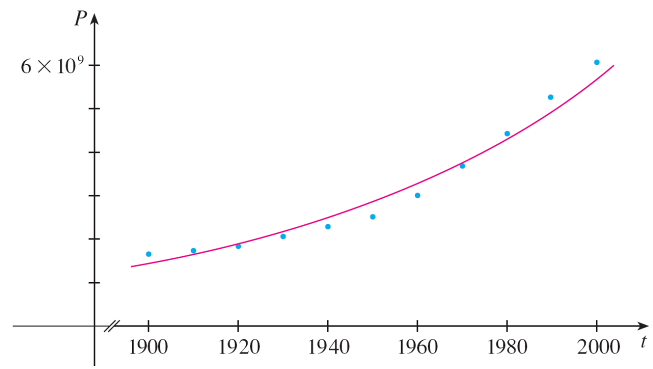
Figure 8

Applications of Exponential Functions

The pattern of the data points in Figure 8 suggests exponential growth, so we use a graphing calculator with exponential regression capability to apply the method of least squares and obtain the exponential model

$$P = (1436.53) \cdot (1.01395)^t$$

where $t = 0$ corresponds to 1900. Figure 9 shows the graph of this exponential function together with the original data points.



Exponential model for population growth

Figure 9

Applications of Exponential Functions

We see that the exponential curve fits the data reasonably well.

The period of relatively slow population growth is explained by the two world wars and the Great Depression of the 1930s.

The Number *e*

The Number e

Of all possible bases for an exponential function, there is one that is most convenient for the purposes of calculus. The choice of a base a is influenced by the way the graph of $y = a^x$ crosses the y-axis. Figures 10 and 11 show the tangent lines to the graphs of $y = 2^x$ and $y = 3^x$ at the point (0, 1).

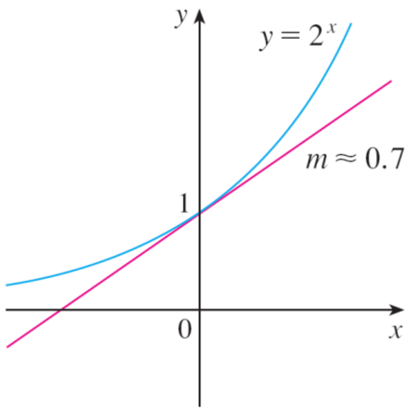


Figure 10

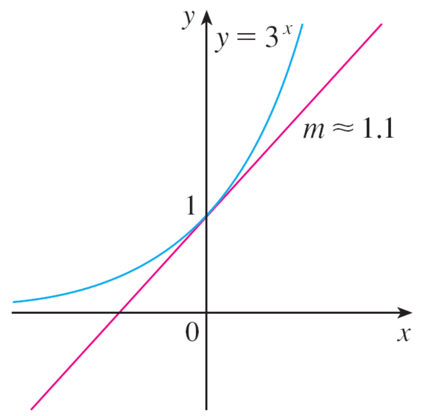


Figure 11

The Number e

(For present purposes, you can think of the tangent line to an exponential graph at a point as the line that touches the graph only at that point.)

If we measure the slopes of these tangent lines at (0, 1), we find that $m \approx 0.7$ for $y = 2^x$ and $m \approx 1.1$ for $y = 3^x$.

The Number e

It turns out that some of the formulas of calculus will be greatly simplified if we choose the base a so that the slope of the tangent line to $y = a^x$ at (0, 1) is *exactly* 1.
(See Figure 12.)

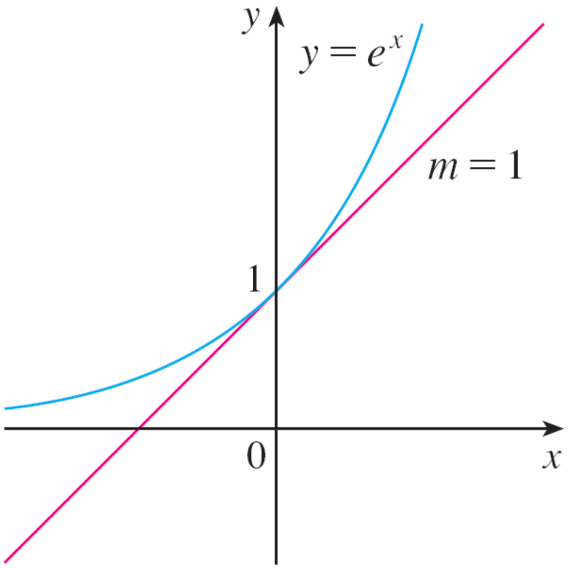


Figure 12

The natural exponential function crosses the y-axis with a slope of 1.

The Number *e*

In fact, there *is* such a number and it is denoted by the letter *e*.

(This notation was chosen by the Swiss mathematician Leonhard Euler in 1727, probably because it is the first letter of the word *exponential*.)

The Number *e*

In view of Figures 10 and 11, it comes as no surprise that the number *e* lies between 2 and 3 and the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.
(See Figure 13.)

We will see that the value of *e*, correct to five decimal places, is

$$e \approx 2.71828$$

We call the function $f(x) = e^x$ the **natural exponential function**.

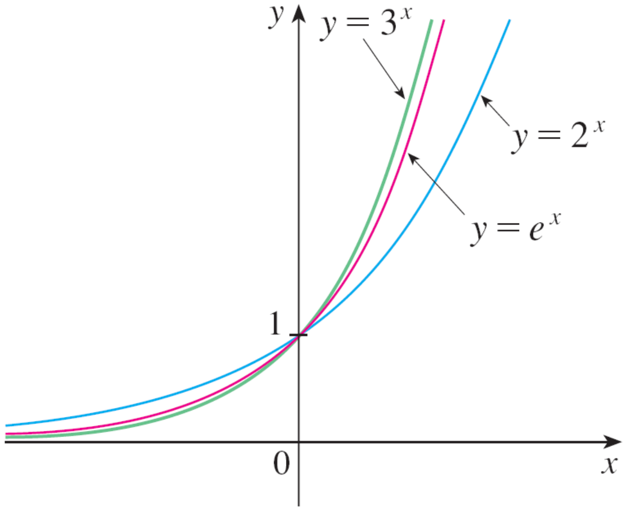


Figure 13

Example 4

Graph the function $y = \frac{1}{2}e^{-x} - 1$ and state the domain and range.

Solution:

We start with the graph of $y = e^x$ from Figures 12 and 14(a) and reflect about the y-axis to get the graph of $y = e^{-x}$ in Figure 14(b). (Notice that the graph crosses the y-axis with a slope of –1).

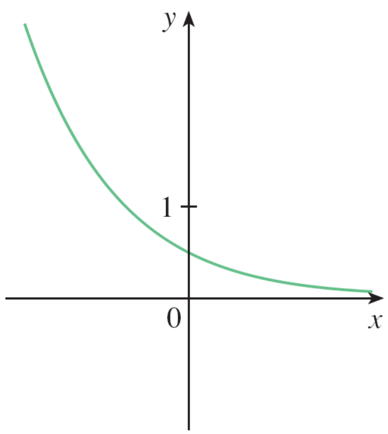
(a) $y = e^x$ **Figure 15** **(b)** $y = e^{-x}$

Example 4 – *Solution*

cont'd

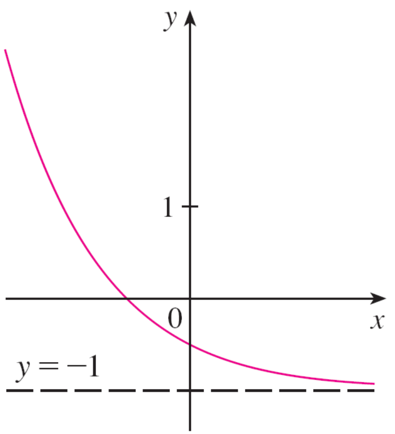
Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ in Figure 14(c).

Finally, we shift the graph downward one unit to get the desired graph in Figure 14(d).



(c) $y = \frac{1}{2}e^{-x}$

Figure 15



(d) $y = \frac{1}{2}e^{-x} - 1$

The domain is $\mathbb{R}$ and the range is $(-1, \infty)$.

1 Functions and Models

1.6 Inverse Functions and Logarithms

Inverse Functions and Logarithms

Table 1 gives data from an experiment in which a bacteria culture started with 100 bacteria in a limited nutrient medium; the size of the bacteria population was recorded at hourly intervals.

The number of bacteria *N* is a function of the time *t*: $N = f(t)$.

Suppose, however, that the biologist changes her point of view and becomes interested in the time required for the population to reach various levels. In other words, she is thinking of *t* as a function of *N*.

t (hours)	$N = f(t)$ = population at time t
0	100
1	168
2	259
3	358
4	445
5	509
6	550
7	573
8	586

N as a function of *t*

Table 1

Inverse Functions and Logarithms

This function is called the *inverse function* of f, denoted by f^{-1}, and read "f inverse." Thus $t = f^{-1}(N)$ is the time required for the population level to reach N.

The values of f^{-1} can be found by reading Table 1 from right to left or by consulting Table 2.

For instance, $f^{-1}(550) = 6$ because $f(6) = 550$.

Not all functions possess inverses.

N	$t = f^{-1}(N)$ = time to reach N bacteria
100	0
168	1
259	2
358	3
445	4
509	5
550	6
573	7
586	8

t as a function of N

Table 2

Inverse Functions and Logarithms

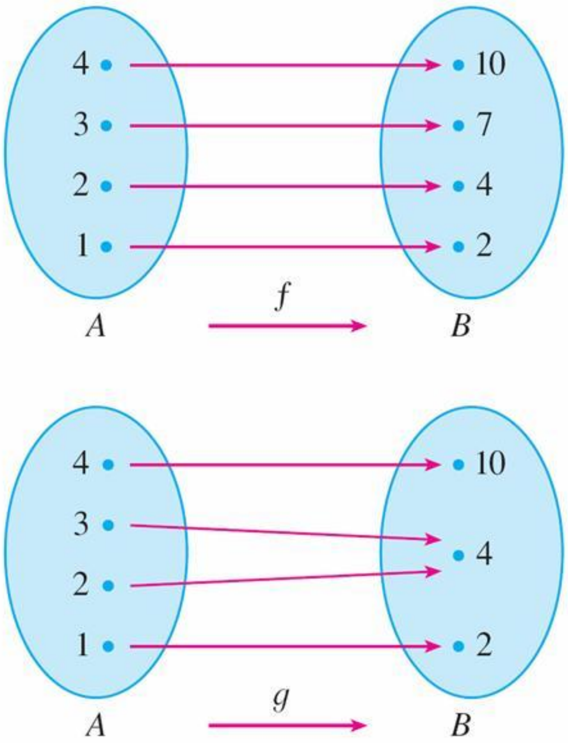
Let's compare the functions *f* and *g* whose arrow diagrams are shown in Figure 1.

Note that *f* never takes on the same value twice (any two inputs in *A* have different outputs), whereas *g* does take on the same value twice (both 2 and 3 have the same output, 4).

In symbols,

$$g(2) = g(3)$$

but $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$



f is one-to-one; *g* is not

Figure 1

Inverse Functions and Logarithms

Functions that share this property with *f* are called *one-to-one functions.*

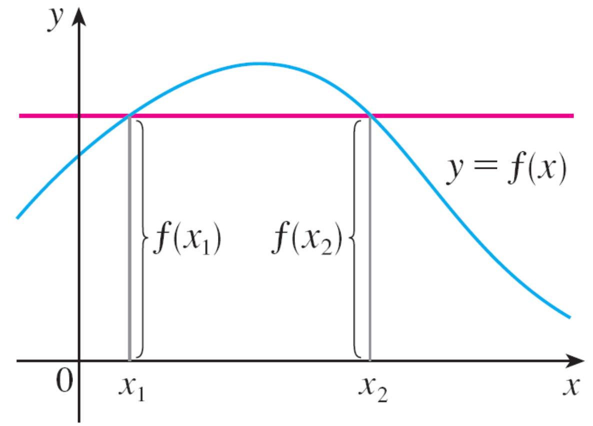
$\boxed{1}$ **Definition** A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Inverse Functions and Logarithms

If a horizontal line intersects the graph of f in more than one point, then we see from Figure 2 that there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$.

This means that f is not one-to-one.



This function is not one-to-one because $f(x_1) = f(x_2)$.

Figure 2

Therefore we have the following geometric method for determining whether a function is one-to-one.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Example 1

Is the function $f(x) = x^3$ one-to-one?

Solution 1:

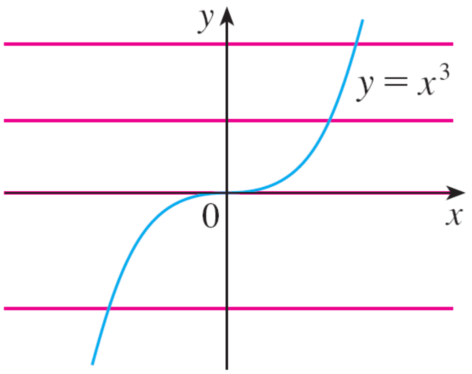
If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers can't have the same cube).

Therefore, by Definition 1, $f(x) = x^3$ is one-to-one.

Solution 2:

From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once.

Therefore, by the Horizontal Line Test, f is one-to-one.



$f(x) = x^3$ is one-to-one.

Figure 3

Inverse Functions and Logarithms

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

$\boxed{2}$ **Definition** Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

This definition says that if *f* maps *x* into *y*, then f^{-1} maps *y* back into *x*. (If *f* were not one-to-one, then f^{-1} would not be uniquely defined.)

Inverse Functions and Logarithms

The arrow diagram in Figure 5 indicates that f^{-1} reverses the effect of f.

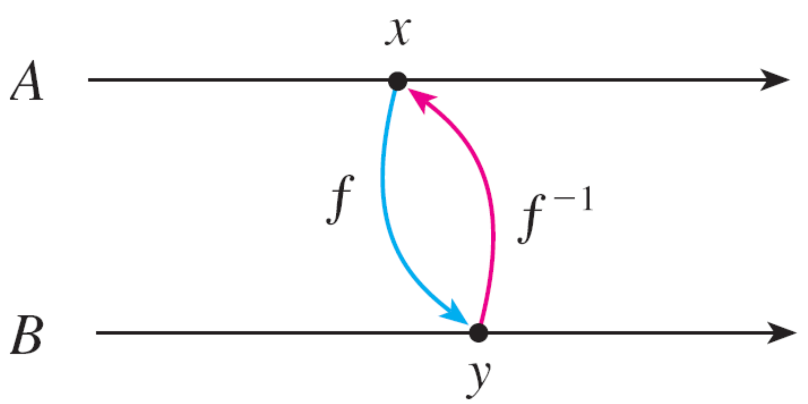


Figure 5

Note that

$$\boxed{\begin{aligned} \text{domain of } f^{-1} &= \text{range of } f \\ \text{range of } f^{-1} &= \text{domain of } f \end{aligned}}$$

Inverse Functions and Logarithms

For example, the inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ because if $y = x^3$, then

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x$$

Caution

Do not mistake the –1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \text{ does } \textit{not} \text{ mean } \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ could, however, be written as $[f(x)]^{-1}$.

Example 3

If $f(1) = 5$, $f(3) = 7$, and $f(8) = –10$, find $f^{–1}(7)$, $f^{–1}(5)$, and $f^{–1}(–10)$.

Solution:

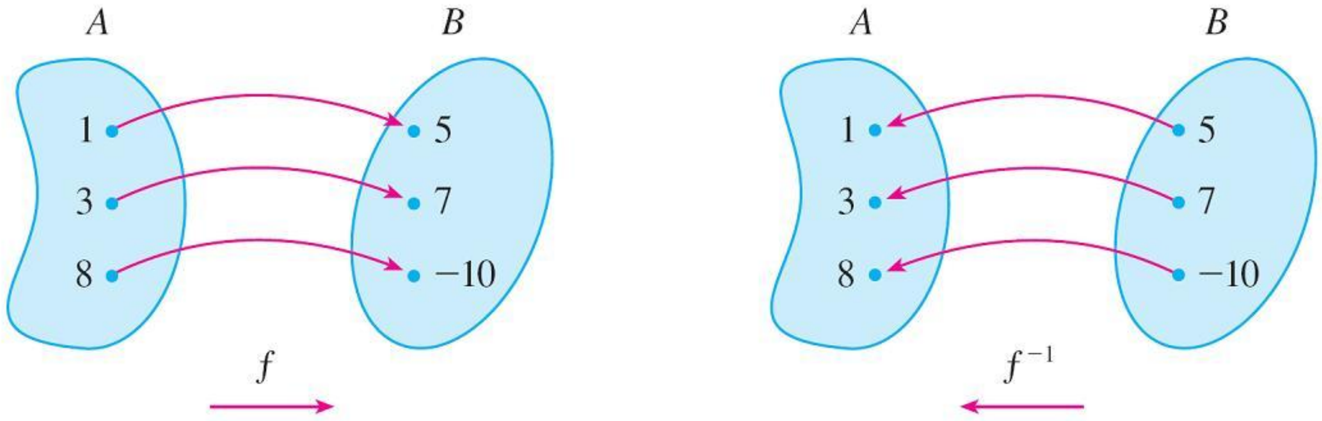
From the definition of $f^{–1}$ we have

$f^{–1}(7) = 3$	because	$f(3) = 7$
$f^{–1}(5) = 1$	because	$f(1) = 5$
$f^{–1}(–10) = 8$	because	$f(8) = –10$

Example 3 – *Solution*

cont'd

The diagram in Figure 6 makes it clear how f^{-1} reverses the effect of f in this case.



The inverse function reverses inputs and outputs.

Figure 6

Inverse Functions and Logarithms

The letter *x* is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on *f*, we usually reverse the roles of *x* and *y* in Definition 2 and write

3

$$f^{-1}(x) = y \iff f(y) = x$$

By substituting for *y* in Definition 2 and substituting for *x* in $\boxed{3}$ we get the following **cancellation equations**:

4

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Inverse Functions and Logarithms

The first cancellation equation says that if we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started (see the machine diagram in Figure 7).

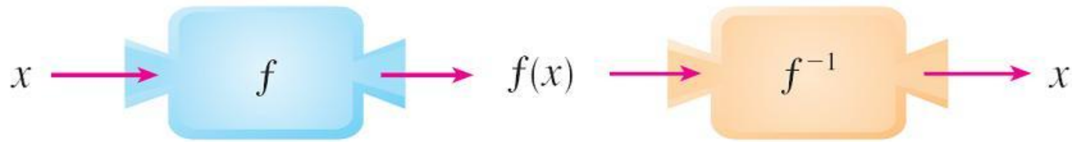


Figure 7

Thus f^{-1} undoes what f does.

The second equation says that f undoes what f^{-1} does.

Inverse Functions and Logarithms

For example, if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$ and so the cancellation equations become

$$f^{-1}(f(x)) = (x^3)^{1/3} = x$$

$$f(f^{-1}(x)) = (x^{1/3})^3 = x$$

These equations simply say that the cube function and the cube root function cancel each other when applied in succession.

Inverse Functions and Logarithms

Now let's see how to compute inverse functions.

If we have a function $y = f(x)$ and are able to solve this equation for x in terms of y, then according to Definition 2 we must have $x = f^{-1}(y)$.

If we want to call the independent variable x, we then interchange x and y and arrive at the equation $y = f^{-1}(x)$.

5 How to Find the Inverse Function of a One-to-One Function f

Step 1 Write $y = f(x)$.

Step 2 Solve this equation for x in terms of y (if possible).

Step 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Inverse Functions and Logarithms

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f.

Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}.

But we get the point (b, a) from (a, b) by reflecting about the line $y = x$. (See Figure 8.)

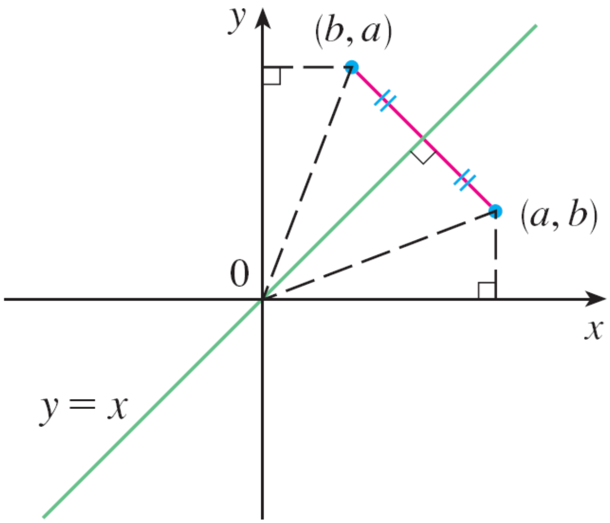


Figure 8

Inverse Functions and Logarithms

Therefore, as illustrated by Figure 9:

> The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

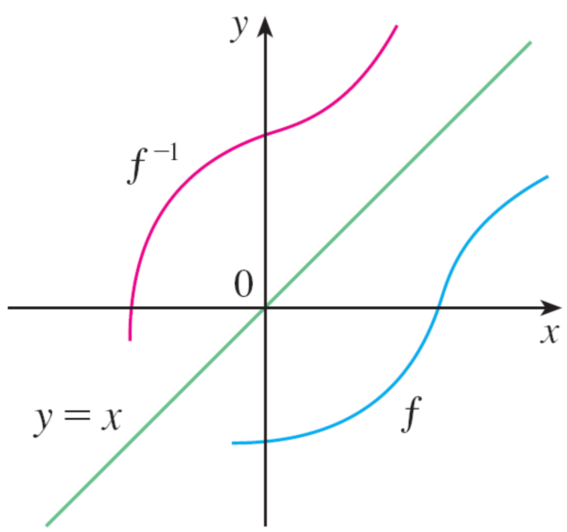


Figure 9

Logarithmic Functions

Logarithmic Functions

If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test. It therefore has an inverse function f^{-1}, which is called the **logarithmic function with base *a*** and is denoted by $\log_a$.

If we use the formulation of an inverse function given by $\boxed{3}$,

$$f^{-1}(x) = y \iff f(y) = x$$

then we have

6 $$\boxed{\log_a x = y \iff a^y = x}$$

Logarithmic Functions

Thus, if $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x.

For example, $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.

The cancellation equations $\boxed{4}$, when applied to the functions $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, become

$\boxed{7}$

$$\boxed{\begin{aligned} \log_a(a^x) &= x \quad \text{for every } x \in \mathbb{R} \\ a^{\log_a x} &= x \quad \text{for every } x > 0 \end{aligned}}$$

Logarithmic Functions

The logarithmic function $\log_a$ has domain $(0, \infty)$ and range $\mathbb{R}$. Its graph is the reflection of the graph of $y = a^x$ about the line $y = x$.

Figure 11 shows the case where $a > 1$. (The most important logarithmic functions have base $a > 1$.)

The fact that $y = a^x$ is a very rapidly increasing function for $x > 0$ is reflected in the fact that $y = \log_a x$ is a very slowly increasing function for $x > 1$.

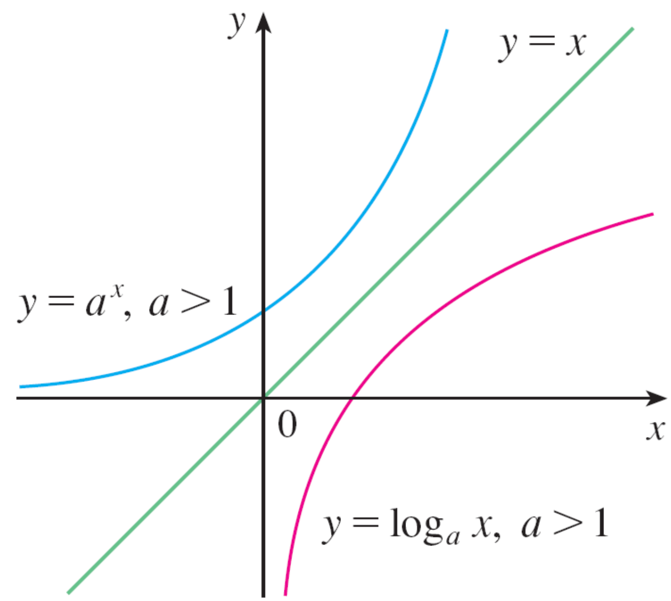


Figure 11

Logarithmic Functions

Figure 12 shows the graphs of $y = \log_a x$ with various values of the base $a > 1$.

Since $\log_a 1 = 0$, the graphs of all logarithmic functions pass through the point (1, 0).

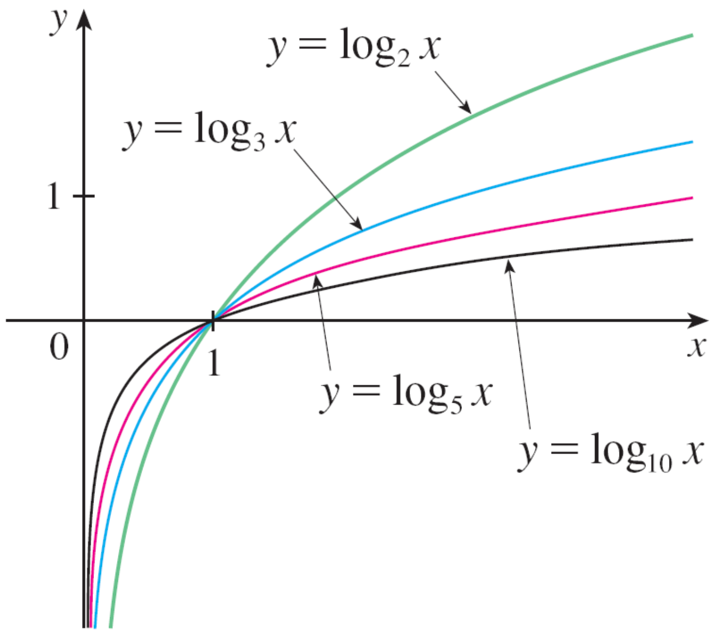


Figure 12

Logarithmic Functions

The following properties of logarithmic functions follow from the corresponding properties of exponential functions.

Laws of Logarithms If x and y are positive numbers, then

1. $\log_a(xy) = \log_a x + \log_a y$

2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

3. $\log_a(x^r) = r\log_a x$ (where r is any real number)

Example 6

Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

Solution:

Using Law 2, we have

$$\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right)$$

$$= \log_2 16$$

$$= 4$$

because $2^4 = 16$.

Natural Logarithms

Natural Logarithms

Of all possible bases *a* for logarithms, we will see that the most convenient choice of a base is the number *e*.

The logarithm with base *e* is called the **natural logarithm** and has a special notation:

$$\log_e x = \ln x$$

If we put $a = e$ and replace $\log_e$ with "ln" in $\boxed{6}$ and $\boxed{7}$, then the defining properties of the natural logarithm function become

$\boxed{8}$

$$\ln x = y \quad \Longleftrightarrow \quad e^y = x$$

Natural Logarithms

9

$$\ln(e^x) = x \qquad x \in \mathbb{R}$$

$$e^{\ln x} = x \qquad x > 0$$

In particular, if we set $x = 1$, we get

$$\ln e = 1$$

Example 7

Find x if $\ln x = 5$.

Solution 1:

From $\boxed{8}$ we see that

$$\ln x = 5 \qquad \text{means} \qquad e^5 = x$$

Therefore, $x = e^5$.

(If you have trouble working with the "ln" notation, just replace it by $\log_e$. Then the equation becomes $\log_e x = 5$; so, by the definition of logarithm, $e^5 = x$.)

Example 7 – *Solution*

cont'd

Solution 2:

Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But the second cancellation equation in $\boxed{9}$ says that $e^{\ln x} = x$.

Therefore $x = e^5$.

Natural Logarithms

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

$\boxed{10}$ **Change of Base Formula** For any positive number a ($a \neq 1$), we have

$$\log_a x = \frac{\ln x}{\ln a}$$

Example 10

Evaluate $\log_8 5$ correct to six decimal places.

Solution:

Formula 10 gives

$$\log_8 5 = \frac{\ln 5}{\ln 8}$$

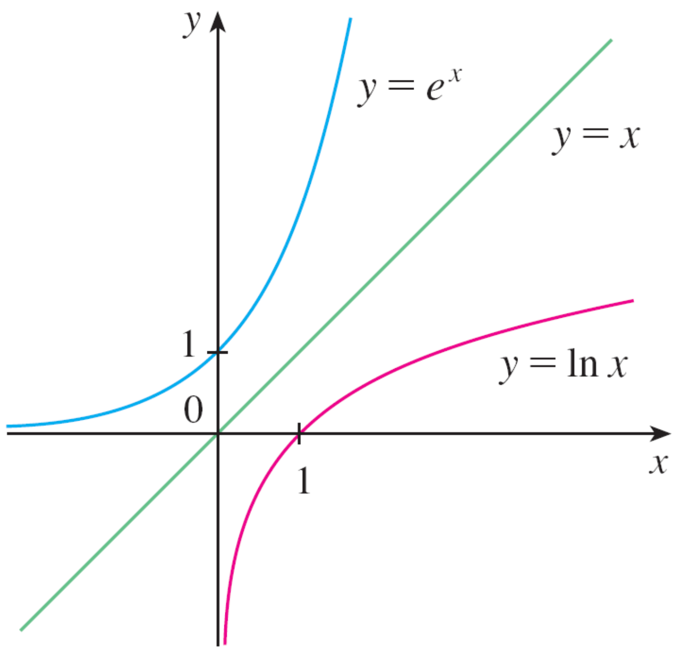
$$\approx 0.773976$$

Graph and Growth of the Natural Logarithm

Graph and Growth of the Natural Logarithm

The graphs of the exponential function $y = e^x$ and its inverse function, the natural logarithm function, are shown in Figure 13.

Because the curve $y = e^x$ crosses the y-axis with a slope of 1, it follows that the reflected curve $y = \ln x$ crosses the x-axis with a slope of 1.



The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ about the line $y = x$

Figure 13

Graph and Growth of the Natural Logarithm

In common with all other logarithmic functions with base greater than 1, the natural logarithm is an increasing function defined on $(0, \infty)$ and the *y*-axis is a vertical asymptote.

(This means that the values of $\ln x$ become very large negative as x approaches 0.)

Example 11

Sketch the graph of the function *y* = ln (*x* – 2) – 1.

Solution:

We start with the graph of *y* = ln *x* as given in Figure 13.

We shift it 2 units to the right to get the graph of *y* = ln (*x* – 2) and then we shift it 1 unit downward to get the graph of *y* = ln (*x* – 2) – 1. (See Figure 14.)

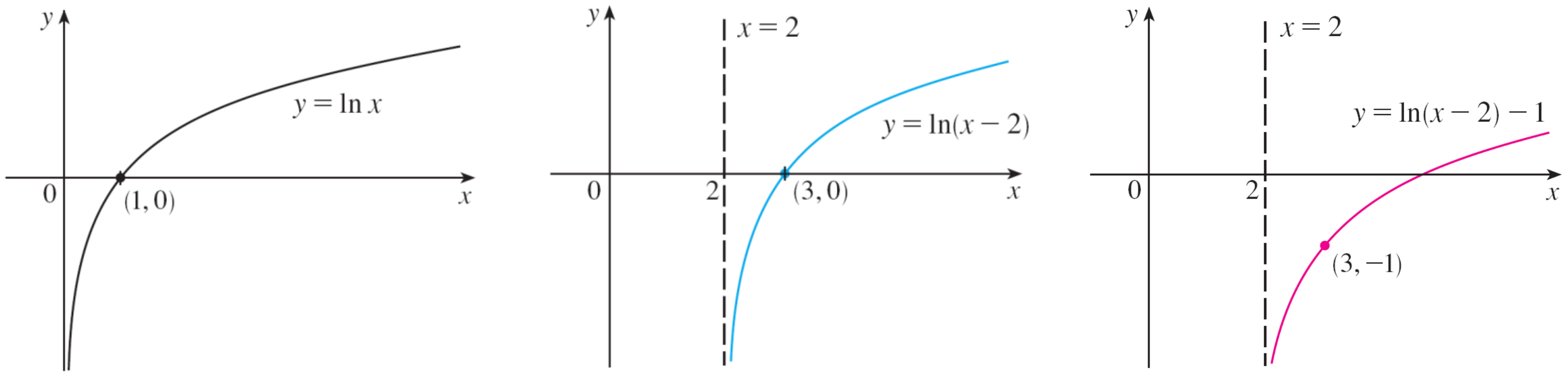


Figure 14

Graph and Growth of the Natural Logarithm

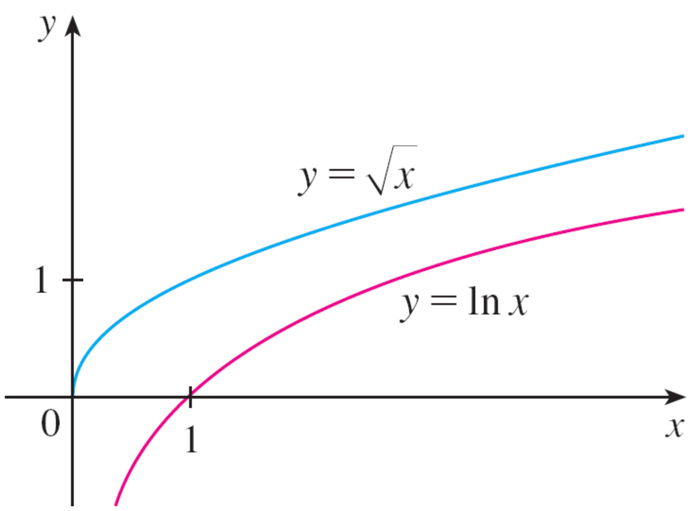
Although ln x is an increasing function, it grows *very* slowly when $x > 1$. In fact, ln x grows more slowly than any positive power of x.

To illustrate this fact, we compare approximate values of the functions $y = \ln x$ and $y = x^{1/2} = \sqrt{x}$ in the following table.

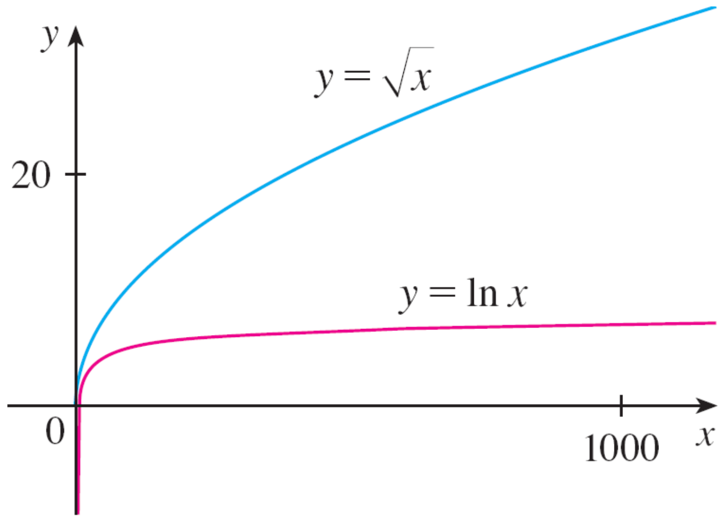
x	1	2	5	10	50	100	500	1000	10,000	100,000
$\ln x$	0	0.69	1.61	2.30	3.91	4.6	6.2	6.9	9.2	11.5
$\sqrt{x}$	1	1.41	2.24	3.16	7.07	10.0	22.4	31.6	100	316
$\dfrac{\ln x}{\sqrt{x}}$	0	0.49	0.72	0.73	0.55	0.46	0.28	0.22	0.09	0.04

Graph and Growth of the Natural Logarithm

We graph them in Figures 15 and 16.



Figures 15



Figures 16

You can see that initially the graphs of y = $\sqrt{x}$ and *y* = ln *x* grow at comparable rates, but eventually the root function far surpasses the logarithm.