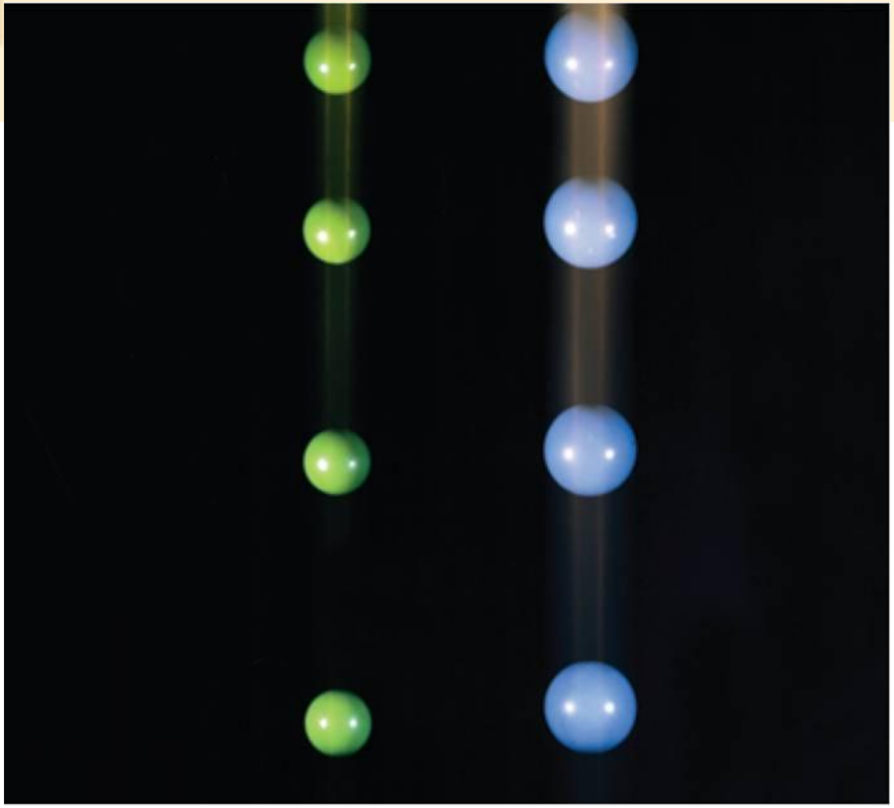
2

Limits and Derivatives



2.1 The Tangent and Velocity Problems

The Tangent Problem

The Tangent Problems

The word *tangent* is derived from the Latin word *tangens*, which means “touching.”

Thus a tangent to a curve is a line that touches the curve.

In other words, a tangent line should have the same direction as the curve at the point of contact.

The Tangent Problems

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once, as in Figure 1(a).

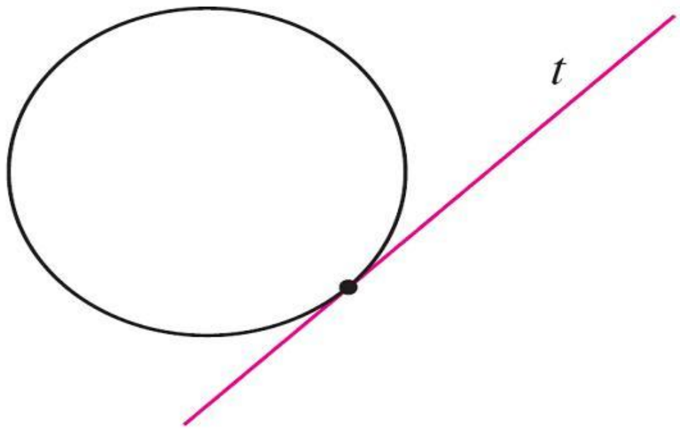


Figure 1(a)

For more complicated curves this definition is inadequate.

The Tangent Problems

Figure 1(b) shows two lines *l* and *t* passing through a point *P* on a curve *C*.

The line *l* intersects *C* only once, but it certainly does not look like what we think of as a tangent.

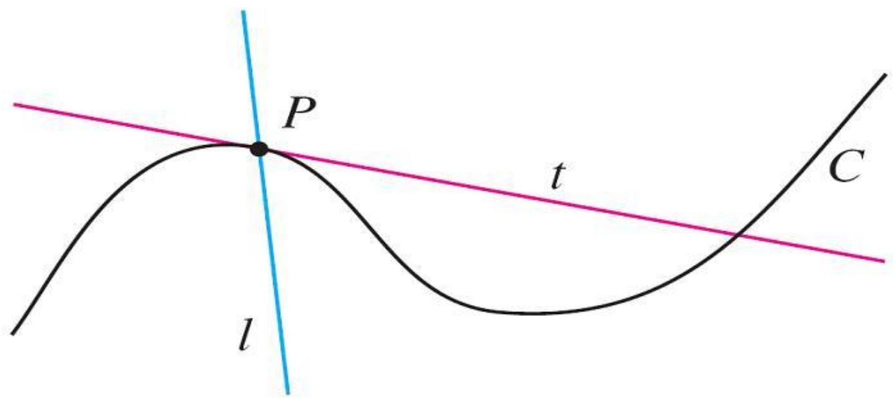


Figure 1(b)

The line *t*, on the other hand, looks like a tangent but it intersects *C* twice.

Example 1

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution:

We will be able to find an equation of the tangent line t as soon as we know its slope m.

The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope.

Example 1 – *Solution*

cont'd

But observe that we can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ. [A **secant line**, from the Latin word *secans*, meaning cutting, is a line that cuts (intersects) a curve more than once.]

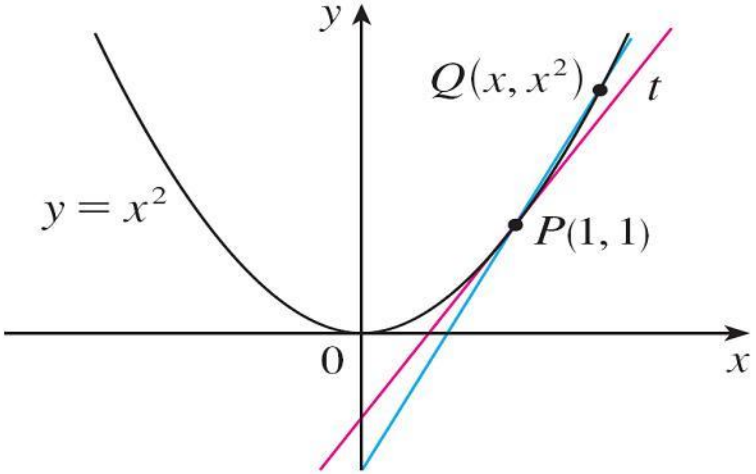


Figure 2

Example 1 – *Solution*

cont'd

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1}$$

$$= \frac{1.25}{0.5}$$

$$= 2.5$$

Example 1 – *Solution*

cont'd

The following tables show the values of m_{PQ} for several values of x close to 1.

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2.

Example 1 – *Solution*

cont'd

This suggests that the slope of the tangent line *t* should be $m = 2$.

We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m \quad \text{and} \quad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

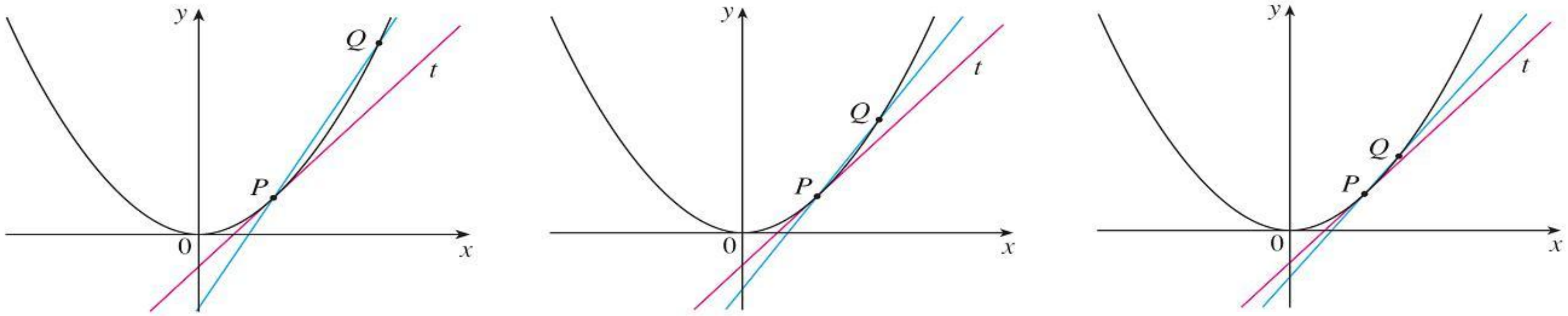
Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line to write the equation of the tangent line through (1, 1) as

$$y - 1 = 2(x - 1) \quad \text{or} \quad y = 2x - 1$$

Example 1 – *Solution*

cont'd

Figure 3 illustrates the limiting process that occurs in this example.

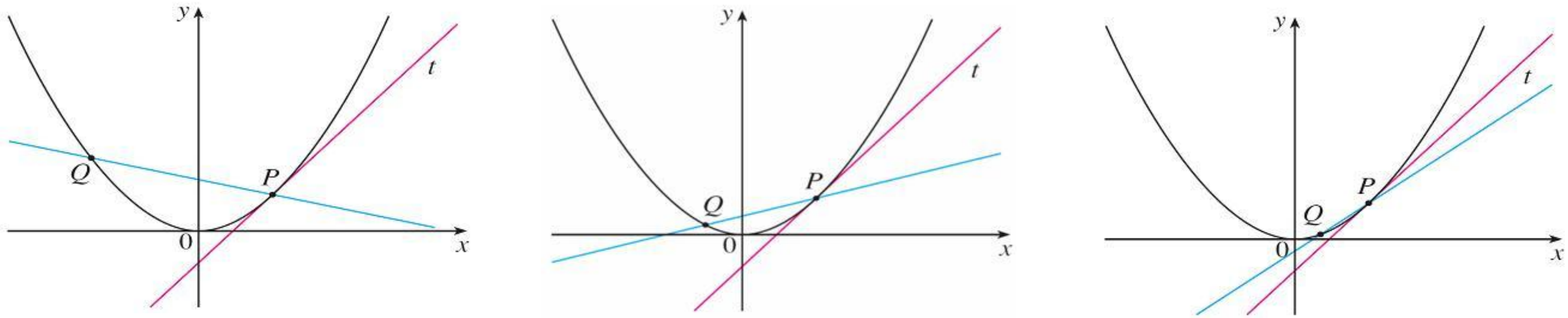


Q approaches *P* from the right

Figure 3

Example 1 – *Solution*

cont'd



Q approaches *P* from the left

Figure 3

As *Q* approaches *P* along the parabola, the corresponding secant lines rotate about *P* and approach the tangent line *t*.

The Velocity Problem

Example 3

Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Solution:

Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.)

Example 3 – *Solution*

cont'd

If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved.

Example 3 – *Solution*

cont'd

However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from t = 5 to t = 5.1:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

$$= \frac{s(5.1) - s(5)}{0.1}$$

$$= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1}$$

$$= 49.49 \text{ m/s}$$

Example 3 – *Solution*

cont'd

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

Example 3 – *Solution*

cont'd

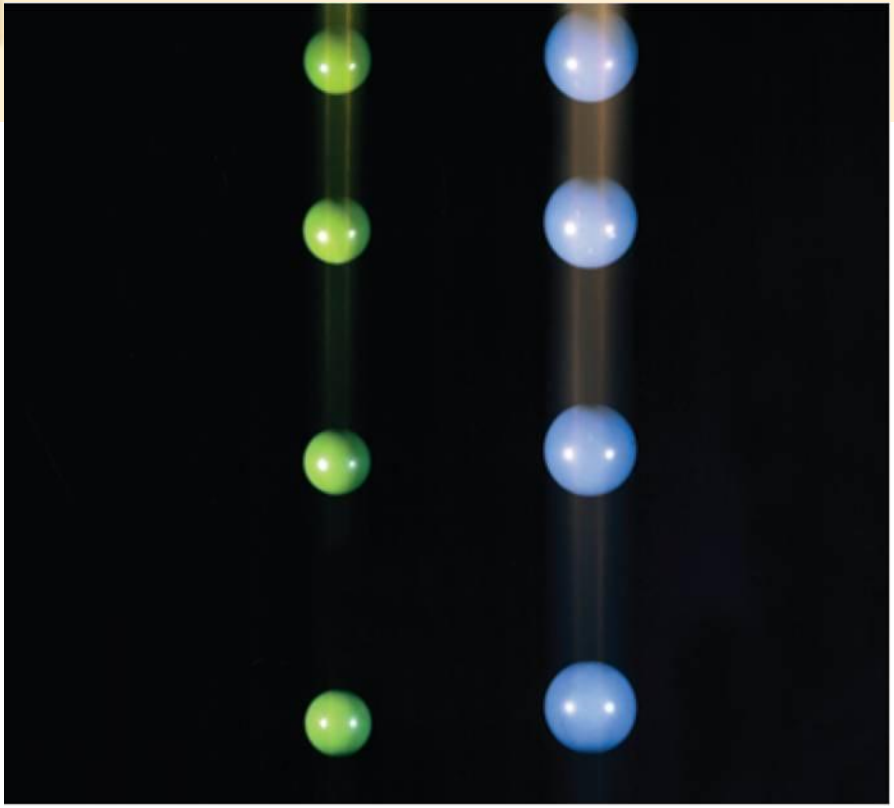
It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s.

The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$.

Thus the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$

2 Limits and Derivatives



2.2 The Limit of a Function

The Limit of a Function

To find the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and numerical and graphical methods for computing them.

Let's investigate the behavior of the function f defined by $f(x) = x^2 - x + 2$ for values of x near 2.

The Limit of a Function

The following table gives values of $f(x)$ for values of x close to 2 but not equal to 2.

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

The Limit of a Function

From the table and the graph of *f* (a parabola) shown in Figure 1 we see that when *x* is close to 2 (on either side of 2), *f*(*x*) is close to 4.

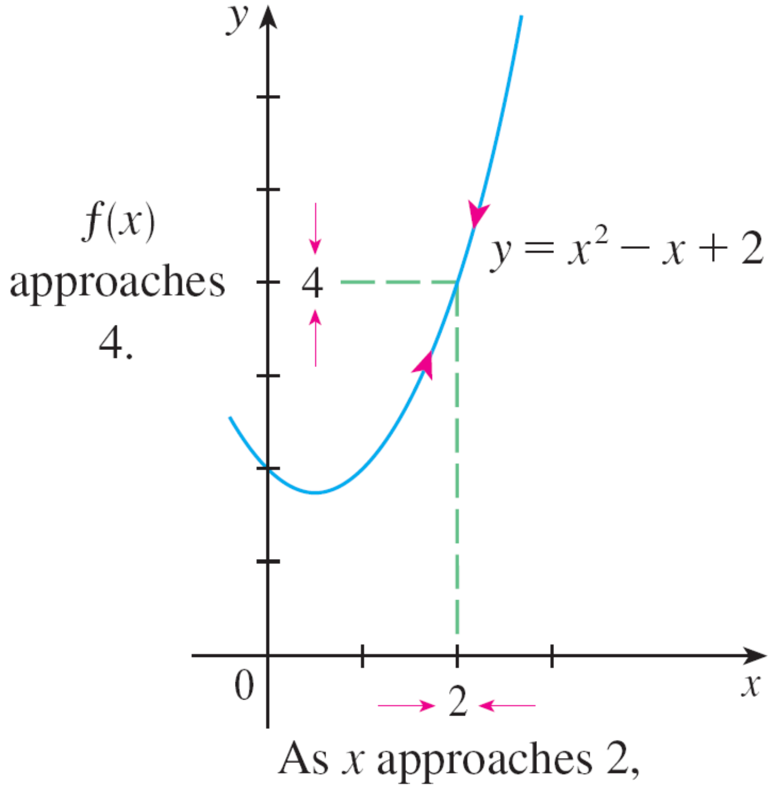


Figure 1

The Limit of a Function

In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2.

We express this by saying "the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4."

The notation for this is

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

The Limit of a Function

In general, we use the following notation.

1 Definition We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

This says that the values of $f(x)$ approach L as x approaches a. In other words, the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$.

The Limit of a Function

An alternative notation for

$$\lim_{x \to a} f(x) = L$$

is $f(x) \to L$ as $x \to a$

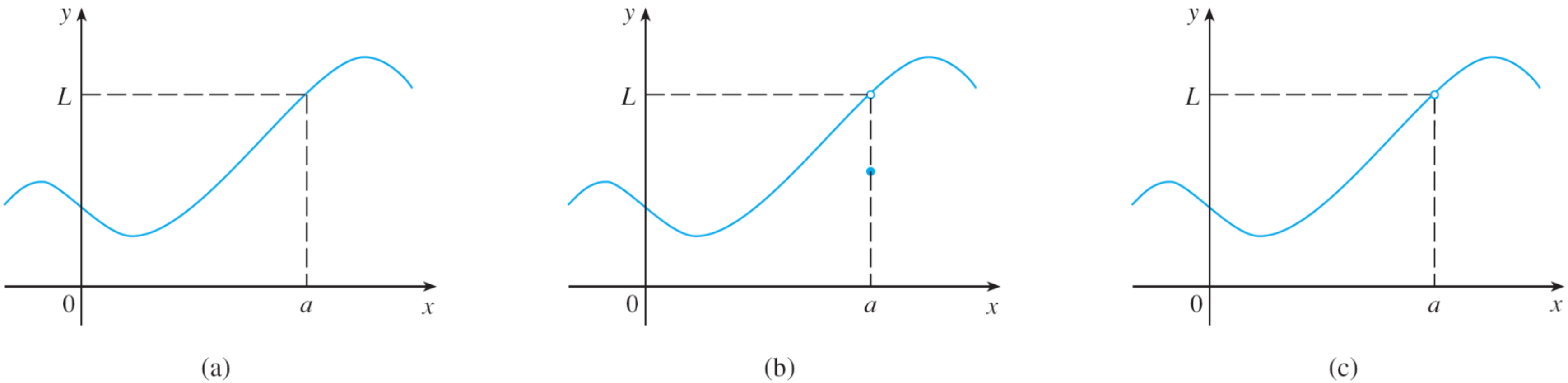
which is usually read "$f(x)$ approaches L as x approaches a."

Notice the phrase "but $x \neq a$" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near a*.

The Limit of a Function

Figure 2 shows the graphs of three functions. Note that in part (c), $f(a)$ is not defined and in part (b), $f(a) \neq L$.

But in each case, regardless of what happens at a, it is true that $\lim_{x \to a} f(x) = L$.



$\lim_{x \to a} f(x) = L$ in all three cases

Figure 2

Example 1

Guess the value of $\lim_{x \to 1} \dfrac{x - 1}{x^2 - 1}$.

Solution:

Notice that the function $f(x) = (x - 1)/(x^2 - 1)$ is not defined when $x = 1$, but that doesn't matter because the definition of $\lim_{x \to a} f(x)$ says that we consider values of x that are close to a but not equal to a.

Example 1 – *Solution*

cont'd

The tables below give values of $f(x)$ (correct to six decimal places) for values of x that approach 1 (but are not equal to 1).

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

On the basis of the values in the tables, we make the guess that

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = 0.5$$

The Limit of a Function

Example 1 is illustrated by the graph of *f* in Figure 3. Now let's change *f* slightly by giving it the value 2 when $x = 1$ and calling the resulting function *g*:

$$g(x) = \begin{cases} \dfrac{x-1}{x^2-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

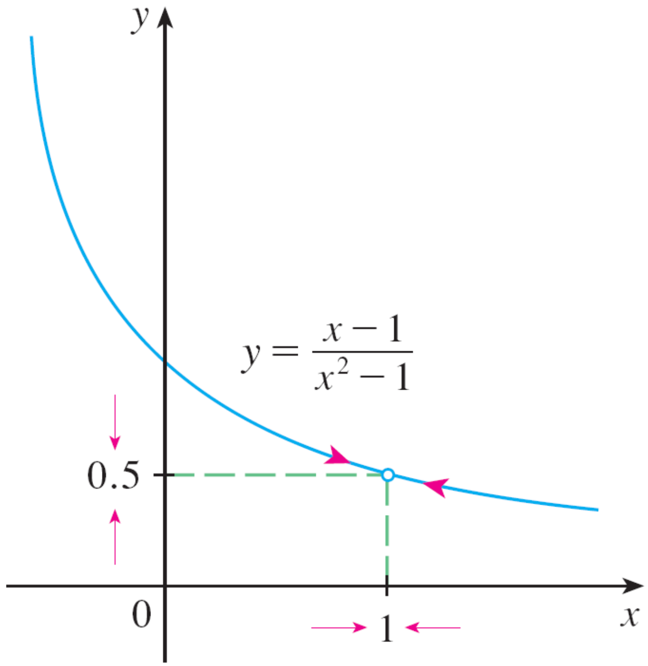


Figure 3

The Limit of a Function

This new function g still has the same limit as x approaches 1. (See Figure 4.)

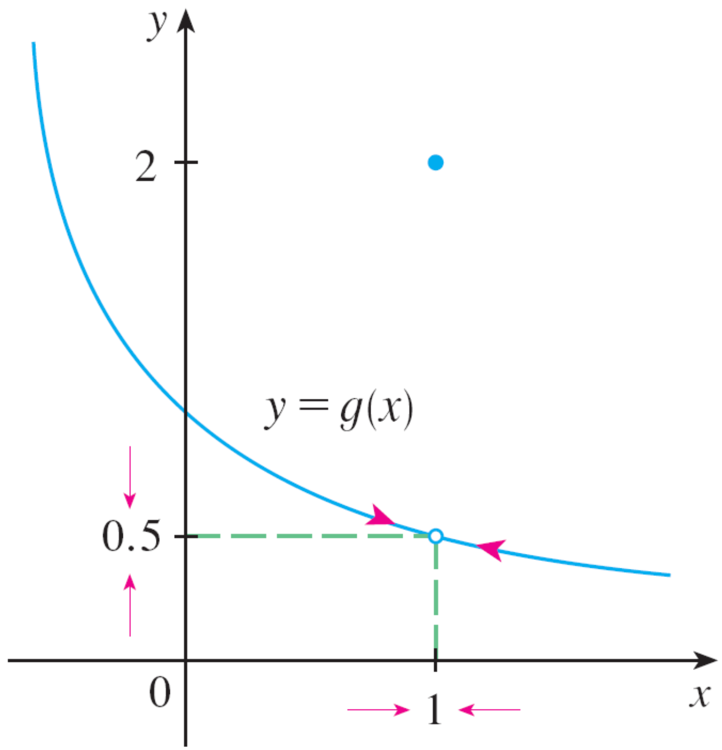


Figure 4

One-Sided Limits

One-Sided Limits

The function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

$H(t)$ approaches 0 as t approaches 0 from the left and $H(t)$ approaches 1 as t approaches 0 from the right.

We indicate this situation symbolically by writing

$$\lim_{t \to 0^-} H(t) = 0 \qquad \text{and} \qquad \lim_{t \to 0^+} H(t) = 1$$

One-Sided Limits

The symbol "$t \to 0^-$" indicates that we consider only values of t that are less than 0.

Likewise, "$t \to 0^+$" indicates that we consider only values of t that are greater than 0.

One-Sided Limits

$\boxed{2}$ **Definition** We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

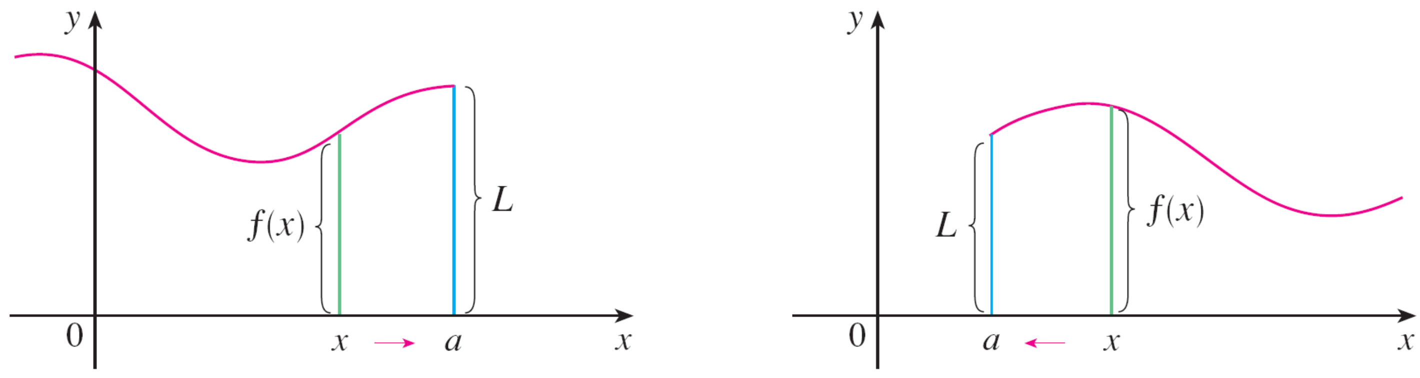
Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a.

One-Sided Limits

Similarly, if we require that x be greater than a, we get "the **right-hand limit of $f(x)$ as x approaches a** is equal to L" and we write

$$\lim_{x \to a^+} f(x) = L$$

Thus the symbol "$x \to a^+$" means that we consider only $x > a$. These definitions are illustrated in Figure 9.



(a) $\lim_{x \to a^-} f(x) = L$

(b) $\lim_{x \to a^+} f(x) = L$

Figure 9

One-Sided Limits

By comparing Definition 1 with the definitions of one-sided limits, we see that the following is true.

$$\boxed{3} \qquad \lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

Example 7

The graph of a function *g* is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

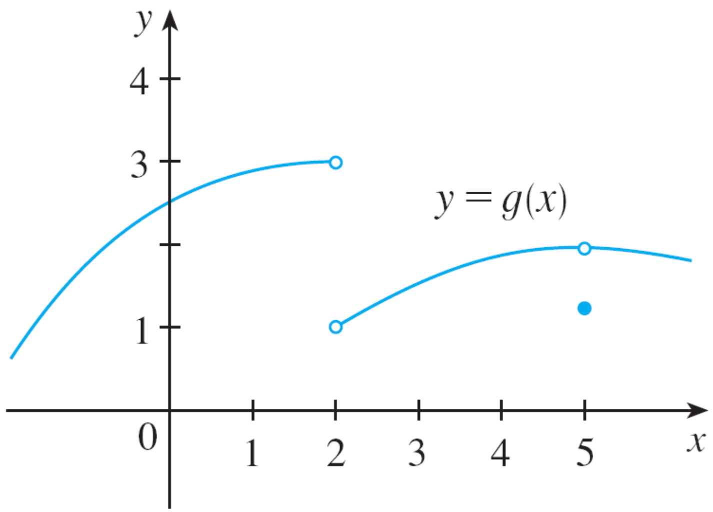


Figure 10

Example 7 – *Solution*

From the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right.

Therefore

$$\text{(a)} \quad \lim_{x \to 2^-} g(x) = 3 \qquad \text{and} \qquad \text{(b)} \quad \lim_{x \to 2^+} g(x) = 1$$

(c) Since the left and right limits are different, we conclude from $\boxed{3}$ that $\lim_{x \to 2} g(x)$ does not exist.

Example 7 – *Solution*

cont'd

The graph also shows that

(d) $\lim_{x \to 5^-} g(x) = 2$ and (e) $\lim_{x \to 5^+} g(x) = 2$

(f) This time the left and right limits are the same and so, by $\boxed{3}$, we have

$$\lim_{x \to 5} g(x) = 2$$

Despite this fact, notice that $g(5) \neq 2$.

Infinite Limits

Infinite Limits

4 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Another notation for $\lim_{x \to a} f(x) = \infty$ is

$$f(x) \to \infty \text{ as } x \to a$$

Infinite Limits

Again, the symbol ∞ is not a number, but the expression $\lim_{x \to a} f(x) = \infty$ is often read as

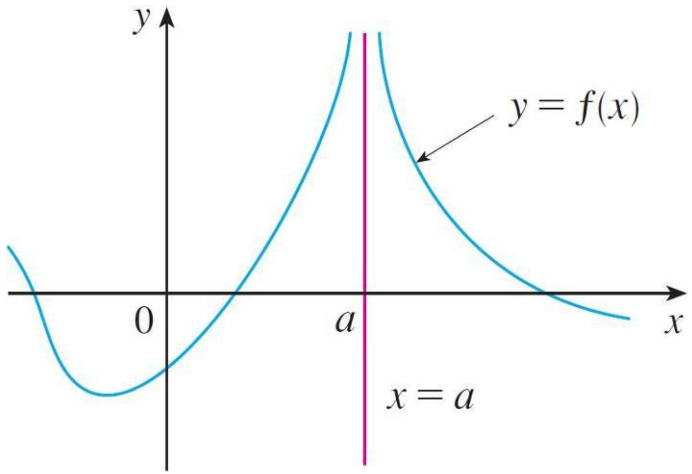
"the limit of $f(x)$, as approaches a, is infinity"

or "$f(x)$ becomes infinite as approaches a"

or "$f(x)$ increases without bound as approaches a"

Infinite Limits

This definition is illustrated graphically in Figure 12.

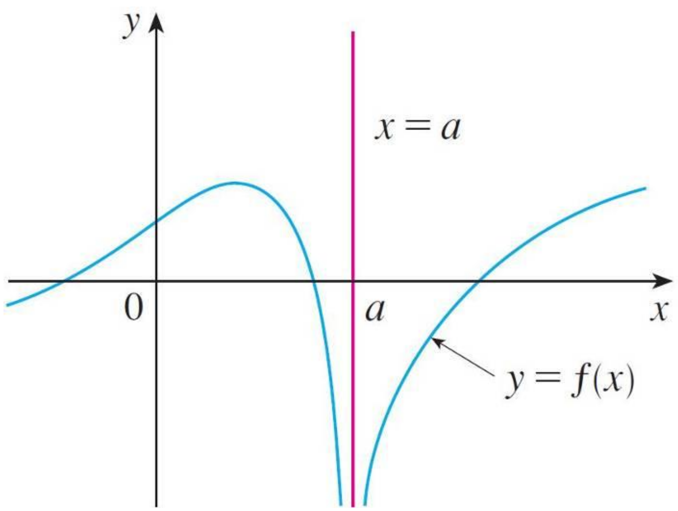


$$\lim_{x \to a} f(x) = \infty$$

Figure 12

Infinite Limits

A similar sort of limit, for functions that become large negative as *x* gets close to *a*, is defined in Definition 5 and is illustrated in Figure 13.



$$\lim_{x \to a} f(x) = -\infty$$

Figure 13

Infinite Limits

5 Definition Let f be defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

The symbol $\lim_{x \to a} f(x) = -\infty$ can be read as "the limit of $f(x)$, as x approaches a, is negative infinity" or "$f(x)$ decreases without bound as x approaches a." As an example we have

$$\lim_{x \to 0} \left(-\frac{1}{x^2} \right) = -\infty$$

Infinite Limits

Similar definitions can be given for the one-sided infinite limits

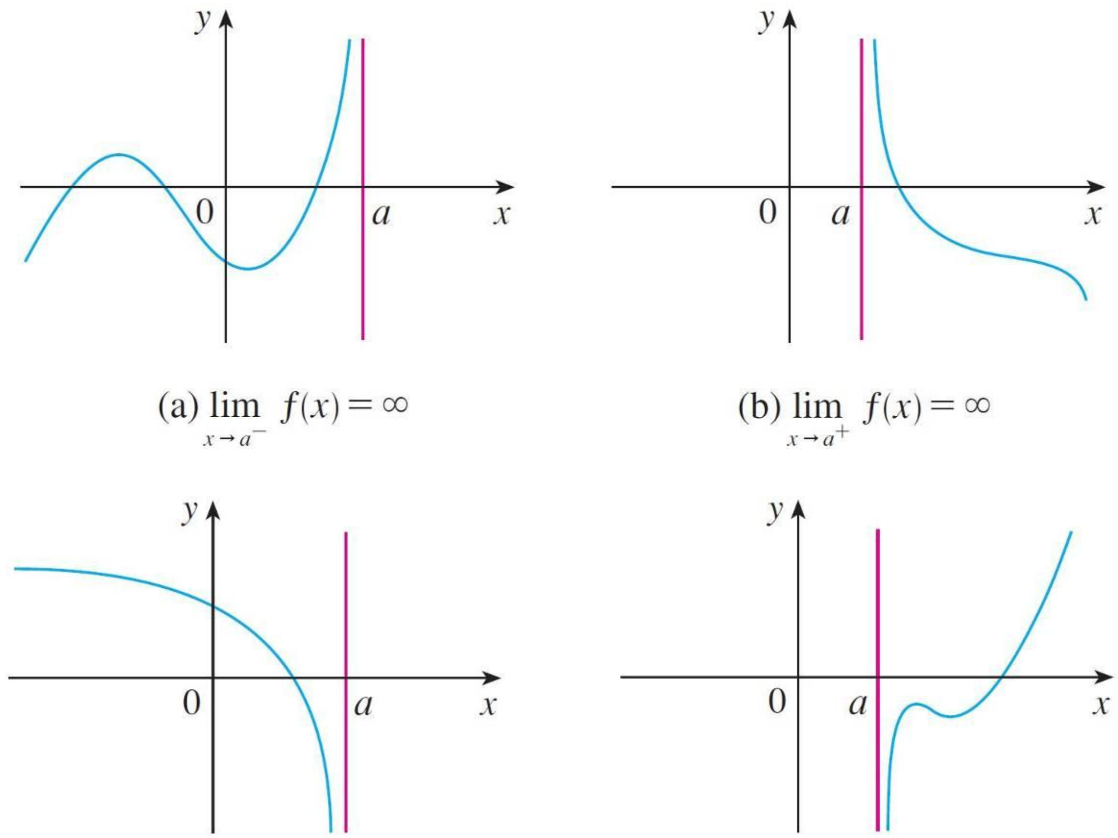
$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

remembering that "$x \to a^-$" means that we consider only values of x that are less than a, and similarly "$x \to a^+$" means that we consider only $x > a$.

Infinite Limits

Illustrations of these four cases are given in Figure 14.



(a) $\lim_{x \to a^-} f(x) = \infty$

(b) $\lim_{x \to a^+} f(x) = \infty$

(c) $\lim_{x \to a^-} f(x) = -\infty$

(d) $\lim_{x \to a^+} f(x) = -\infty$

Figure 14

Infinite Limits

$\boxed{6}$ **Definition** The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Example 10

Find the vertical asymptotes of $f(x) = \tan x$.

Solution:

Because

$$\tan x = \frac{\sin x}{\cos x}$$

there are potential vertical asymptotes where $\cos x = 0$.

In fact, since $\cos x \to 0^+$ as $x \to (\pi/2)^-$ and $\cos x \to 0^-$ as $x \to (\pi/2)^+$, whereas $\sin x$ is positive when x is near $\pi/2$, we have

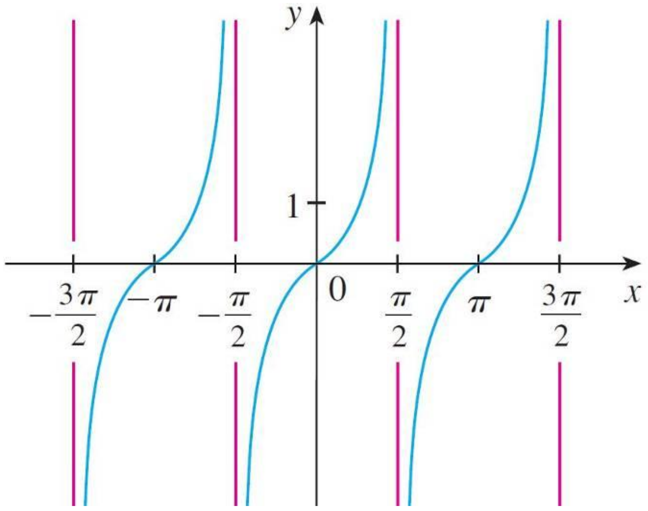
$$\lim_{x \to (\pi/2)^-} \tan x = \infty \qquad \text{and} \qquad \lim_{x \to (\pi/2)^+} \tan x = -\infty$$

Example 10 – *Solution*

cont'd

This shows that the line $x = \pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = (2n + 1)\pi/2$, where n is an integer, are all vertical asymptotes of $f(x) = \tan x$.

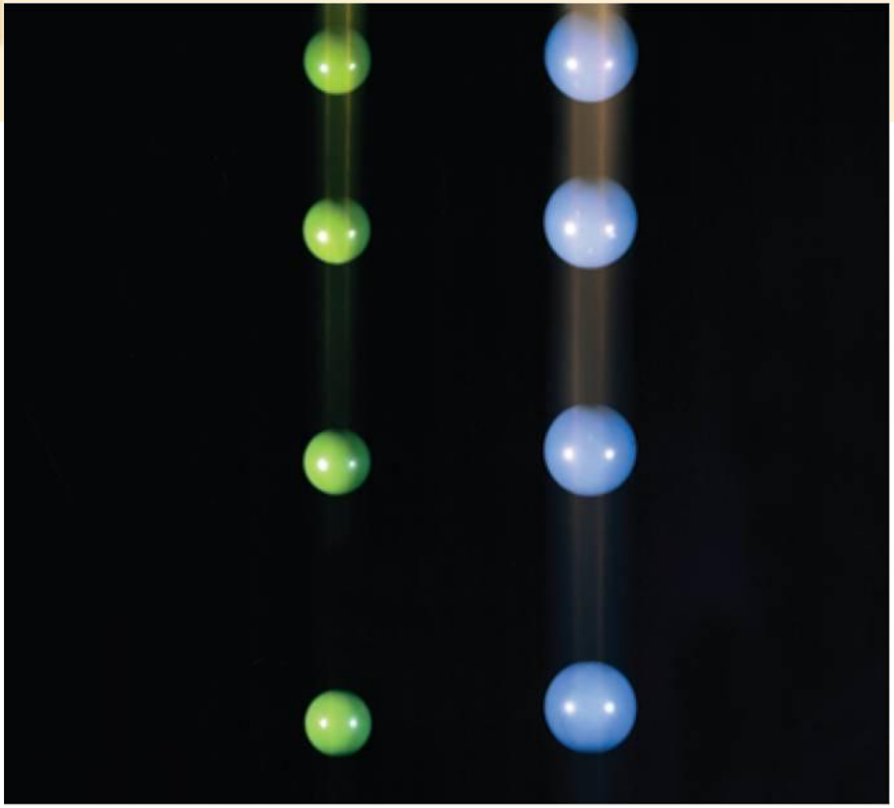
The graph in Figure 16 confirms this.



$y = \tan x$

Figure 16

2 Limits and Derivatives



2.3 Calculating Limits Using the Limit Laws

Calculating Limits Using the Limit Laws

In this section we use the following properties of limits, called the *Limit Laws,* to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$

Calculating Limits Using the Limit Laws

These five laws can be stated verbally as follows:

Sum Law

1. The limit of a sum is the sum of the limits.

Difference Law

2. The limit of a difference is the difference of the limits.

Constant Multiple Law

3. The limit of a constant times a function is the constant times the limit of the function.

Calculating Limits Using the Limit Laws

Product Law

4. The limit of a product is the product of the limits.

Quotient Law

5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$.

Example 1

Use the Limit Laws and the graphs of *f* and *g* in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \frac{f(x)}{g(x)}$

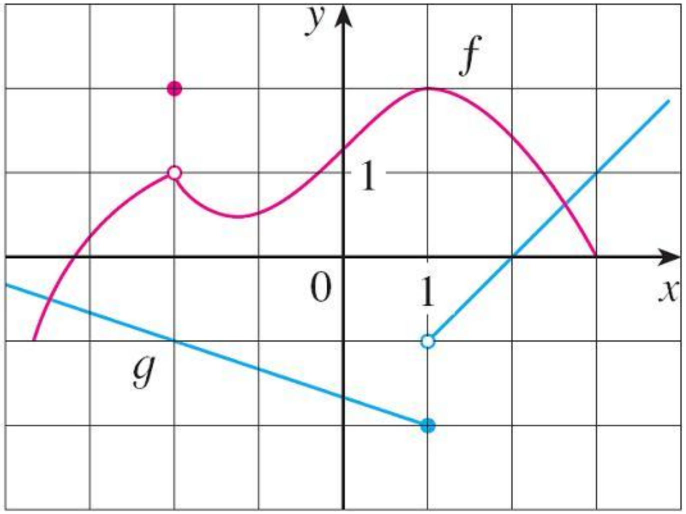


Figure 1

Example 1(a) – *Solution*

From the graphs of *f* and *g* we see that

$$\lim_{x \to -2} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to -2} g(x) = -1$$

Therefore we have

$$\lim_{x \to -2} [f(x) + 5g(x)] = \lim_{x \to -2} f(x) + \lim_{x \to -2} [5g(x)] \qquad \text{(by Law 1)}$$

$$= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x) \qquad \text{(by Law 3)}$$

$$= 1 + 5(-1)$$

$$= -4$$

Example 1(b) – *Solution*

cont'd

We see that $\lim_{x \to 1} f(x) = 2$. But $\lim_{x \to 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x \to 1^-} g(x) = -2 \qquad \lim_{x \to 1^+} g(x) = -1$$

So we can't use Law 4 for the desired limit. But we *can* use Law 4 for the one-sided limits:

$$\lim_{x \to 1^-} [f(x)g(x)] = 2 \cdot (-2) = -4 \qquad \lim_{x \to 1^+} [f(x)g(x)] = 2 \cdot (-1) = -2$$

The left and right limits aren't equal, so $\lim_{x \to 1} [f(x)g(x)]$ does not exist.

Example 1(c) – *Solution*

cont'd

The graphs show that

$$\lim_{x \to 2} f(x) \approx 1.4 \quad \text{and} \quad \lim_{x \to 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5.

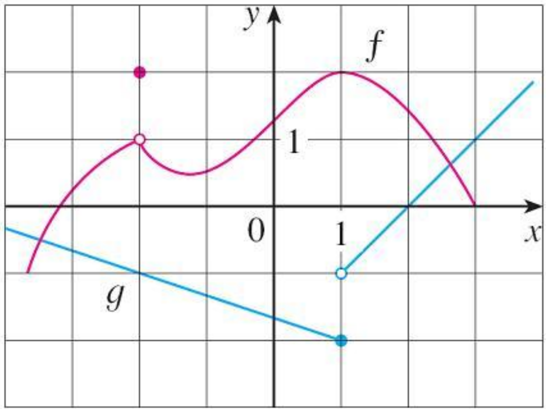


Figure 1

The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

Calculating Limits Using the Limit Laws

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

In applying these six limit laws, we need to use two special limits:

7. $\lim_{x \to a} c = c$

8. $\lim_{x \to a} x = a$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$).

Calculating Limits Using the Limit Laws

If we now put *f*(*x*) = *x* in Law 6 and use Law 8, we get another useful special limit.

9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

A similar limit holds for roots as follows.

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

More generally, we have the following law.

Root Law

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x \to a} f(x) > 0$.]

Calculating Limits Using the Limit Laws

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Functions with the Direct Substitution Property are called *continuous at a.*

In general, we have the following useful fact.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

Calculating Limits Using the Limit Laws

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

Calculating Limits Using the Limit Laws

The next two theorems give two additional properties of limits.

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then
$$\lim_{x \to a} g(x) = L$$

Calculating Limits Using the Limit Laws

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7.

It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

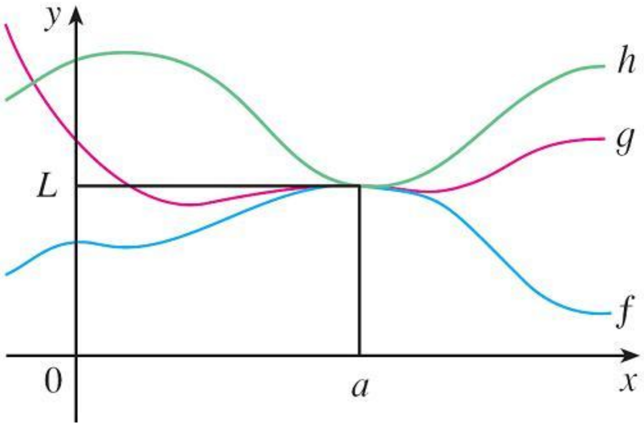
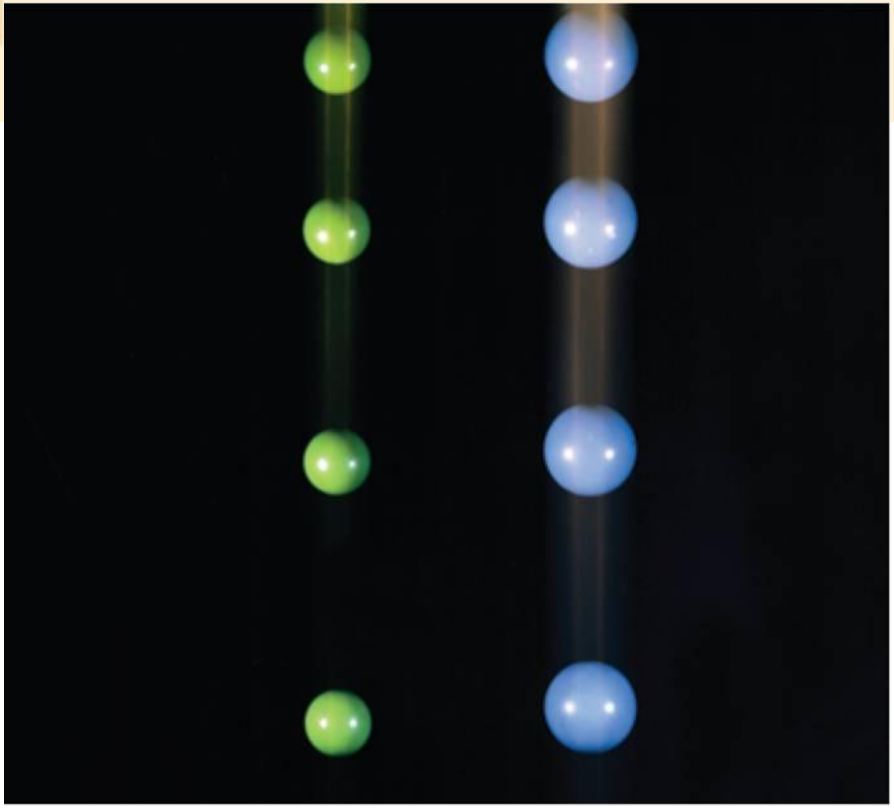


Figure 7

2 Limits and Derivatives



2.4

The Precise Definition of a Limit

The Precise Definition of a Limit

The intuitive definition of a limit is inadequate for some purposes because such phrases as “*x* is close to 2” and “*f*(*x*) gets closer and closer to *L*” are vague.

In order to be able to prove conclusively that

$$\lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10{,}000} \right) = 0.0001 \qquad \text{or} \qquad \lim_{x \to 0} \frac{\sin x}{x} = 1$$

we must make the definition of a limit precise.

The Precise Definition of a Limit

To motivate the precise definition of a limit, let's consider the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Intuitively, it is clear that when x is close to 3 but $x \neq 3$, then $f(x)$ is close to 5, and so $\lim_{x \to 3} f(x) = 5$.

To obtain more detailed information about how $f(x)$ varies when x is close to 3, we ask the following question:
How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.1?

The Precise Definition of a Limit

The distance from x to 3 is $|x - 3|$ and the distance from $f(x)$ to 5 is $|f(x) - 5|$, so our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad |x - 3| < \delta \qquad \text{but } x \neq 3$$

If $|x - 3| > 0$, then $x \neq 3$, so an equivalent formulation of our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad 0 < |x - 3| < \delta$$

The Precise Definition of a Limit

Notice that if $0 < |x - 3| < (0.1)/2 = 0.05$ then

$$|f(x) - 5| = |(2x - 1) - 5| = |2x - 6|$$

$$= 2|x - 3| < 2(0.05) = 0.1$$

that is,

$$|f(x) - 5| < 0.1 \quad \text{if} \quad 0 < |x - 3| < 0.05$$

Thus an answer to the problem is given by $\delta = 0.05$; that is, if x is within a distance of 0.05 from 3, then $f(x)$ will be within a distance of 0.1 from 5.

The Precise Definition of a Limit

If we change the number 0.1 in our problem to the smaller number 0.01, then by using the same method we find that $f(x)$ will differ from 5 by less than 0.01 provided that x differs from 3 by less than $(0.01)/2 = 0.005$:

$$|f(x) - 5| < 0.01 \qquad \text{if} \qquad 0 < |x - 3| < 0.005$$

Similarly,

$$|f(x) - 5| < 0.001 \qquad \text{if} \qquad 0 < |x - 3| < 0.0005$$

The numbers 0.1, 0.01 and 0.001 that we have considered are *error tolerances* that we might allow.

The Precise Definition of a Limit

For 5 to be the precise limit of $f(x)$ as x approaches 3, we must not only be able to bring the difference between $f(x)$ and 5 below each of these three numbers; we must be able to bring it below *any* positive number.

And, by the same reasoning, we can! If we write ε (the Greek letter epsilon) for an arbitrary positive number, then we find as before that

$$\boxed{1} \qquad |f(x) - 5| < \varepsilon \quad \text{if} \quad 0 < |x - 3| < \delta = \frac{\varepsilon}{2}$$

The Precise Definition of a Limit

This is a precise way of saying that $f(x)$ is close to 5 when x is close to 3 because $\boxed{1}$ says that we can make the values of $f(x)$ within an arbitrary distance ε from 5 by taking the values of x within a distance $\varepsilon/2$ from 3 (but $x \neq 3$).

Note that $\boxed{1}$ can be rewritten as follows: if

$$3 - \delta < x < 3 + \delta \qquad (x \neq 3)$$

then

$$5 - \varepsilon < f(x) < 5 + \varepsilon$$

and this is illustrated in Figure 1.

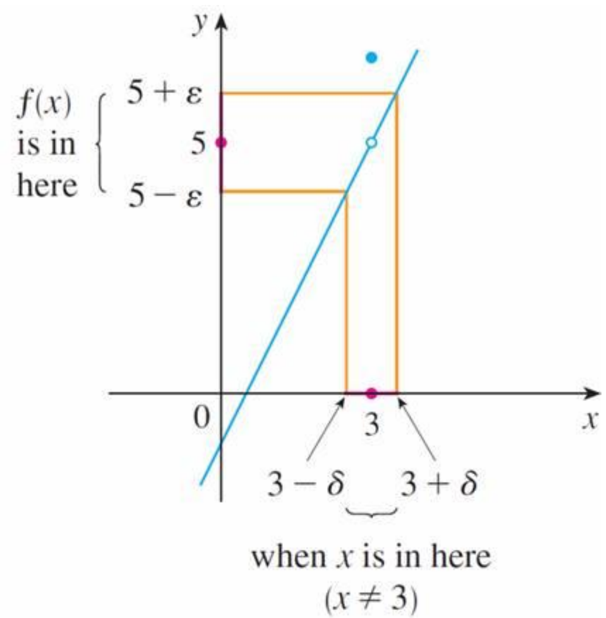


Figure 1

The Precise Definition of a Limit

By taking the values of x ($\neq 3$) to lie in the interval $(3 - \delta, 3 + \delta)$ we can make the values of $f(x)$ lie in the interval $(5 - \varepsilon, 5 + \varepsilon)$.

Using $\boxed{1}$ as a model, we give a precise definition of a limit.

$\boxed{2}$ **Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

The Precise Definition of a Limit

Since $|x - a|$ is the distance from x to a and $|f(x) - L|$ is the distance from $f(x)$ to L, and since ε can be arbitrarily small, the definition of a limit can be expressed in words as follows:

$$\lim_{x \to a} f(x) = L$$

means that the distance between $f(x)$ and L can be made arbitrarily small by taking the distance from x to a sufficiently small (but not 0).

The Precise Definition of a Limit

Alternatively,

$$\lim_{x \to a} f(x) = L$$

the values of $f(x)$ can be made as close as we please to L by taking x close enough to a (but not equal to a).

The Precise Definition of a Limit

We can also reformulate Definition 2 in terms of intervals by observing that the inequality $|x - a| < \delta$ is equivalent to $-\delta < x - a < \delta$, which in turn can be written as $a - \delta < x < a + \delta$.

Also $0 < |x - a|$ is true if and only if $x - a \neq 0$, that is, $x \neq a$.

The Precise Definition of a Limit

Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to the pair of inequalities $L - \varepsilon < f(x) < L + \varepsilon$. Therefore, in terms of intervals, Definition 2 can be stated as follows:

$$\lim_{x \to a} f(x) = L$$

means that for every $\varepsilon > 0$ (no matter how small ε is) we can find $\delta > 0$ such that if x lies in the open interval $(a - \delta, a + \delta)$ and $x \neq a$, then $f(x)$ lies in the open interval $(L - \varepsilon, L + \varepsilon)$.

The Precise Definition of a Limit

We interpret this statement geometrically by representing a function by an arrow diagram as in Figure 2, where *f* maps a subset of $\mathbb{R}$ onto another subset of $\mathbb{R}$.

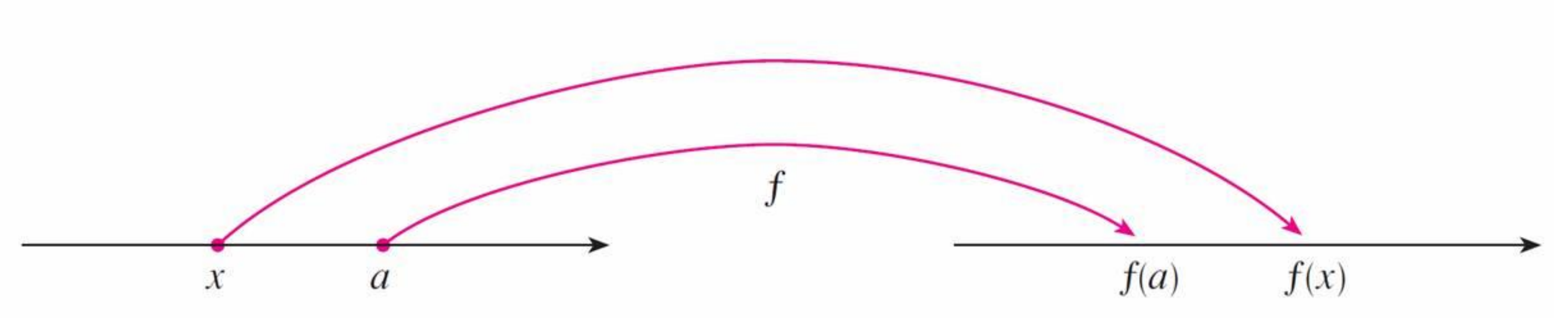


Figure 2

The Precise Definition of a Limit

The definition of limit says that if any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L, then we can find an interval $(a - \delta, a + \delta)$ around a such that f maps all the points in $(a - \delta, a + \delta)$ (except possibly a) into the interval $(L - \varepsilon, L + \varepsilon)$. (See Figure 3.)

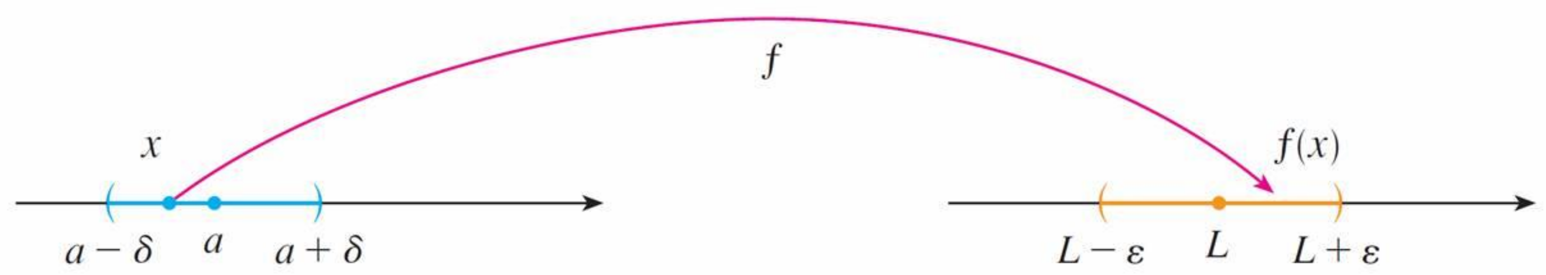


Figure 3

The Precise Definition of a Limit

Another geometric interpretation of limits can be given in terms of the graph of a function. If $\varepsilon > 0$ is given, then we draw the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ and the graph of f. (See Figure 4.)

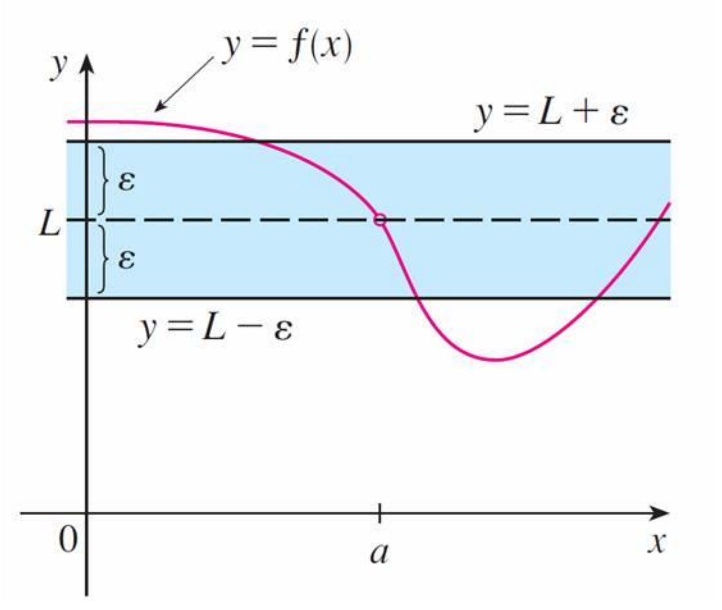


Figure 4

The Precise Definition of a Limit

If $\lim_{x \to a} f(x) = L$, then we can find a number $\delta > 0$ such that if we restrict x to lie in the interval $(a - \delta, a + \delta)$ and take $x \neq a$, then the curve $y = f(x)$ lies between the lines $y = L - \varepsilon$ and $y = L + \varepsilon$ (See Figure 5.) You can see that if such a δ has been found, then any smaller δ will also work.

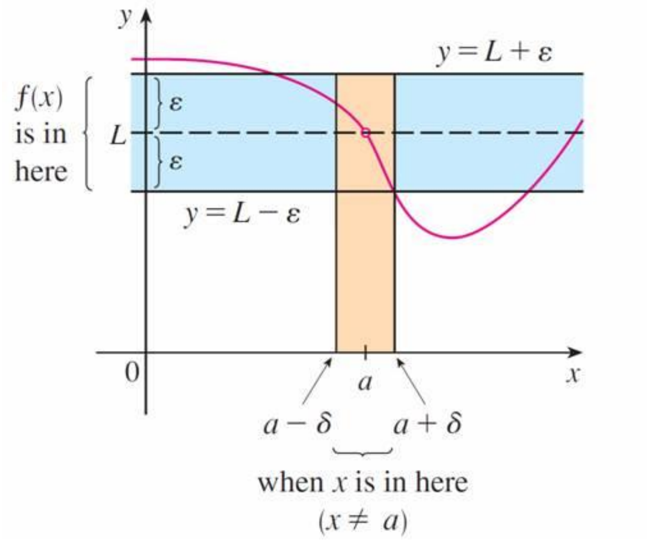


Figure 5

The Precise Definition of a Limit

It is important to realize that the process illustrated in Figures 4 and 5 must work for *every* positive number ε, no matter how small it is chosen. Figure 6 shows that if a smaller ε is chosen, then a smaller δ may be required.

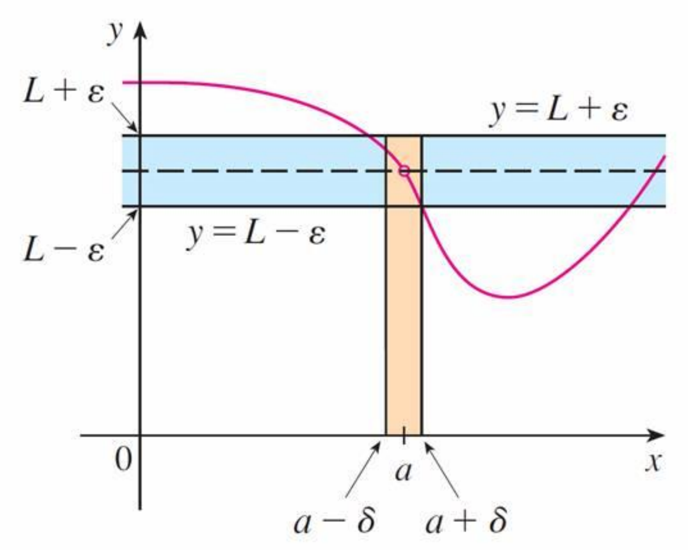


Figure 6

Example 1

Use a graph to find a number δ such that

$$\text{if } |x - 1| < \delta \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.2$$

In other words, find a number δ that corresponds to $\varepsilon = 0.2$ in the definition of a limit for the function $f(x) = x^3 - 5x + 6$ with $a = 1$ and $L = 2$.

Example 1 – *Solution*

A graph of *f* is shown in Figure 7; we are interested in the region near the point (1, 2).

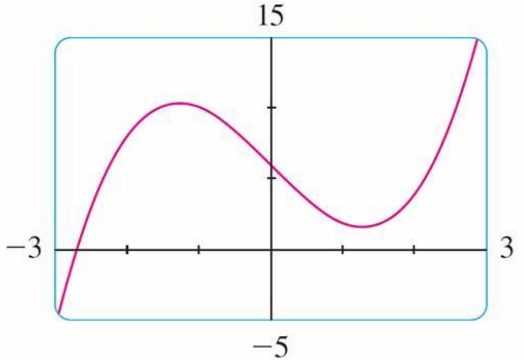


Figure 7

Notice that we can rewrite the inequality

$$|(x^3 - 5x + 6) - 2| < 0.2$$

as
$$1.8 < x^3 - 5x + 6 < 2.2$$

Example 1 – *Solution*

cont'd

So we need to determine the values of for x which the curve $y = x^3 - 5x + 6$ lies between the horizontal lines $y = 1.8$ and $y = 2.2$.

Therefore we graph the curves $y = x^3 - 5x + 6$, $y = 1.8$, and $y = 2.2$ near the point (1,2) in Figure 8.

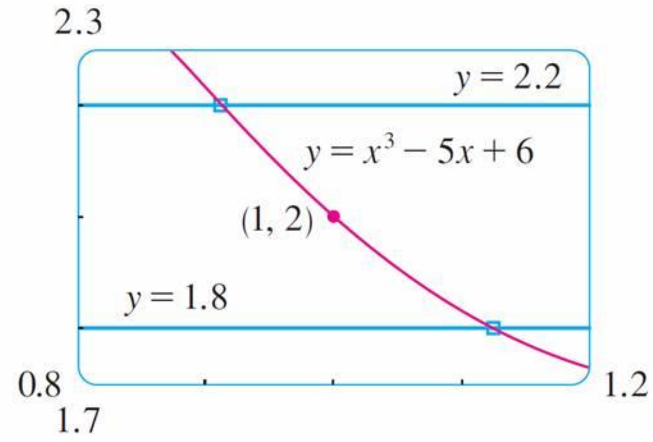


Figure 8

Example 1 – *Solution*

cont'd

Then we use the cursor to estimate that the x-coordinate of the point of intersection of the line $y = 2.2$ and the curve $y = x^3 - 5x + 6$ is about 0.911.

Similarly, $y = x^3 - 5x + 6$ intersects the line $y = 1.8$ when $x \approx 1.124$. So, rounding to be safe, we can say that

$$\text{if} \quad 0.92 < x < 1.12 \quad \text{then} \quad 1.8 < x^3 - 5x + 6 < 2.2$$

This interval $(0.92, 1.12)$ is not symmetric about $x = 1$. The distance from $x = 1$ to the left endpoint is $1 - 0.92 = 0.08$ and the distance to the right endpoint is 0.12.

Example 1 – *Solution*

cont'd

We can choose δ to be the smaller of these numbers, that is, $\delta = 0.08$.

Then we can rewrite our inequalities in terms of distances as follows:

$$\text{if} \quad |x - 1| < 0.08 \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.2$$

This just says that by keeping x within 0.08 of 1, we are able to keep $f(x)$ within 0.2 of 2.

Although we chose $\delta = 0.08$, any smaller positive value of δ would also have worked.

Example 2

Prove that $\lim_{x \to 3} (4x - 5) = 7$.

Solution:

1. *Preliminary analysis of the problem* (*guessing a value for* δ).

Let ε be a given positive number. We want to find a number δ such that

if $\quad 0 < |x - 3| < \delta \quad$ then $\quad |(4x - 5) - 7| < \varepsilon$

But $|(4x - 5) - 7| = |4x - 12| = |4(x - 3)| = 4|x - 3|$.

Example 2 – *Solution*

cont'd

Therefore we want δ such that

$$\text{if} \quad 0 < |x - 3| < \delta \qquad \text{then} \qquad 4|x - 3| < \varepsilon$$

that is, $\quad$ if $\quad 0 < |x - 3| < \delta \quad$ then $\quad |x - 3| < \dfrac{\varepsilon}{4}$

This suggests that we should choose $\delta = \varepsilon/4$.

Example 2 – *Solution*

cont'd

2. *Proof* (*showing that this δ works*). Given $\varepsilon > 0$, choose $\delta = \varepsilon/4$. If $0 < |x - 3| < \delta$, then

$$|(4x - 5) - 7| = |4x - 12| = 4|x - 3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$

Thus

if $\quad 0 < |x - 3| < \delta \qquad$ then $\qquad |(4x - 5) - 7| < \varepsilon$

Example 2 – *Solution*

Therefore, by the definition of a limit,

$$\lim_{x \to 3} (4x - 5) = 7$$

This example is illustrated by Figure 9.

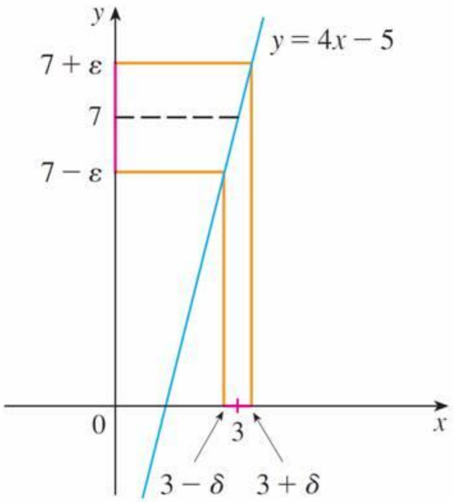


Figure 9

The Precise Definition of a Limit

The intuitive definitions of one-sided limits can be precisely reformulated as follows.

3 Definition of Left-Hand Limit

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \varepsilon$$

The Precise Definition of a Limit

$\boxed{4}$ **Definition of Right-Hand Limit**

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \qquad a < x < a + \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Example 3

Use Definition 4 to prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Example 3 – *Solution*

1. *Guessing a value for δ.* Let ε be a given positive number. Here $a = 0$ and $L = 0$, so we want to find a number δ such that

$$\text{if} \quad 0 < x < \delta \qquad \text{then} \qquad |\sqrt{x} - 0| < \varepsilon$$

that is,

$$\text{if} \quad 0 < x < \delta \qquad \text{then} \qquad \sqrt{x} < \varepsilon$$

or, squaring both sides of the inequality $\sqrt{x} < \varepsilon$, we get

$$\text{if} \quad 0 < x < \delta \qquad \text{then} \qquad x < \varepsilon^2$$

This suggests that we should choose $\delta = \varepsilon^2$.

Example 3 – *Solution*

cont'd

2. *Showing that this δ works.* Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. If $0 < x < \delta$, then

$$\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

so $$|\sqrt{x} - 0| < \varepsilon$$

According to Definition 4, this shows that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

The Precise Definition of a Limit

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ both exist, then

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

Infinite Limits

Infinite Limits

Infinite limits can also be defined in a precise way.

$\boxed{6}$ **Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) > M$$

Infinite Limits

This says that the values of $f(x)$ can be made arbitrarily large (larger than any given number M) by taking x close enough to a (within a distance δ, where δ depends on M, but with $x \neq a$). A geometric illustration is shown in Figure 10.

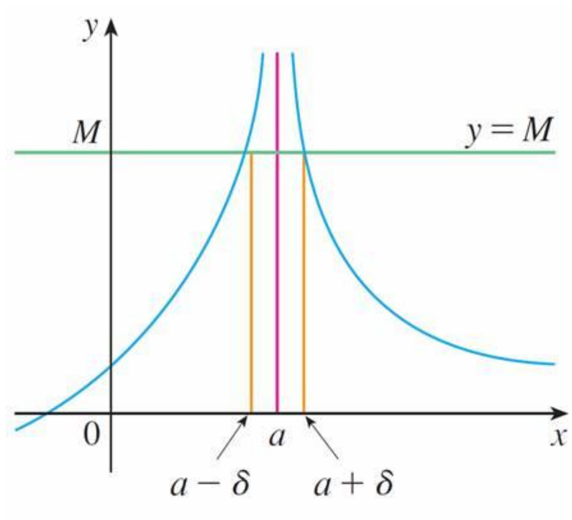


Figure 10

Infinite Limits

Given any horizontal line $y = M$, we can find a number $\delta > 0$ such that if we restrict to x lie in the interval $(a - \delta, a + \delta)$ but $x \neq a$, then the curve $y = f(x)$ lies above the line $y = M$.

You can see that if a larger M is chosen, then a smaller δ may be required.

Example 5

Use Definition 6 to prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution:

Let M be a given positive number. We want to find a number δ such that

if $0 < |x| < \delta$ then $1/x^2 > M$

But $\frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff |x| < \frac{1}{\sqrt{M}}$

So if we choose $\delta = 1/\sqrt{M}$ and $0 < |x| < \delta = 1/\sqrt{M}$, then $1/x^2 > M$. This shows that as $1/x^2 \to \infty$ as $x \to 0$.

Infinite Limits

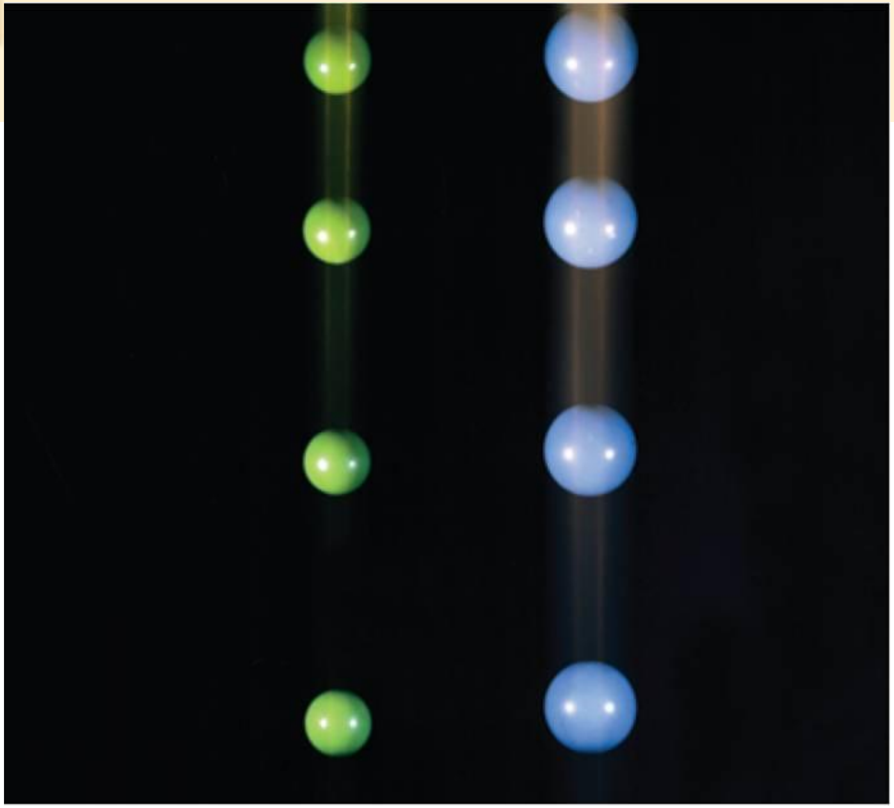
$\boxed{7}$ **Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) < N$$

2 Limits and Derivatives



2.5 Continuity

Continuity

The limit of a function as *x* approaches *a* can often be found simply by calculating the value of the function at *a*. Functions with this property are called *continuous at a*.

We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

1 Definition A function f is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$

Continuity

Notice that Definition 1 implicitly requires three things if *f* is continuous at *a*:

1. *f*(*a*) is defined (that is, *a* is in the domain of *f*)

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

The definition says that *f* is continuous at *a* if *f*(*x*) approaches *f*(*a*) as *x* approaches *a*. Thus a continuous function *f* has the property that a small change in *x* produces only a small change in *f*(*x*).

Continuity

In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous at *a*** (or f has a **discontinuity** at a) if f is not continuous at a.

Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents.

Continuity

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper.

Example 1

Figure 2 shows the graph of a function *f*. At which numbers is *f* discontinuous? Why?

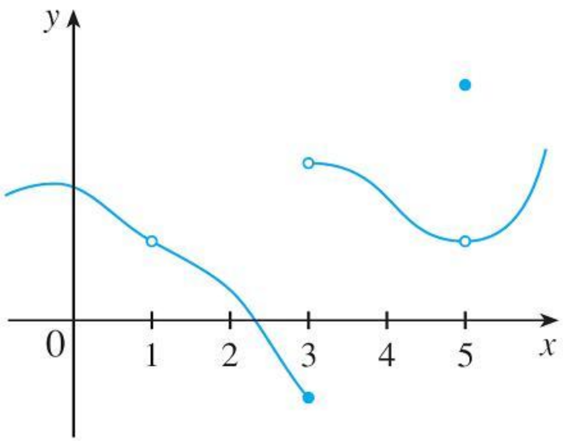


Figure 2

Solution:

It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The official reason that *f* is discontinuous at 1 is that $f(1)$ is not defined.

Example 1 – *Solution*

cont'd

The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x \to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3.

What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x \to 5} f(x)$ exists (because the left and right limits are the same).

But

$$\lim_{x \to 5} f(x) \neq f(5)$$

So f is discontinuous at 5.

Example 2

Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

Solution:

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why f is continuous at all other numbers.

Example 2 – *Solution*

cont'd

(b) Here $f(0) = 1$ is defined but

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2}$$

does not exist. So f is discontinuous at 0.

(c) Here $f(2) = 1$ is defined and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2}$$

Example 2 – *Solution*

cont'd

$$= \lim_{x \to 2} (x + 1)$$

= 3 exists.

But

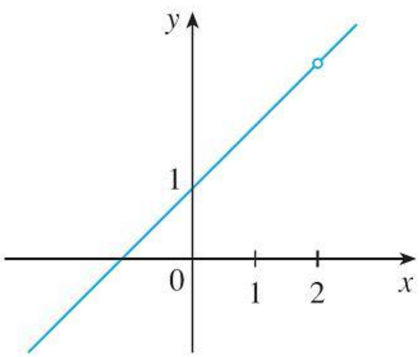
$$\lim_{x \to 2} f(x) \neq f(2)$$

so *f* is not continuous at 2.

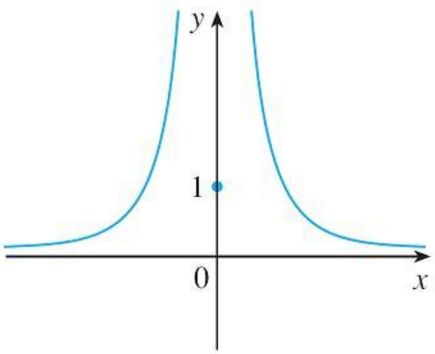
(d) The greatest integer function *f*(*x*) = $[\![x]\!]$ has discontinuities at all of the integers because $\lim_{x \to n} [\![x]\!]$ does not exist if *n* is an integer.

Continuity

Figure 3 shows the graphs of the functions in Example 2.



(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

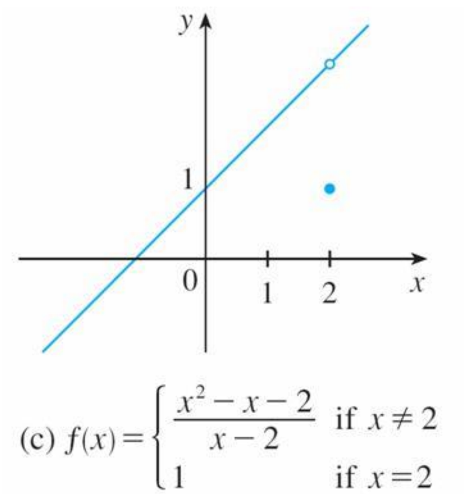


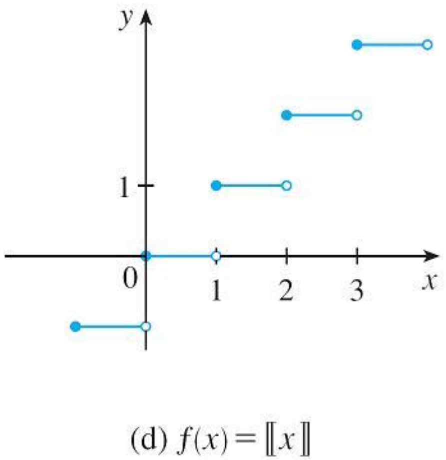
(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Graphs of the functions in Example 2

Figure 3

Continuity





Graphs of the functions in Example 2

Figure 3

Continuity

In each case the graph can’t be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph.

The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.]

The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function “jumps” from one value to another.

Continuity

2 Definition A function f is **continuous from the right at a number *a*** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at *a*** if

$$\lim_{x \to a^-} f(x) = f(a)$$

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

Continuity

Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones.

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

Continuity

It follows from Theorem 4 and Definition 3 that if f and g are continuous on an interval, then so are the functions $f + g$, $f - g$, cf, fg, and (if g is never 0) f/g.

The following theorem was stated as the Direct Substitution Property.

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Continuity

As an illustration of Theorem 5, observe that the volume of a sphere varies continuously with its radius because the formula $V(r) = \frac{4}{3}\pi r^3$ shows that V is a polynomial function of r.

Likewise, if a ball is thrown vertically into the air with a velocity of 50 ft/s, then the height of the ball in feet t seconds later is given by the formula $h = 50t - 16t^2$.

Again this is a polynomial function, so the height is a continuous function of the elapsed time.

Continuity

It turns out that most of the familiar functions are continuous at every number in their domains.

From the appearance of the graphs of the sine and cosine functions, we would certainly guess that they are continuous.

We know from the definitions of $\sin\theta$ and $\cos\theta$ that the coordinates of the point P in Figure 5 are $(\cos\theta, \sin\theta)$. As $\theta \to 0$, we see that P approaches the point (1, 0) and so $\cos\theta \to 1$ and $\sin\theta \to 0$.

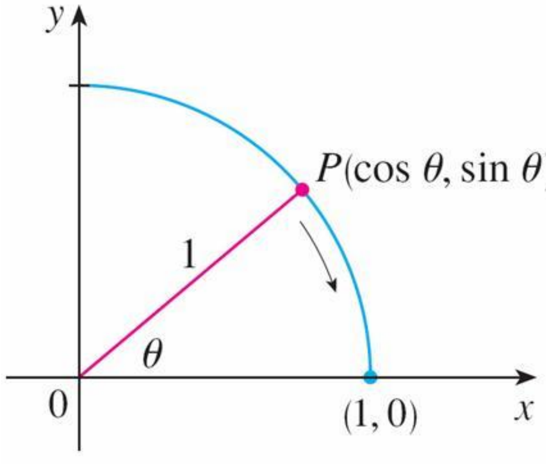


Figure 5

Continuity

Thus

$$\boxed{6} \qquad \lim_{\theta \to 0} \cos \theta = 1 \qquad \lim_{\theta \to 0} \sin \theta = 0$$

Since cos 0 = 1 and sin 0 = 0, the equations in $\boxed{6}$ assert that the cosine and sine functions are continuous at 0.

The addition formulas for cosine and sine can then be used to deduce that these functions are continuous everywhere.

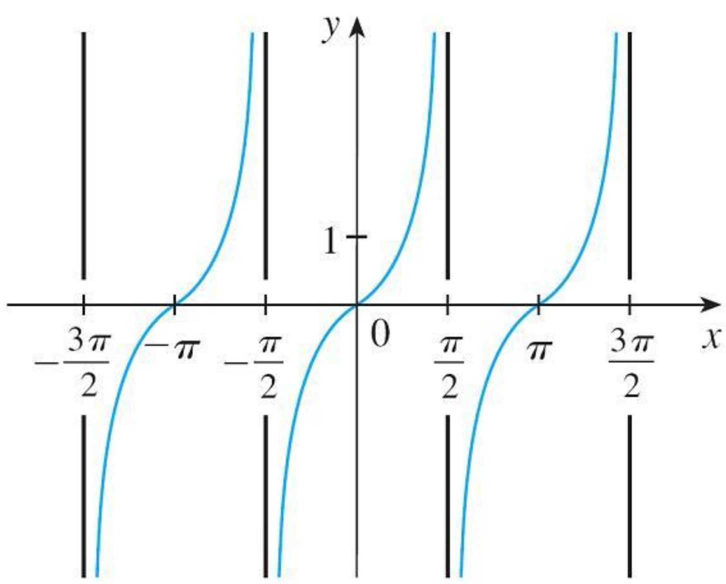
It follows from part 5 of Theorem 4 that

$$\tan x = \frac{\sin x}{\cos x}$$

is continuous except where cos x = 0.

Continuity

This happens when x is an odd integer multiple of $\pi/2$, so $y = \tan x$ has infinite discontinuities when $x = \pm\pi/2, \pm3\pi/2, \pm5\pi/2$, and so on (see Figure 6).



$y = \tan x$

Figure 6

Continuity

7 Theorem The following types of functions are continuous at every number in their domains:

polynomials	rational functions
root functions	trigonometric functions

Another way of combining continuous functions *f* and *g* to get a new continuous function is to form the composite function $f \circ g$. This fact is a consequence of the following theorem.

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Continuity

Intuitively, Theorem 8 is reasonable because if x is close to a, then $g(x)$ is close to b, and since f is continuous at b, if $g(x)$ is close to b, then $f(g(x))$ is close to $f(b)$.

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Continuity

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 8.

Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].

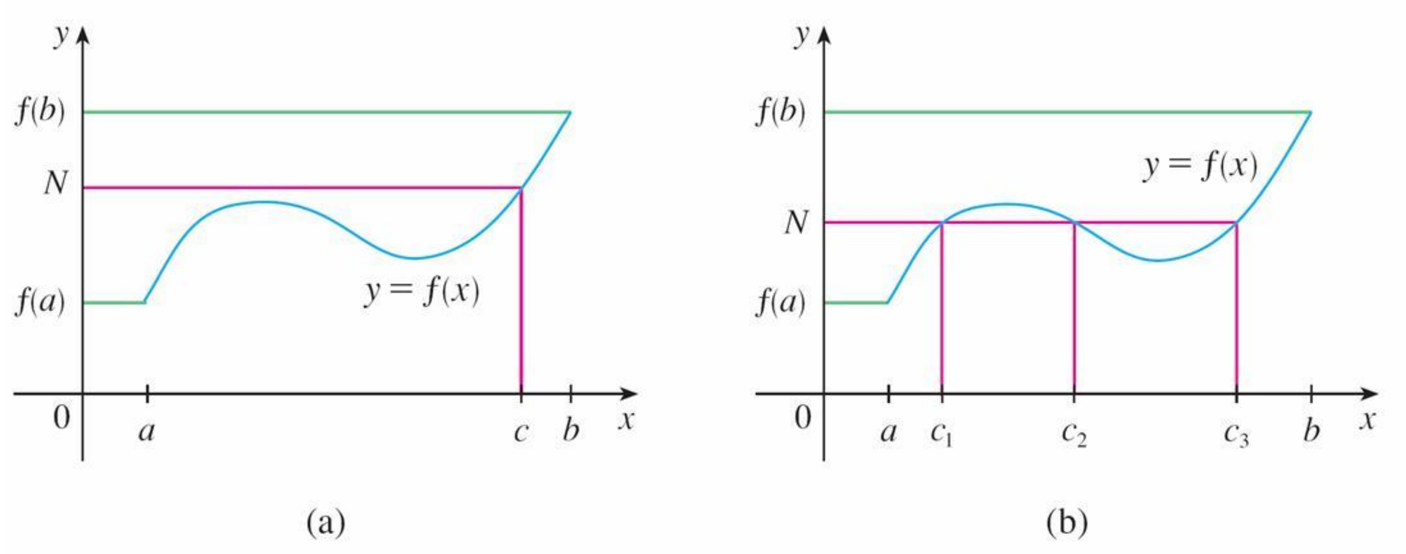


Figure 8

Continuity

If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true.

In geometric terms it says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ as in Figure 9, then the graph of f can't jump over the line. It must intersect $y = N$ somewhere.

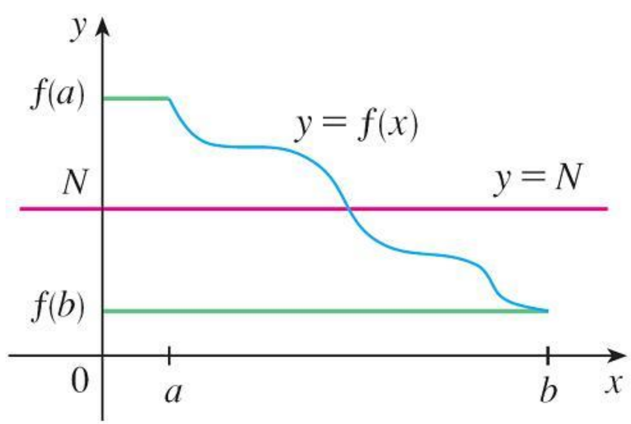


Figure 9

Continuity

It is important that the function *f* in Theorem 10 be continuous. The Intermediate Value Theorem is not true in general for discontinuous functions.

We can use a graphing calculator or computer to illustrate the use of the Intermediate Value Theorem.

Figure 10 shows the graph of *f* in the viewing rectangle [–1, 3] by [–3, 3] and you can see that the graph crosses the *x*-axis between 1 and 2.

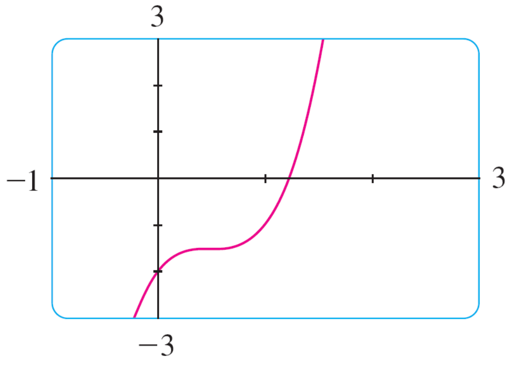


Figure 10

Continuity

Figure 11 shows the result of zooming in to the viewing rectangle [1.2, 1.3] by [–0.2, 0.2].

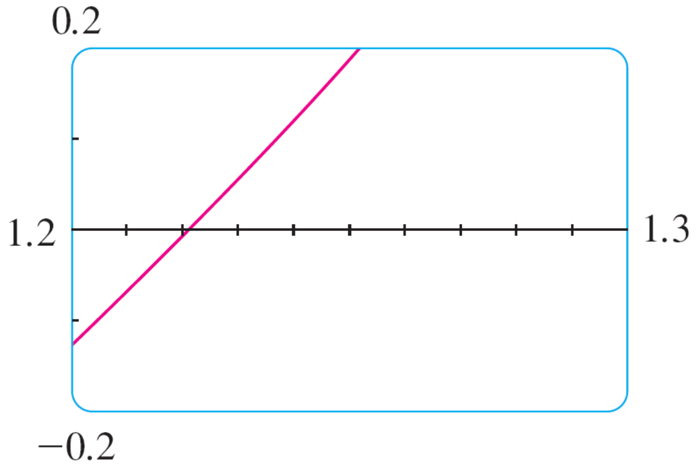


Figure 11

In fact, the Intermediate Value Theorem plays a role in the very way these graphing devices work.

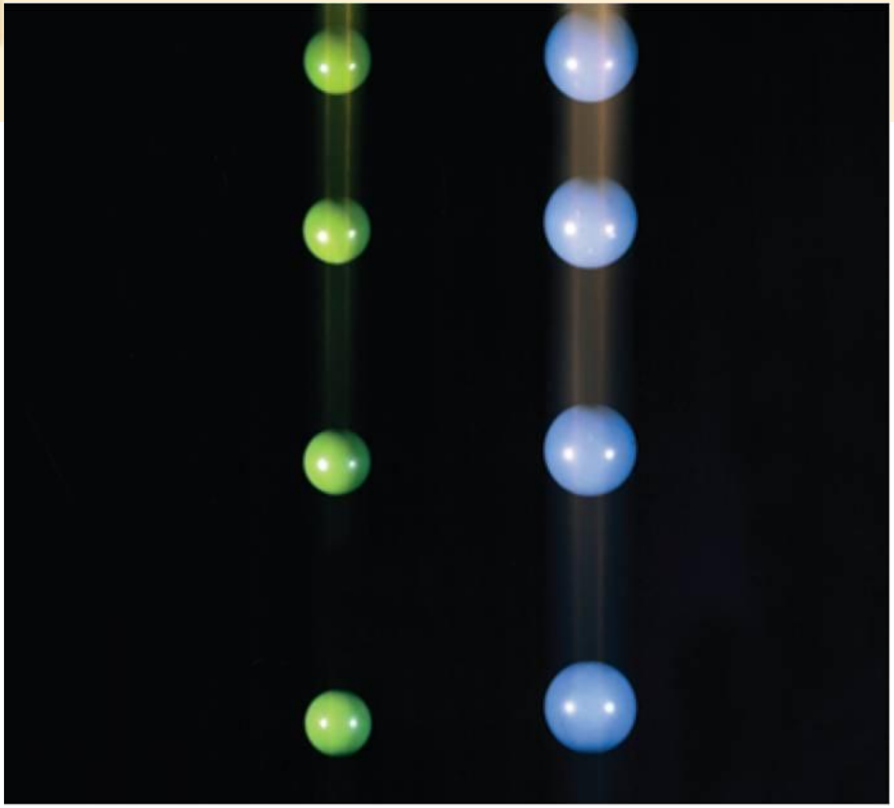
Continuity

A computer calculates a finite number of points on the graph and turns on the pixels that contain these calculated points.

It assumes that the function is continuous and takes on all the intermediate values between two consecutive points.

The computer therefore connects the pixels by turning on the intermediate pixels.

2 Limits and Derivatives



2.6 Limits at Infinity; Horizontal Asymptotes

Limits at Infinity; Horizontal Asymptotes

In this section we let *x* become arbitrarily large (positive or negative) and see what happens to *y*.

Let's begin by investigating the behavior of the function *f* defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as *x* becomes large.

Limits at Infinity; Horizontal Asymptotes

The table gives values of this function correct to six decimal places, and the graph of *f* has been drawn by a computer in Figure 1.

x	$f(x)$
0	-1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998

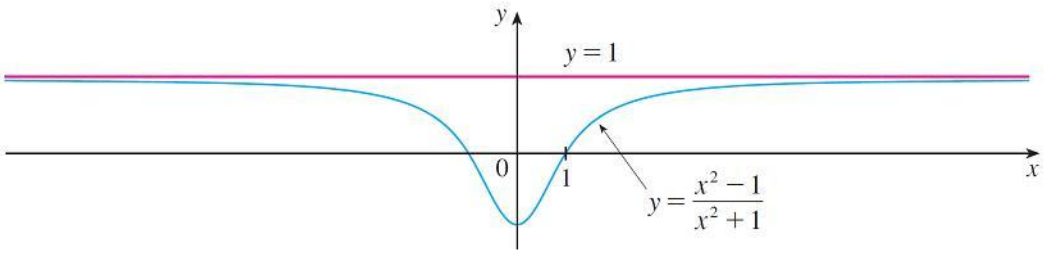


Figure 1

Limits at Infinity; Horizontal Asymptotes

As x grows larger and larger you can see that the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large.

This situation is expressed symbolically by writing

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

Limits at Infinity; Horizontal Asymptotes

In general, we use the notation

$$\lim_{x \to \infty} f(x) = L$$

to indicate that the values of *f*(*x*) approach *L* as *x* becomes larger and larger.

1 Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

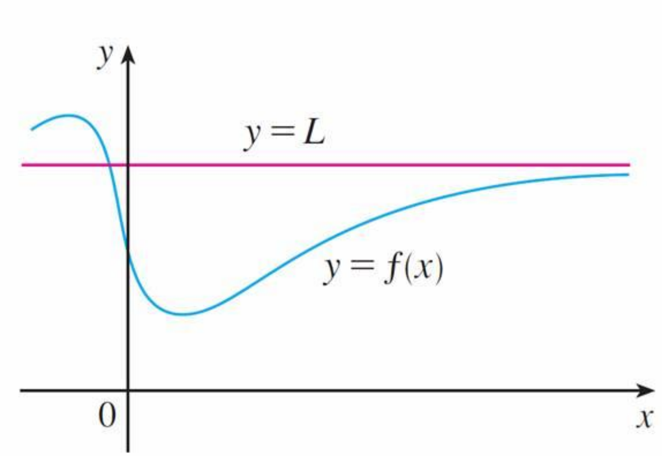
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Limits at Infinity; Horizontal Asymptotes

Another notation for $\lim_{x\to\infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

Geometric illustrations of Definition 1 are shown in Figure 2.

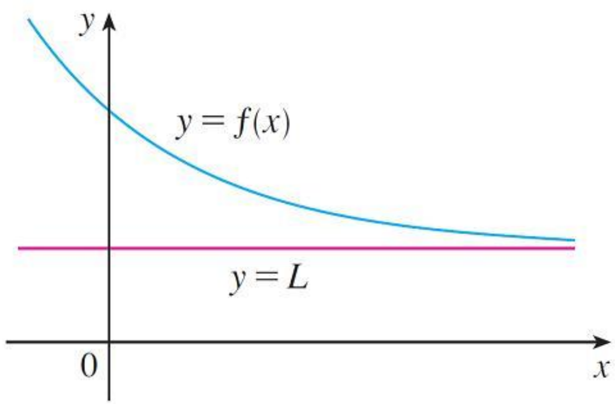


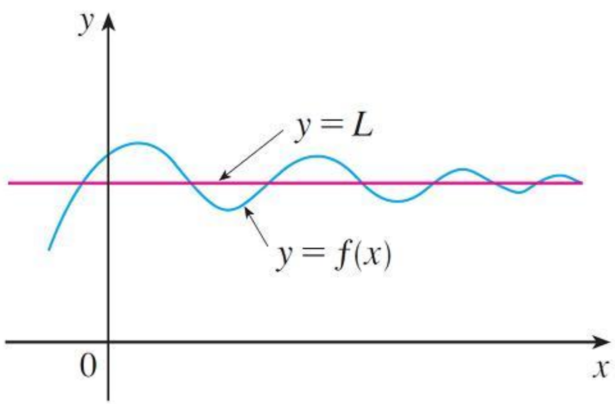
Examples illustrating $\lim_{x\to\infty} f(x) = L$

Figure 2

Limits at Infinity; Horizontal Asymptotes

Notice that there are many ways for the graph of *f* to approach the line *y* = *L* (which is called a *horizontal asymptote*) as we look to the far right of each graph.





Limits at Infinity; Horizontal Asymptotes

Referring back to Figure 1, we see that for numerically large negative values of *x*, the values of *f*(*x*) are close to 1.

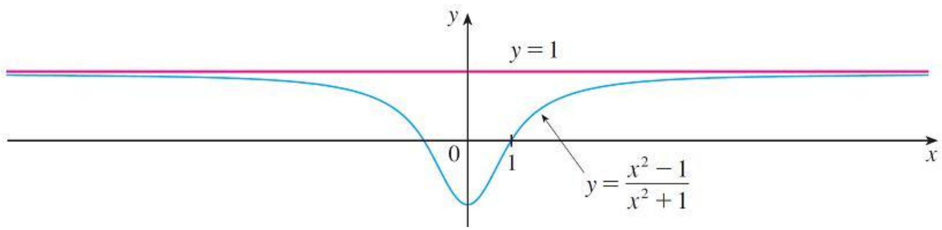


Figure 1

By letting *x* decrease through negative values without bound, we can make *f*(*x*) as close to 1 as we like.

This is expressed by writing

$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

Limits at Infinity; Horizontal Asymptotes

The general definition is as follows.

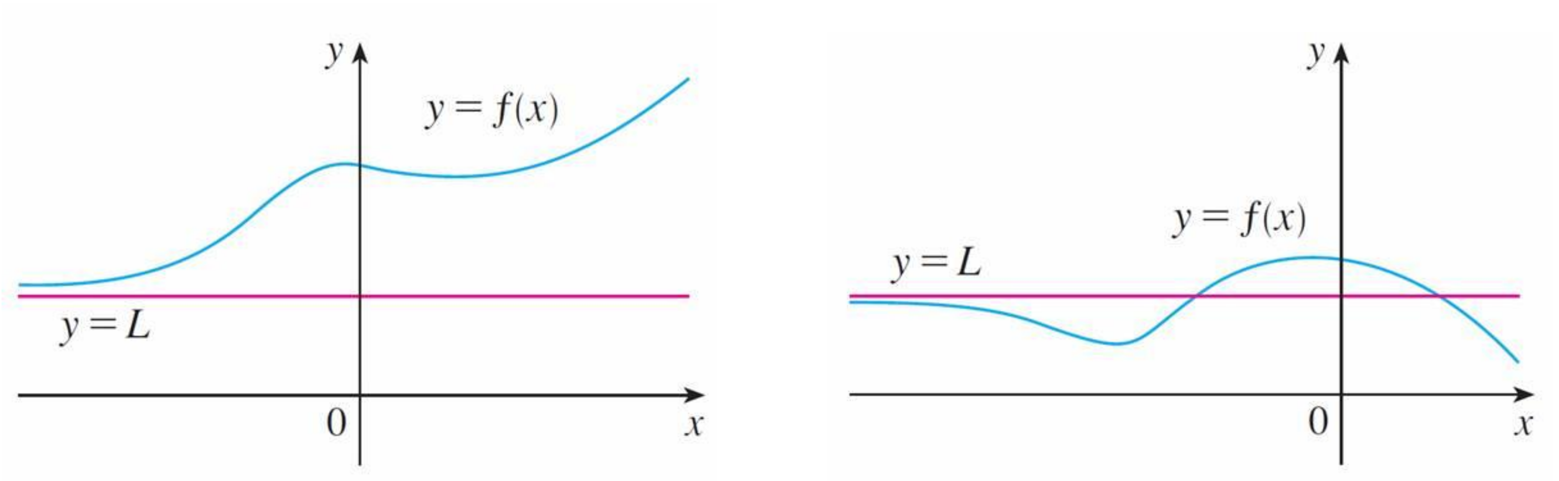
2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

Limits at Infinity; Horizontal Asymptotes

Definition 2 is illustrated in Figure 3. Notice that the graph approaches the line $y = L$ as we look to the far left of each graph.



Examples illustrating $\lim_{x \to \infty} f(x) = L$

Figure 3

Limits at Infinity; Horizontal Asymptotes

$\boxed{3}$ **Definition** The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

Example 2

Find $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$.

Solution:

Observe that when is large, 1/*x* is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking *x* large enough, we can make 1/*x* as close to 0 as we please.

Example 2 – *Solution*

cont'd

Therefore, according to Definition 1, we have

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

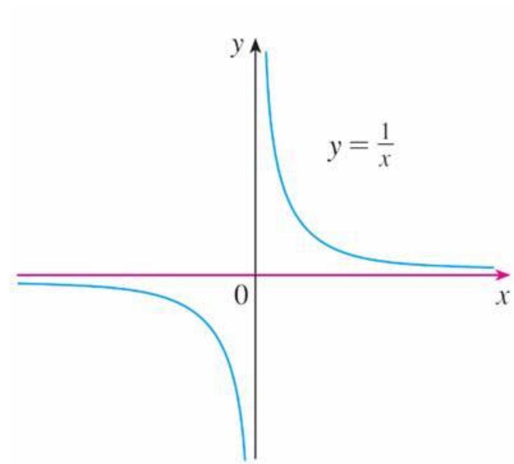
Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

Example 2 – *Solution*

cont'd

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is an equilateral hyperbola; see Figure 6.)



$$\lim_{x\to\infty} \frac{1}{x} = 0, \quad \lim_{x\to-\infty} \frac{1}{x} = 0$$

Figure 6

Limits at Infinity; Horizontal Asymptotes

$\boxed{5}$ **Theorem** If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Example 3

Evaluate $\lim_{x \to \infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$ and indicate which properties of limits are used at each stage.

Solution:

As *x* becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio. We need to do some preliminary algebra.

To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of *x* that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of *x*.)

Example 3 – *Solution*

cont'd

In this case the highest power of x in the denominator is x^2, so we have

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x \to \infty} \frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}}$$

Example 3 – *Solution*

cont'd

$$= \frac{\lim_{x\to\infty}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x\to\infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

(by Limit Law 5)

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty} \frac{1}{x} - 2\lim_{x\to\infty} \frac{1}{x^2}}{\lim_{x\to\infty} 5 + 4\lim_{x\to\infty} \frac{1}{x} + \lim_{x\to\infty} \frac{1}{x^2}}$$

(by 1, 2, and 3)

$$= \frac{3 - 0 - 0}{5 + 0 + 0}$$

(by 7 and Theorem 4)

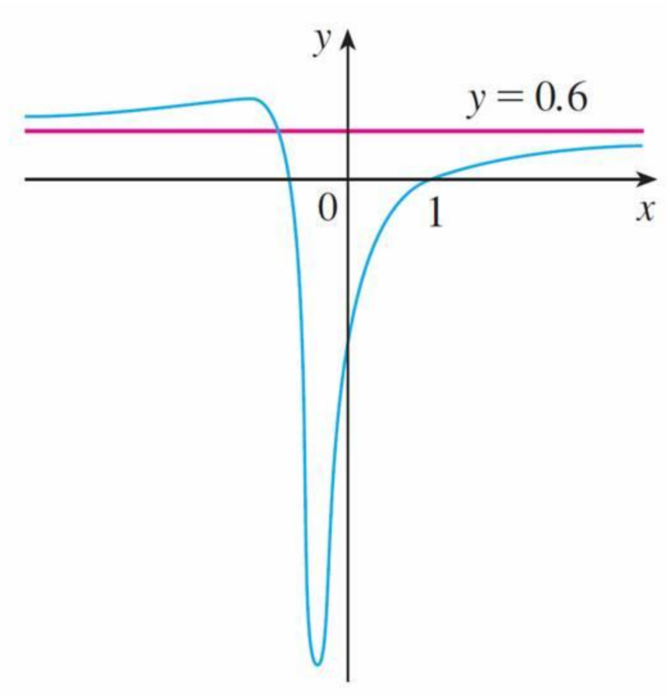
$$= \frac{3}{5}$$

Example 3 – *Solution*

cont'd

A similar calculation shows that the limit as $x \to -\infty$ is also $\frac{3}{5}$.

Figure 7 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5}$.



$$y = \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

Figure 7

Example 4

Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

Solution:

Dividing both numerator and denominator by x and using the properties of limits, we have

$$\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to \infty} \frac{\sqrt{2 + \dfrac{1}{x^2}}}{3 - \dfrac{5}{x}}$$

(since $\sqrt{x^2}$ – x for $x > 0$)

Example 4 – *Solution*

cont'd

$$= \frac{\lim_{x \to \infty} \sqrt{2 + \frac{1}{x^2}}}{\lim_{x \to \infty} \left(3 - \frac{5}{x}\right)}$$

$$= \frac{\sqrt{\lim_{x \to \infty} 2 + \lim_{x \to \infty} \frac{1}{x^2}}}{\lim_{x \to \infty} 3 - 5 \lim_{x \to \infty} \frac{1}{x}}$$

$$= \frac{\sqrt{2 + 0}}{3 - 5 \cdot 0}$$

Example 4 – *Solution*

cont'd

$$= \frac{\sqrt{2}}{3}$$

Therefore the line $y = \sqrt{2}/3$ is a horizontal asymptote of the graph of f.

In computing the limit as $x \to -\infty$, we must remember that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$.

Example 4 – *Solution*

cont'd

So when we divide the numerator by x, for $x < 0$ we get

$$\frac{1}{x}\sqrt{2x^2 + 1} = -\frac{1}{\sqrt{x^2}}\sqrt{2x^2 + 1}$$

$$= -\sqrt{2 + \frac{1}{x^2}}$$

Example 4 – *Solution*

cont'd

Therefore

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to -\infty} \frac{-\sqrt{2 + \dfrac{1}{x^2}}}{3 - \dfrac{5}{x}}$$

$$= \frac{-\sqrt{2 + \displaystyle\lim_{x \to -\infty} \frac{1}{x^2}}}{3 - 5 \displaystyle\lim_{x \to -\infty} \frac{1}{x}}$$

$$= -\frac{\sqrt{2}}{3}$$

Example 4 – *Solution*

cont'd

Thus the line $y = -\sqrt{2}/3$ is also a horizontal asymptote.

A vertical asymptote is likely to occur when the denominator, $3x - 5$, is 0, that is, when $x = \frac{5}{3}$.

If x is close to $\frac{5}{3}$ and $x > \frac{5}{3}$, then the denominator is close to 0 and $3x - 5$ is positive. The numerator $\sqrt{2x^2 + 1}$ is always positive, so $f(x)$ is positive.

Therefore

$$\lim_{x \to (5/3)^+} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \infty$$

Example 4 – *Solution*

cont'd

If is close to $\frac{5}{3}$ but $x < \frac{5}{3}$, then $3x - 5 < 0$ and so $f(x)$ is large negative. Thus

$$\lim_{x \to (5/3)^-} \frac{\sqrt{2x^2 + 1}}{3x - 5} = -\infty$$

The vertical asymptote is $x = \frac{5}{3}$.
All three asymptotes are shown in Figure 8.

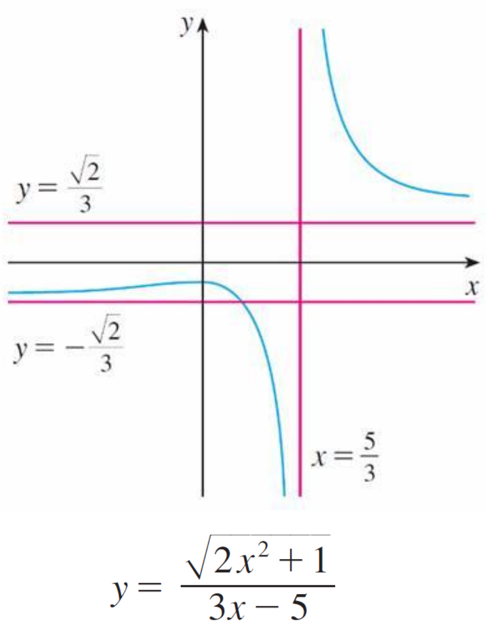


Figure 8

Infinite Limits at Infinity

Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

Example 9

Find $\lim_{x \to \infty} x^3$ and $\lim_{x \to -\infty} x^3$.

Solution:

When becomes large, x^3 also becomes large.

For instance,

$$10^3 = 1000 \qquad 100^3 = 1{,}000{,}000 \qquad 1000^3 = 1{,}000{,}000{,}000$$

In fact, we can make x^3 as big as we like by taking x large enough. Therefore we can write

$$\lim_{x \to \infty} x^3 = \infty$$

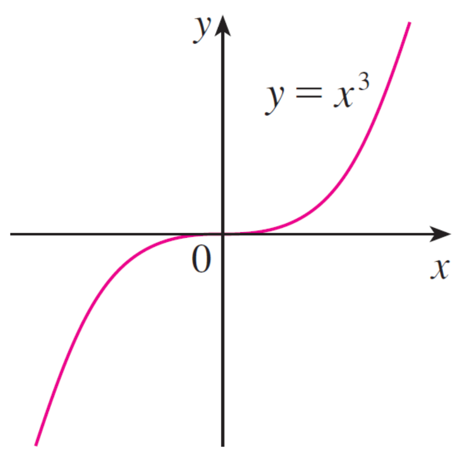
Example 9 – *Solution*

cont'd

Similarly, when x is large negative, so is x^3. Thus

$$\lim_{x \to -\infty} x^3 = -\infty$$

These limit statements can also be seen from the graph of $y = x^3$ in Figure 11.



$$\lim_{x \to \infty} x^3 = \infty, \quad \lim_{x \to -\infty} x^3 = -\infty$$

Figure 11

Precise Definitions

Precise Definitions

Definition 1 can be stated precisely as follows.

$\boxed{7}$ **Definition** Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

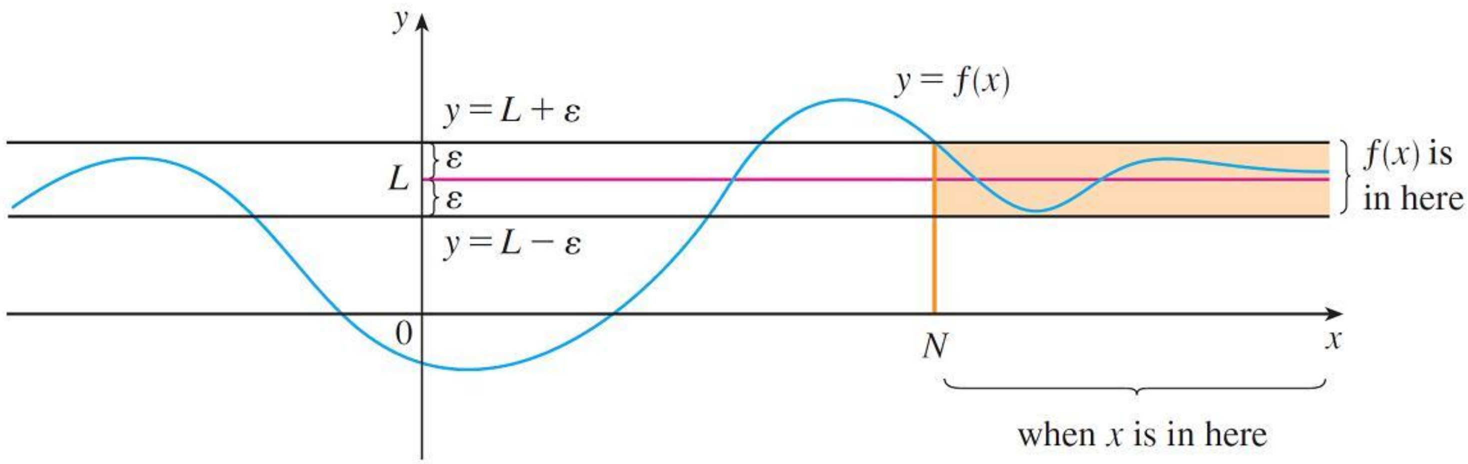
means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \qquad x > N \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

In words, this says that the values of $f(x)$ can be made arbitrarily close to L (within a distance ε, where ε is any positive number) by taking x sufficiently large (larger than N, where depends on ε).

Precise Definitions

Graphically it says that by choosing *x* large enough (larger than some number *N*) we can make the graph of *f* lie between the given horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$ as in Figure 14.

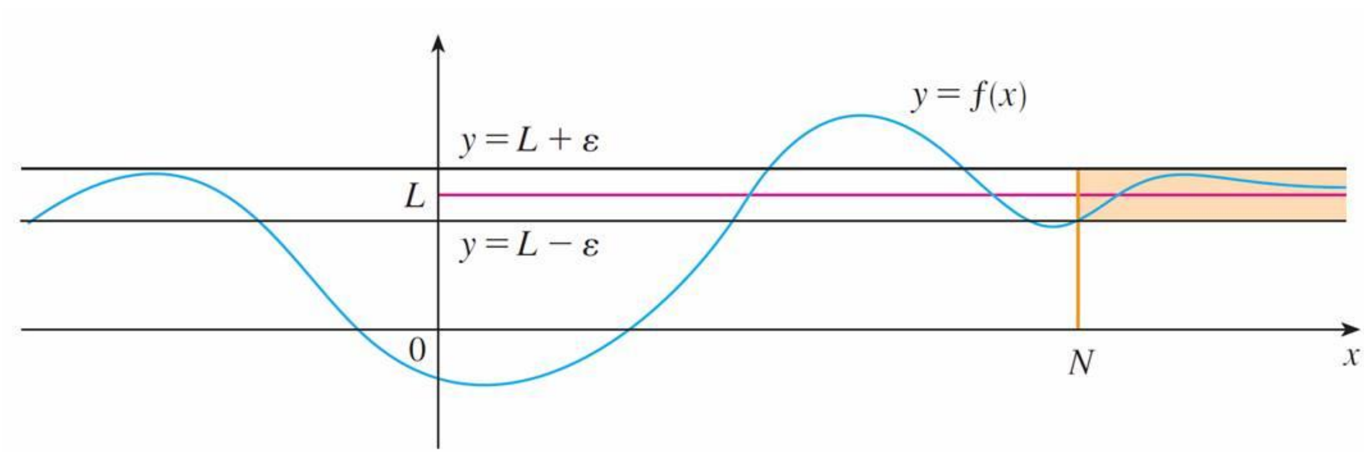


$$\lim_{x \to \infty} f(x) = L$$

Figure 14

Precise Definitions

This must be true no matter how small we choose *ε*. Figure 15 shows that if a smaller value of *ε* is chosen, then a larger value of *N* may be required.



$$\lim_{x \to \infty} f(x) = L$$

Figure 15

Precise Definitions

8 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x < N \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Example 14

Use Definition 7 to prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

Solution:

Given $\varepsilon > 0$, we want to find N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{1}{x} - 0 \right| < \varepsilon$$

In computing the limit we may assume that $x > 0$.

Then

$$1/x < \varepsilon \iff x > 1/\varepsilon.$$

Example 14 – *Solution*

cont'd

Let's choose N = 1/*ε.* So

If $x > N = \dfrac{1}{\varepsilon}$ then $\left|\dfrac{1}{x} - 0\right| = \dfrac{1}{x} < \varepsilon$

Therefore, by Definition 7,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Example 14 – *Solution*

cont'd

Figure 18 illustrates the proof by showing some values of ε and the corresponding values of N.

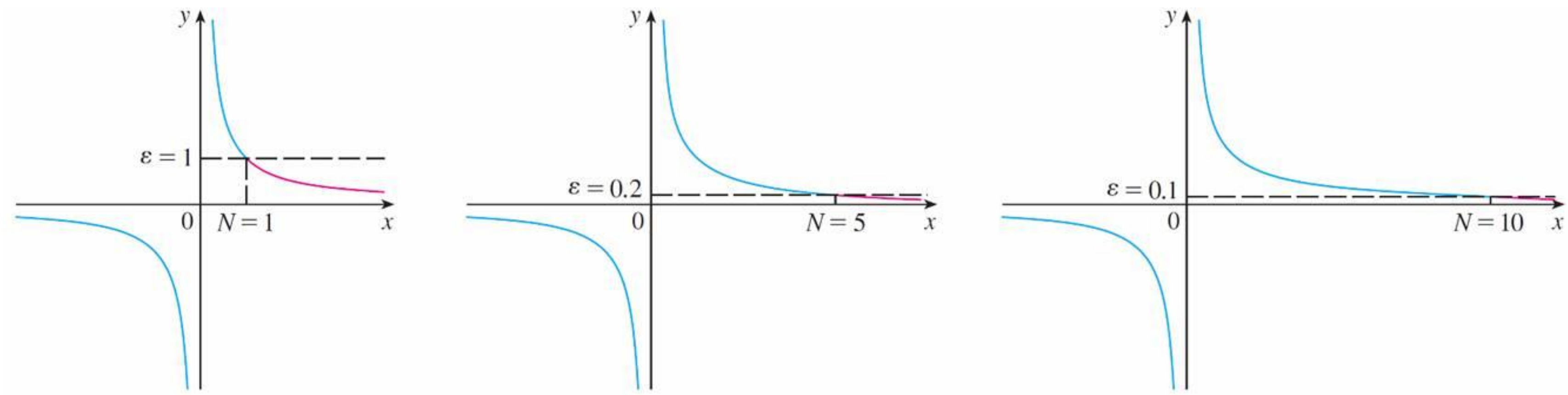
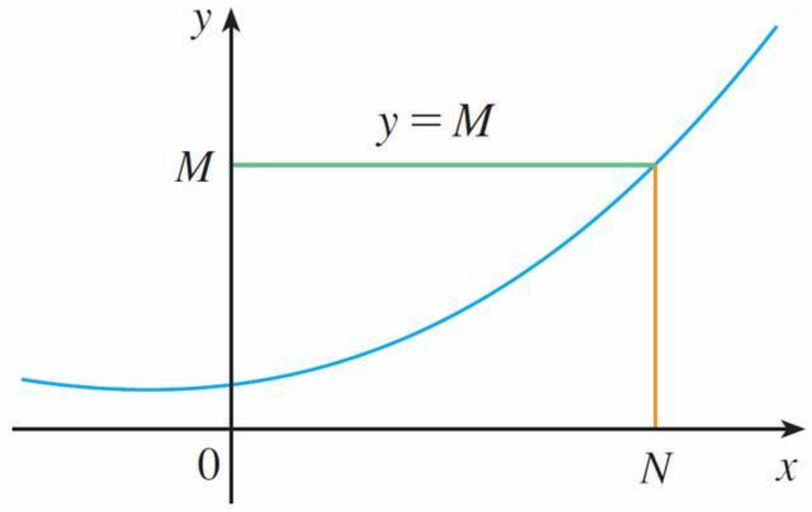


Figure 18

Precise Definitions

Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 19.



$$\lim_{x \to \infty} f(x) = \infty$$

Figure 19

Precise Definitions

9 Definition Let f be a function defined on some interval (a, ∞). Then

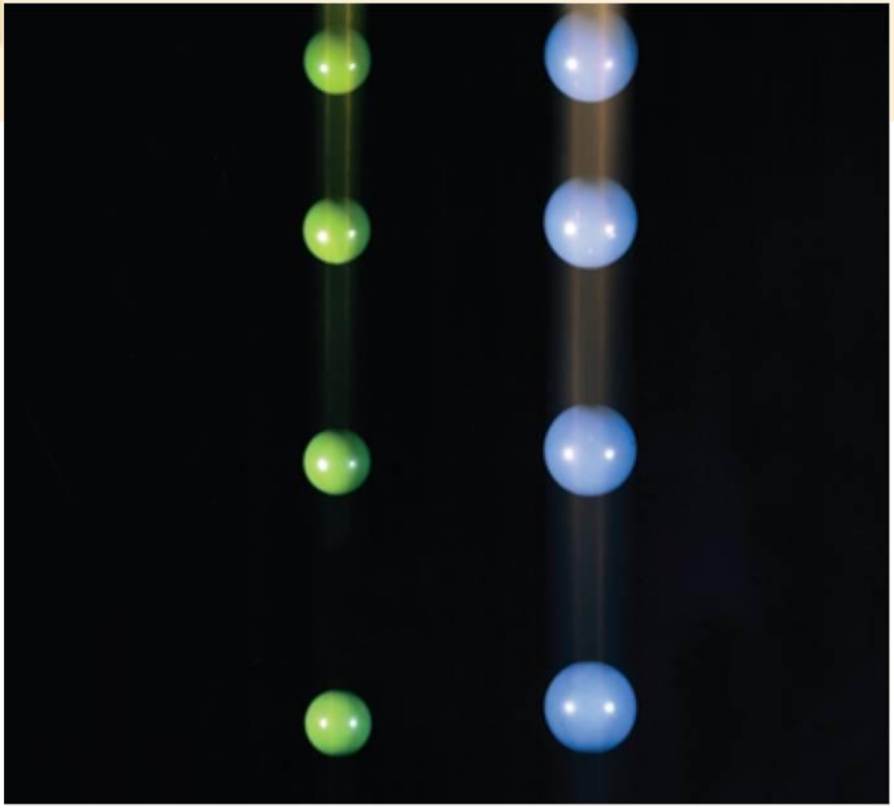
$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad f(x) > M$$

Similar definitions apply when the symbol ∞ is replaced by $-\infty$.

2 Limits and Derivatives



2.7 Derivatives and Rates of Change

Derivatives and Rates of Change

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit.

This special type of limit is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the sciences or engineering.

Tangents

Tangents

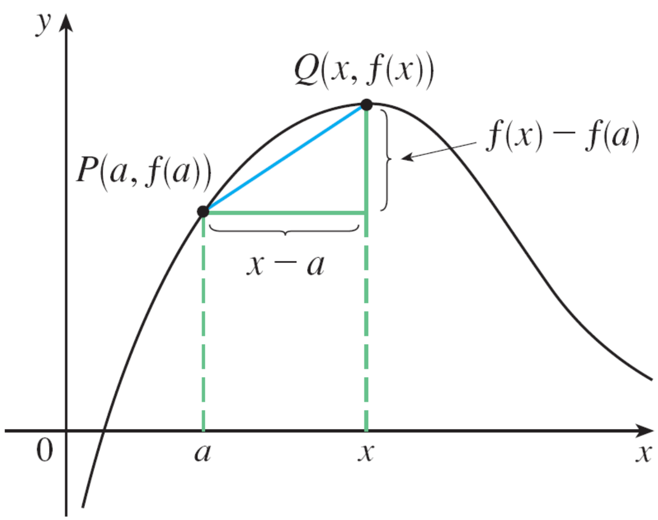
If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a.

Tangents

If m_{PQ} approaches a number m, then we define the *tangent* t to be the line through P with slope m. (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P. See Figure 1.)



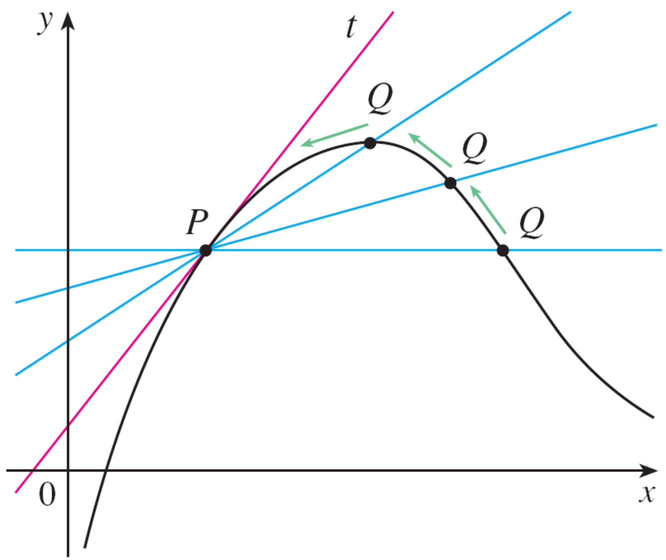


Figure 1

Tangents

1 **Definition** The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Example 1

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution:

Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

Example 1 – *Solution*

cont'd

$$= \lim_{x \to 1} (x + 1)$$

$$= 1 + 1$$

$$= 2$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at (1, 1) is

$y - 1 = 2(x - 1)$ or $y = 2x - 1$

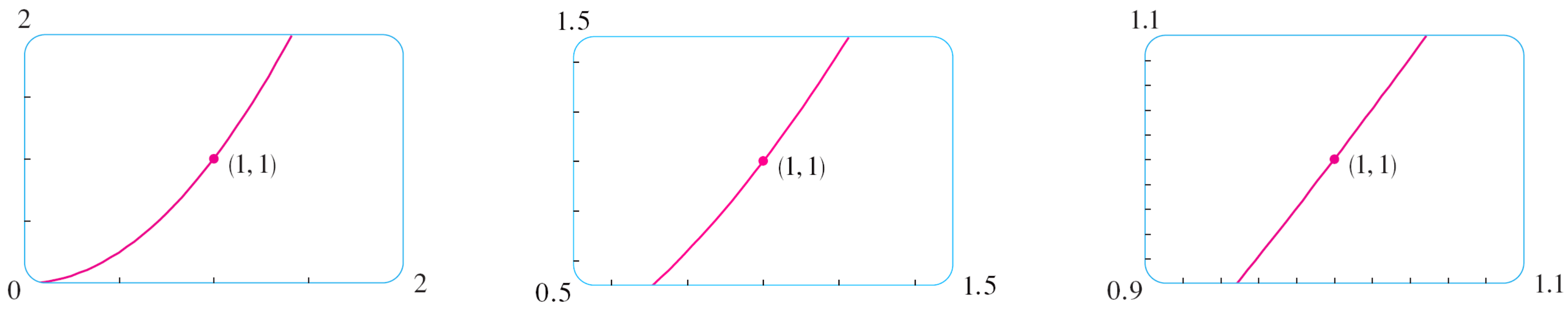
Tangents

We sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point.

The idea is that if we zoom in far enough toward the point, the curve looks almost like a straight line.

Tangents

Figure 2 illustrates this procedure for the curve $y = x^2$ in Example 1.



Zooming in toward the point (1, 1) on the parabola $y = x^2$

Figure 2

Tangents

The more we zoom in, the more the parabola looks like a line.

In other words, the curve becomes almost indistinguishable from its tangent line.

There is another expression for the slope of a tangent line that is sometimes easier to use.

Tangents

If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

(See Figure 3 where the case $h > 0$ is illustrated and Q is to the right of P. If it happened that $h < 0$, however, Q would be to the left of P.)

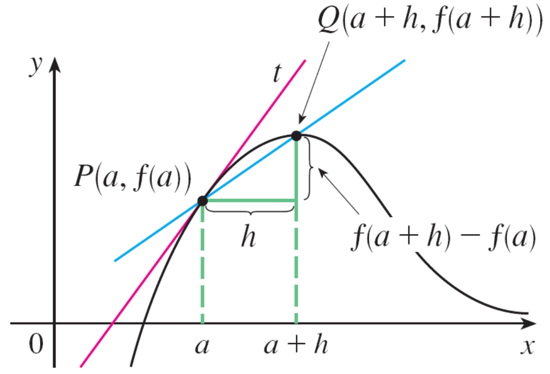


Figure 3

Tangents

Notice that as x approaches a, h approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 1 becomes

2
$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Velocities

Velocities

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t.

The function f that describes the motion is called the **position function** of the object.

In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$.

Velocities

See Figure 5.

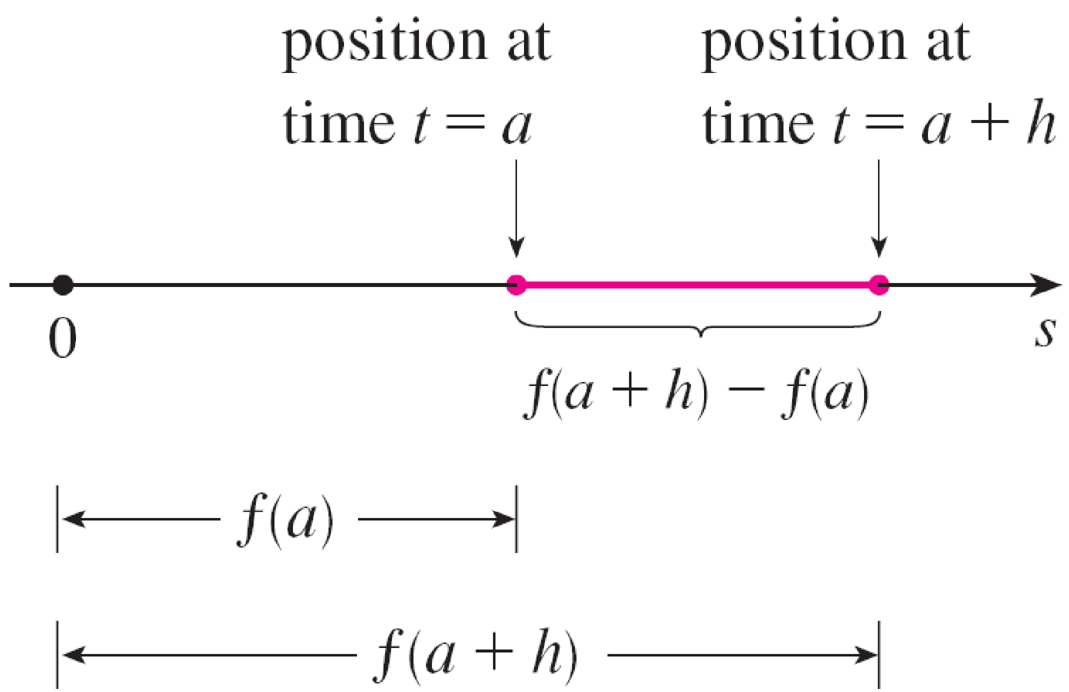


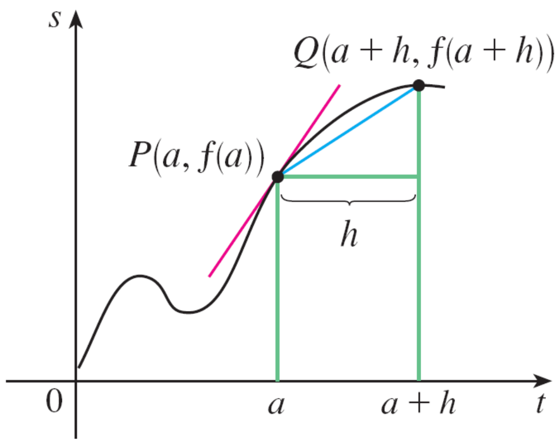
Figure 5

Velocities

The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h)-f(a)}{h}$$

which is the same as the slope of the secant line *PQ* in Figure 6.



$$m_{PQ} = \frac{f(a+h)-f(a)}{h}$$
$$= \text{average velocity}$$

Figure 6

Velocities

Now suppose we compute the average velocities over shorter and shorter time intervals [*a*, *a* + *h*].

In other words, we let *h* approach 0. As in the example of the falling ball, we define the **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ to be the limit of these average velocities:

3

$$\boxed{v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}}$$

This means that the velocity at time $t = a$ is equal to the slope of the tangent line at *P*.

Example 3

Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Solution:

We will need to find the velocity both when $t = 5$ and when the ball hits the ground, so it's efficient to start by finding the velocity at a general time $t = a$.

Example 3 – *Solution*

cont'd

Using the equation of motion $s = f(t) = 4.9t^2$, we have

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{4.9(a + h)^2 - 4.9a^2}{h}$$

$$= \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2 - a^2)}{h}$$

$$= \lim_{h \to 0} \frac{4.9(2ah + h^2)}{h}$$

Example 3 – *Solution*

cont'd

$$= \lim_{h \to 0} 4.9(2a + h)$$

$$= 9.8a$$

(a) The velocity after 5 s is $v(5) = (9.8)(5) = 49$ m/s.

Example 3 – *Solution*

cont'd

(b) Since the observation deck is 450 m above the ground, the ball will hit the ground at the time t_1 when $s(t_1) = 450$, that is,

$$4.9t_1^2 = 450$$

This gives

$$t_1^2 = \frac{450}{4.9} \qquad \text{and} \qquad t_1 = \sqrt{\frac{450}{4.9}} \approx 9.6 \text{ s}$$

Example 3 – *Solution*

cont'd

The velocity of the ball as it hits the ground is therefore

$$v(t_1) = 9.8t_1$$

$$= 9.8\sqrt{\frac{450}{4.9}}$$

$$\approx 94 \text{ m/s}$$

Derivatives

Derivatives

We have seen that the same type of limit arises in finding the slope of a tangent line (Equation 2) or the velocity of an object (Equation 3).

In fact, limits of the form

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics.

Since this type of limit occurs so widely, it is given a special name and notation.

Derivatives

4 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

If we write $x = a + h$, then we have $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines, is

5

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

Example 4

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

Solution:

From Definition 4 we have

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{[(a + h)^2 - 8(a + h) + 9] - [a^2 - 8a + 9]}{h}$$

$$= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$

Example 4 – *Solution*

cont'd

$$= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h}$$

$$= \lim_{h \to 0} (2a + h - 8)$$

$$= 2a - 8$$

Derivatives

We defined the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ to be the line that passes through P and has slope m given by Equation 1 or 2.

Since, by Definition 4, this is the same as the derivative $f'(a)$, we can now say the following.

> The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

Derivatives

If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

Rates of Change

Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$.

If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

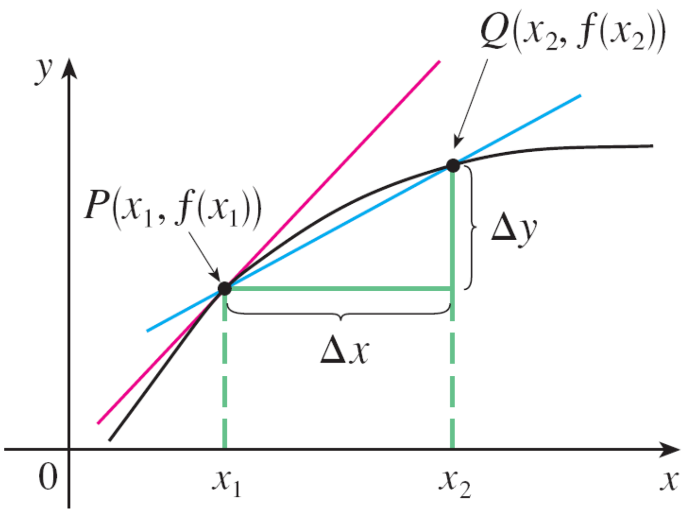
$$\Delta y = f(x_2) - f(x_1)$$

Rates of Change

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of *y* with respect to *x*** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line *PQ* in Figure 8.



average rate of change = m_{PQ}
instantaneous rate of change = slope of tangent at P

Figure 8

Rates of Change

By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0.

The limit of these average rates of change is called the (**instantaneous**) **rate of change of *y* with respect to *x*** at $x = x_1$, which is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$:

$$\boxed{6} \quad \text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We recognize this limit as being the derivative $f'(x_1)$.

Rates of Change

We know that one interpretation of the derivative $f'(a)$ is as the slope of the tangent line to the curve $y = f(x)$ when $x = a$. We now have a second interpretation:

> The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

The connection with the first interpretation is that if we sketch the curve $y = f(x)$, then the instantaneous rate of change is the slope of the tangent to this curve at the point where $x = a$.

Rates of Change

This means that when the derivative is large (and therefore the curve is steep, as at the point *P* in Figure 9), the *y*-values change rapidly.

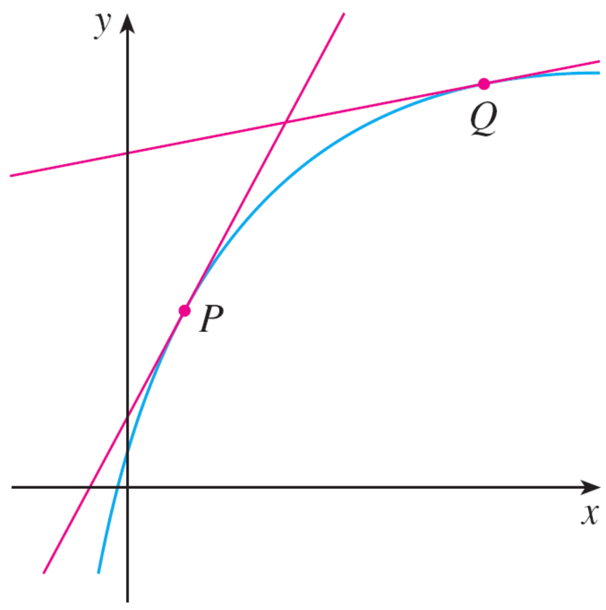


Figure 9

The *y*-values are changing rapidly at *P* and slowly at *Q*.

Rates of Change

When the derivative is small, the curve is relatively flat (as at point Q) and the y-values change slowly.

In particular, if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the rate of change of the displacement s with respect to the time t.

In other words, $f'(a)$ *is the velocity of the particle at time* $t = a$.

The **speed** of the particle is the absolute value of the velocity, that is, $|f'(a)|$.

Example 6

A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ dollars.

(a) What is the meaning of the derivative $f'(x)$? What are its units?

(b) In practical terms, what does it mean to say that $f'(1000) = 9$?

(c) Which do you think is greater, $f'(50)$ or $f'(500)$? What about $f'(5000)$?

Example 6(a) – *Solution*

The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x; that is, $f'(x)$ means the rate of change of the production cost with respect to the number of yards produced.

Because

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$$

the units for $f'(x)$ are the same as the units for the difference quotient $\Delta C/\Delta x$.

Since ΔC is measured in dollars and Δx in yards, it follows that the units for $f'(x)$ are dollars per yard.

Example 6(b) – *Solution*

cont'd

The statement that $f'(1000) = 9$ means that, after 1000 yards of fabric have been manufactured, the rate at which the production cost is increasing is \$9/yard.
(When $x = 1000$, C is increasing 9 times as fast as x.)

Since $\Delta x = 1$ is small compared with $x = 1000$, we could use the approximation

$$f'(1000) \approx \frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \Delta C$$

and say that the cost of manufacturing the 1000th yard (or the 1001st) is about \$9.

Example 6(c) – *Solution*

cont'd

The rate at which the production cost is increasing (per yard) is probably lower when $x = 500$ than when $x = 50$ (the cost of making the 500th yard is less than the cost of the 50th yard) because of economies of scale.

(The manufacturer makes more efficient use of the fixed costs of production.)

So

$$f'(50) > f'(500)$$

Example 6(c) – *Solution*

cont'd

But, as production expands, the resulting large-scale operation might become inefficient and there might be overtime costs.

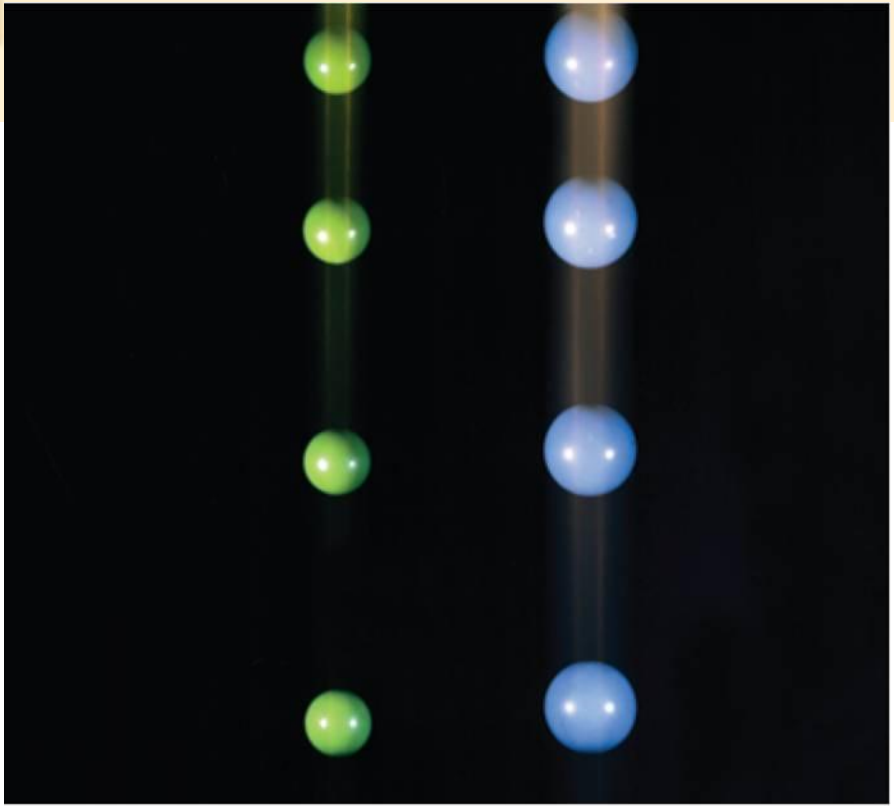
Thus it is possible that the rate of increase of costs will eventually start to rise.

So it may happen that

$$f'(5000) > f'(500)$$

2

Limits and Derivatives



2.8 The Derivative as a Function

The Derivative as a Function

We have considered the derivative of a function *f* at a fixed number *a*:

1 $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Here we change our point of view and let the number *a* vary. If we replace *a* in Equation 1 by a variable *x*, we obtain

2 $$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

The Derivative as a Function

Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we can regard f' as a new function, called the **derivative of *f*** and defined by Equation 2.

We know that the value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

The function f' is called the derivative of f because it has been "derived" from f by the limiting operation in Equation 2. The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$ and may be smaller than the domain of f.

Example 1

The graph of a function f is given in Figure 1. Use it to sketch the graph of the derivative f'.

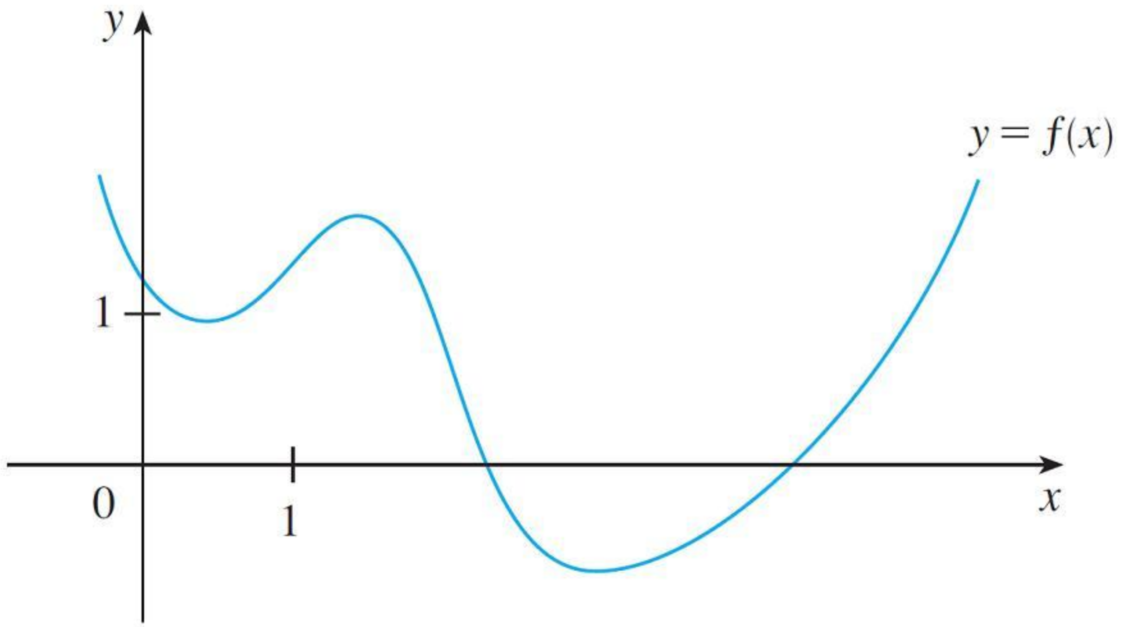


Figure 1

Example 1 – *Solution*

We can estimate the value of the derivative at any value of x by drawing the tangent at the point $(x, f(x))$ and estimating its slope. For instance, for $x = 5$ we draw the tangent at P in Figure 2(a) and estimate its slope to be about $\frac{3}{2}$, so $f'(5) \approx 1.5$.

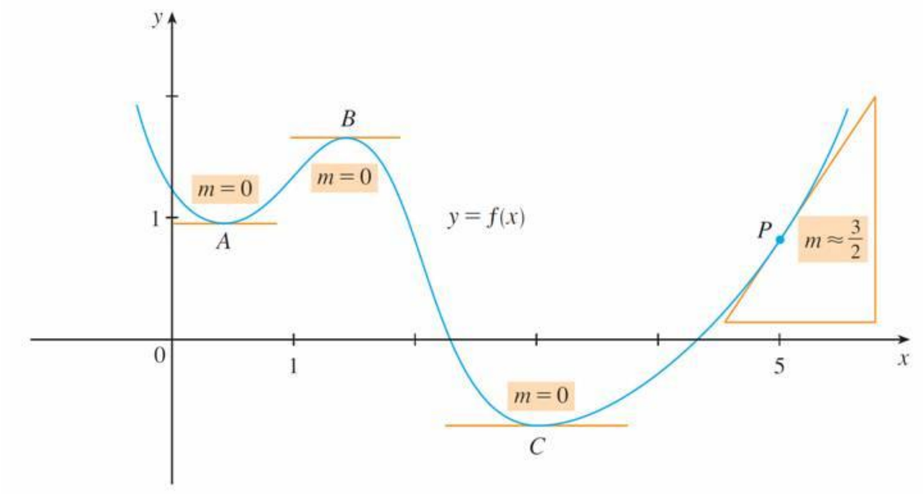


Figure 2(a)

Example 1 – *Solution*

cont'd

This allows us to plot the point $P'(5, 1.5)$ on the graph of f' directly beneath P. Repeating this procedure at several points, we get the graph shown in Figure 2(b).

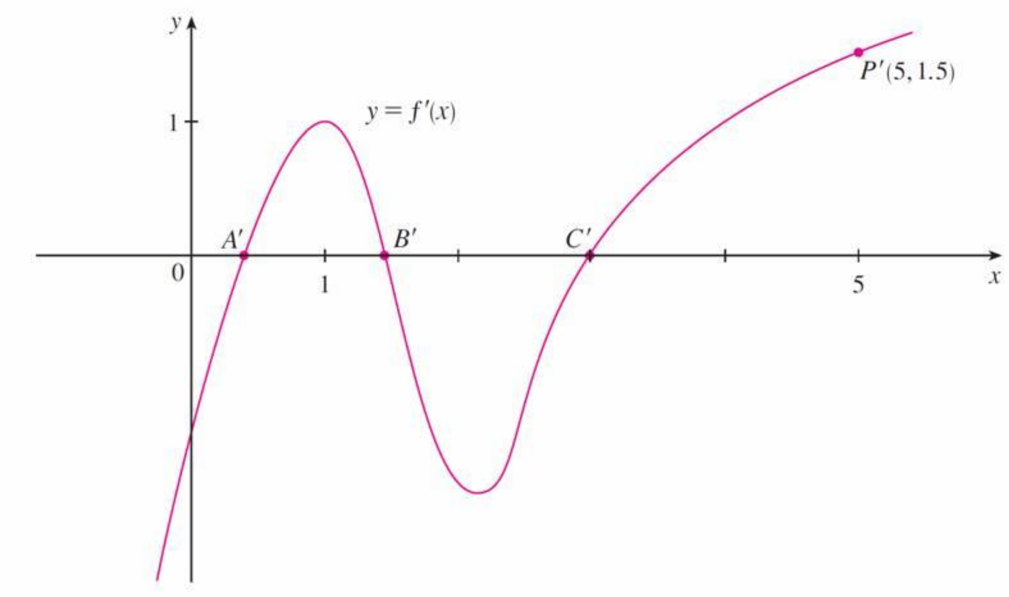


Figure 2(b)

Example 1 – *Solution*

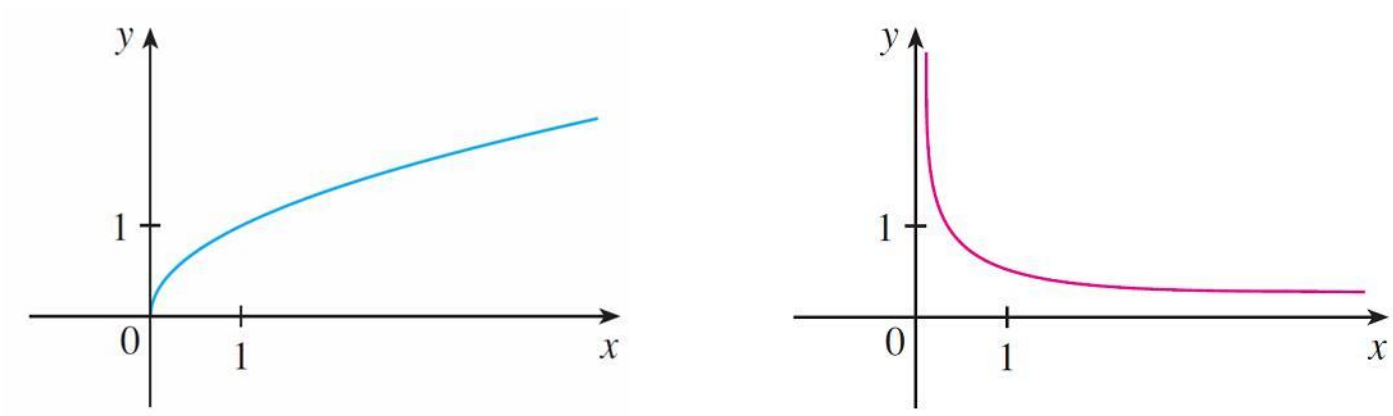
cont'd

Notice that the tangents at A, B, and C are horizontal, so the derivative is 0 there and the graph of f' crosses the x-axis at the points A', B', and C', directly beneath A, B, and C.

Between A and B the tangents have positive slope, so $f'(x)$ is positive there. But between B and C the tangents have negative slope, so $f'(x)$ is negative there.

The Derivative as a Function

When x is close to 0, $\sqrt{x}$ is also close to 0, so $f'(x) = 1/(2\sqrt{x})$ is very large and this corresponds to the steep tangent lines near (0, 0) in Figure 4(a) and the large values of $f'(x)$ just to the right of 0 in Figure 4(b).



(a) $f(x) = \sqrt{x}$

(b) $f'(x) = \dfrac{1}{2\sqrt{x}}$

Figure 4

The Derivative as a Function

When x is large, $f'(x)$ is very small and this corresponds to the flatter tangent lines at the far right of the graph of f and the horizontal asymptote of the graph of f'.

Other Notations

Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

Other Notations

The symbol *dy*/*dx*, which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is simply a synonym for $f'(x)$. Nonetheless, it is a very useful and suggestive notation, especially when used in conjunction with increment notation.

We can rewrite the definition of derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Other Notations

If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$.

$\boxed{3}$ **Definition** A function f is **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example 5

Where is the function $f(x) = |x|$ differentiable?

Solution:

If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$, we have

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h} = \lim_{h \to 0} \frac{(x + h) - x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

and so f is differentiable for any $x > 0$.

Example 5 – *Solution*

cont'd

Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small enough that $x + h < 0$ and so $|x + h| = -(x + h)$.

Therefore, for $x < 0$,

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{-(x + h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

and so f is differentiable for any $x < 0$.

Example 5 – *Solution*

cont'd

For x = 0 we have to investigate

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0 + h| - |0|}{h} \qquad \text{(if it exists)}$$

Let's compute the left and right limits separately:

$$\lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

and

$$\lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

Example 5 – *Solution*

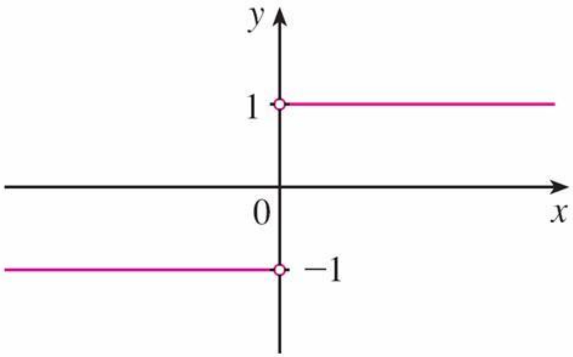
cont'd

Since these limits are different, $f'(0)$ does not exist. Thus f is differentiable at all x except 0.

A formula for f' is given by

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and its graph is shown in Figure 5(b).



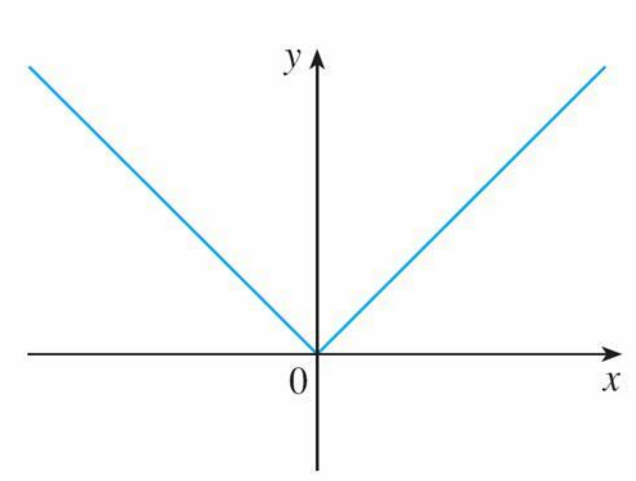
$y = f'(x)$

Figure 5(b)

Example 5 – *Solution*

cont'd

The fact that $f'(0)$ does not exist is reflected geometrically in the fact that the curve $y = |x|$ does not have a tangent line at (0, 0). [See Figure 5(a).]



$y = f(x) = |x|$

Figure 5(a)

Other Notations

Both continuity and differentiability are desirable properties for a function to have. The following theorem shows how these properties are related.

$\boxed{4}$ **Theorem** If f is differentiable at a, then f is continuous at a.

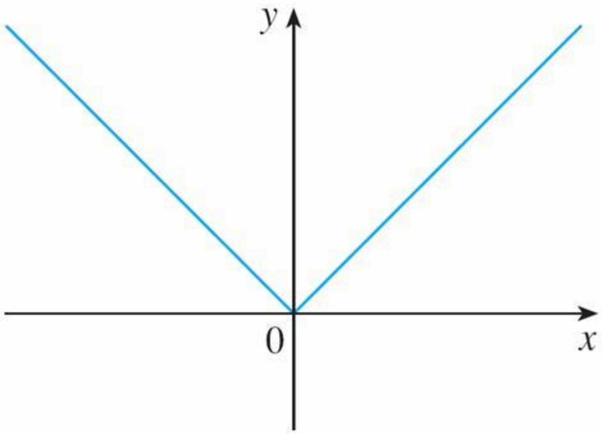
Note: The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable.

How Can a Function Fail to Be Differentiable?

How Can a Function Fail to Be Differentiable?

We saw that the function $y = |x|$ in Example 5 is not differentiable at 0 and Figure 5(a) shows that its graph changes direction abruptly when $x = 0$.

In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]



$y = f(x) = |x|$

Figure 5(a)

How Can a Function Fail to Be Differentiable?

Theorem 4 gives another way for a function not to have a derivative. It says that if f is not continuous at a, then f is not differentiable at a. So at any discontinuity (for instance, a jump discontinuity) f fails to be differentiable.

A third possibility is that the curve has a **vertical tangent line** when $x = a$; that is, f is continuous at a and

$$\lim_{x \to a} |f'(x)| = \infty$$

How Can a Function Fail to Be Differentiable?

This means that the tangent lines become steeper and steeper as $x \to a$. Figure 6 shows one way that this can happen; Figure 7(c) shows another.

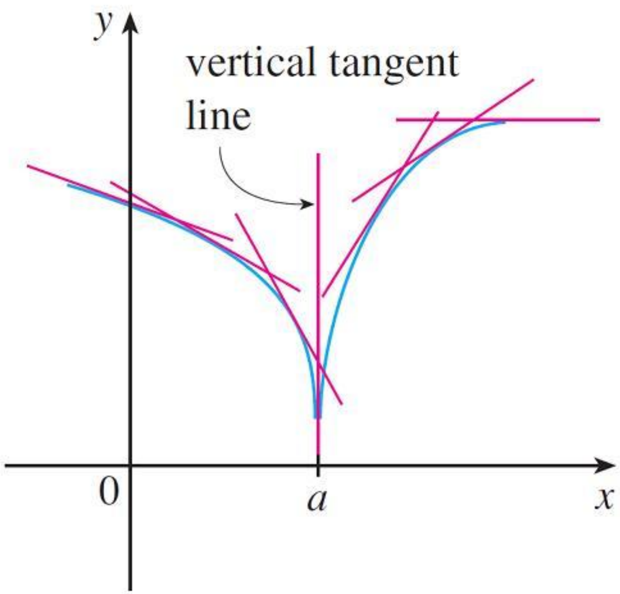
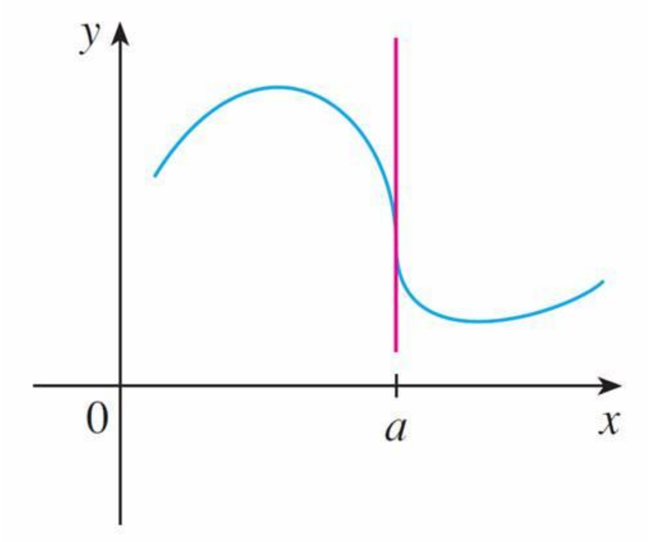


Figure 6

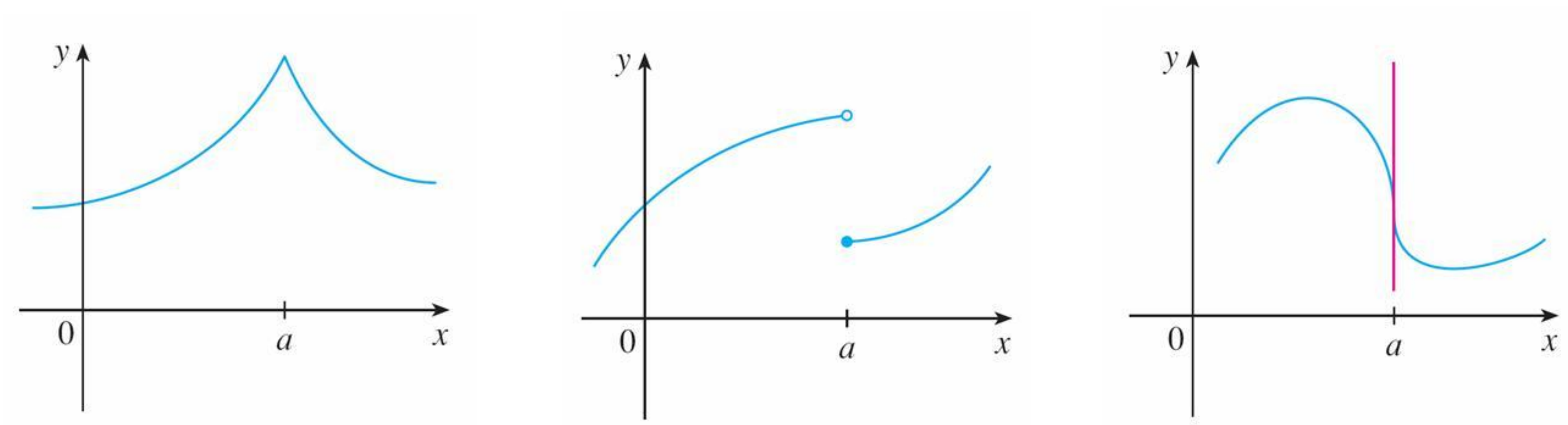


A vertical tangent

Figure 7(c)

How Can a Function Fail to Be Differentiable?

Figure 7 illustrates the three possibilities that we have discussed.



(a) A corner (b) A discontinuity (c) A vertical tangent

Three ways for *f* not to be differentiable at *a*

Figure 7

Higher Derivatives

Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f.

Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Example 6

If $f(x) = x^3 - x$, find and interpret $f''(x)$.

Solution:

The first derivative of $f(x) = x^3 - x$ is $f'(x) = 3x^2 - 1$.

So the second derivative is

$$f''(x) = (f')'(x)$$

$$= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$= \lim_{h \to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h}$$

Example 6 – *Solution*

cont'd

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h}$$

$$= \lim_{h \to 0} (6x + 3h)$$

$$= 6x$$

The graphs of *f*, *f*′, and *f*″ are shown in Figure 10.

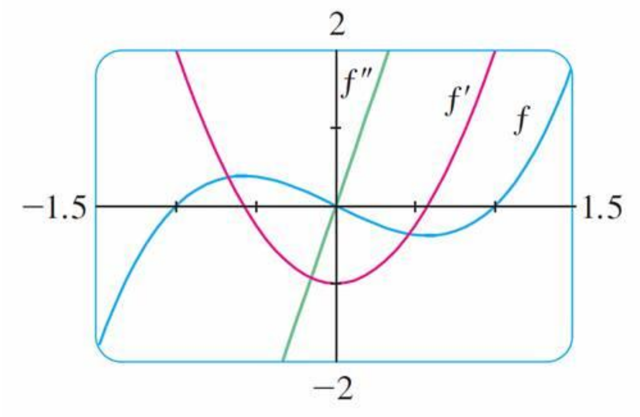


Figure 10

Example 6 – *Solution*

cont'd

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

Notice from Figure 10 that $f''(x)$ is negative when $y = f'(x)$ has negative slope and positive when $y = f'(x)$ has positive slope. So the graphs serve as a check on our calculations.

Higher Derivatives

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration,* which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

Higher Derivatives

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Higher Derivatives

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$.

If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Higher Derivatives

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$.

In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times.

If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Higher Derivatives

We can also interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line.

Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3 s}{dt^3}$$

Higher Derivatives

Thus the jerk j is the rate of change of acceleration.

It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.