3

Differentiation Rules

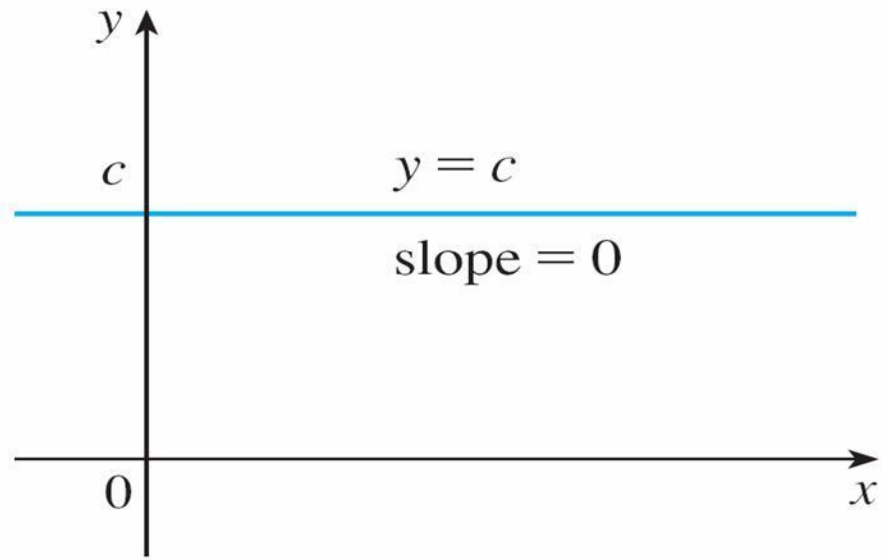
3.1 Derivatives of Polynomials and Exponential Functions

Derivatives of Polynomials and Exponential Functions

In this section we learn how to differentiate constant functions, power functions, polynomials, and exponential functions.

Let's start with the simplest of all functions, the constant function $f(x) = c$.

The graph of this function is the horizontal line $y = c$, which has slope 0, so we must have $f'(x) = 0$. (See Figure 1.)



The graph of $f(x) = c$ is the line $y = c$, so $f'(x) = 0$.

Figure 1

Derivatives of Polynomials and Exponential Functions

A formal proof, from the definition of a derivative, is also easy:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

In Leibniz notation, we write this rule as follows.

Derivative of a Constant Function

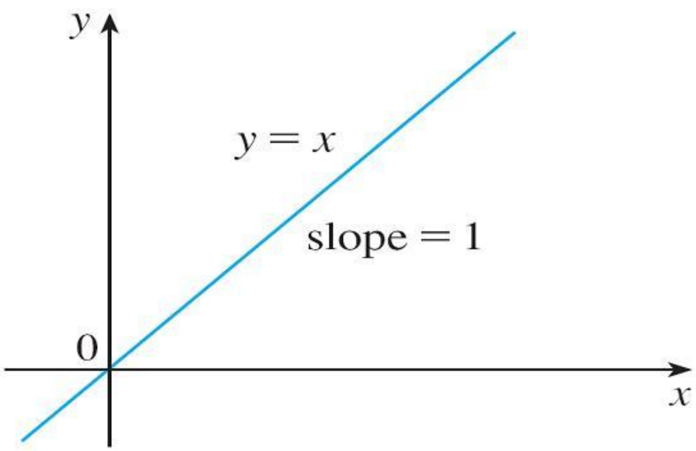
$$\frac{d}{dx}(c) = 0$$

Power Functions

Power Functions

We next look at the functions $f(x) = x^n$, where n is a positive integer.

If $n = 1$, the graph of $f(x) = x$ is the line $y = x$, which has slope 1. (See Figure 2.)



The graph of $f(x) = x$ is the line $y = x$, so $f'(x) = 1$.

Figure 2

Power Functions

So

1

$$\boxed{\frac{d}{dx}(x) = 1}$$

(You can also verify Equation 1 from the definition of a derivative.)

We have already investigated the cases $n = 2$ and $n = 3$. We found that

2

$$\frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dx}(x^3) = 3x^2$$

Power Functions

For $n = 4$ we find the derivative of $f(x) = x^4$ as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h}$$

$$= \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$

$$= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$

$$= \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3)$$

$$= 4x^3$$

Power Functions

Thus

$$\boxed{3} \qquad \frac{d}{dx}(x^4) = 4x^3$$

Comparing the equations in $\boxed{1}$, $\boxed{2}$, and $\boxed{3}$, we see a pattern emerging.

It seems to be a reasonable guess that, when *n* is a positive integer, $(d/dx)(x^n) = nx^{n-1}$. This turns out to be true.

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example 1

(a) If $f(x) = x^6$, then $f'(x) = 6x^5$.

(b) If $y = x^{1000}$, then $y' = 1000x^{999}$.

(c) If $y = t^4$, then $\dfrac{dy}{dt} = 4t^3$.

(d) $\dfrac{d}{dr}(r^3) = 3r^2$

Power Functions

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Power Rule enables us to find tangent lines without having to resort to the definition of a derivative. It also enables us to find *normal lines*.

The **normal line** to a curve *C* at a point *P* is the line through *P* that is perpendicular to the tangent line at *P*.

New Derivatives from Old

New Derivatives from Old

When new functions are formed from old functions by addition, subtraction, or multiplication by a constant, their derivatives can be calculated in terms of derivatives of the old functions.

In particular, the following formula says that *the derivative of a constant times a function is the constant times the derivative of the function.*

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Example 4

(a) $$\frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4)$$

$$= 3(4x^3)$$

$$= 12x^3$$

(b) $$\frac{d}{dx}(-x) = \frac{d}{dx}[(-1)x]$$

$$= (-1)\frac{d}{dx}(x)$$

$$= -1(1)$$

$$= -1$$

New Derivatives from Old

The next rule tells us that *the derivative of a sum of functions is the sum of the derivatives.*

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Sum Rule can be extended to the sum of any number of functions. For instance, using this theorem twice, we get

$$(f + g + h)' = [(f + g) + h)]' = (f + g)' + h' = f' + g' + h'$$

New Derivatives from Old

By writing $f - g$ as $f + (-1)g$ and applying the Sum Rule and the Constant Multiple Rule, we get the following formula.

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

The Constant Multiple Rule, the Sum Rule, and the Difference Rule can be combined with the Power Rule to differentiate any polynomial, as the following examples demonstrate.

Exponential Functions

Exponential Functions

Let's try to compute the derivative of the exponential function $f(x) = a^x$ using the definition of a derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$

The factor a^x doesn't depend on h, so we can take it in front of the limit:

$$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

Exponential Functions

Notice that the limit is the value of the derivative of *f* at 0, that is,

$$\lim_{h \to 0} \frac{a^h - 1}{h} = f'(0)$$

Therefore we have shown that if the exponential function $f(x) = a^x$ is differentiable at 0, then it is differentiable everywhere and

$$\boxed{4} \qquad f'(x) = f'(0)\,a^x$$

This equation says that *the rate of change of any exponential function is proportional to the function itself.* (The slope is proportional to the height.)

Exponential Functions

Numerical evidence for the existence of $f'(0)$ is given in the table at the right for the cases $a = 2$ and $a = 3$. (Values are stated correct to four decimal places.)

It appears that the limits exist and

h	$\dfrac{2^h - 1}{h}$	$\dfrac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

for $a = 2$, $$f'(0) = \lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.69$$

for $a = 3$, $$f'(0) = \lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.10$$

Exponential Functions

In fact, it can be proved that these limits exist and, correct to six decimal places, the values are

$$\frac{d}{dx}(2^x)\bigg|_{x=0} \approx 0.693147 \qquad \frac{d}{dx}(3^x)\bigg|_{x=0} \approx 1.098612$$

Thus, from Equation 4, we have

$$\boxed{5} \qquad \frac{d}{dx}(2^x) \approx (0.69)2^x \qquad \frac{d}{dx}(3^x) \approx (1.10)3^x$$

Of all possible choices for the base *a* in Equation 4, the simplest differentiation formula occurs when $f'(0) = 1$.

Exponential Functions

In view of the estimates of $f'(0)$ for $a = 2$ and $a = 3$, it seems reasonable that there is a number a between 2 and 3 for which $f'(0) = 1$.

It is traditional to denote this value by the letter e. Thus we have the following definition.

Definition of the Number *e*

$$e \text{ is the number such that } \quad \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Exponential Functions

Geometrically, this means that of all the possible exponential functions $y = a^x$, the function $f(x) = e^x$ is the one whose tangent line at (0, 1) has a slope $f'(0)$ that is exactly 1. (See Figures 6 and 7.)

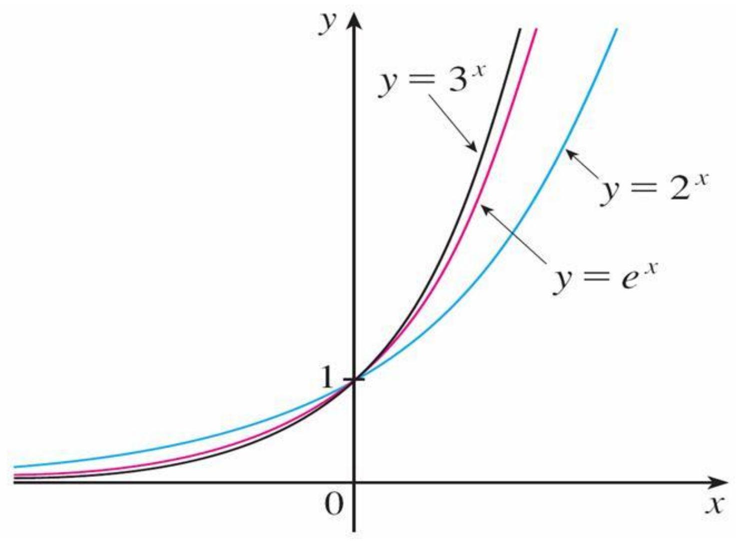


Figure 6

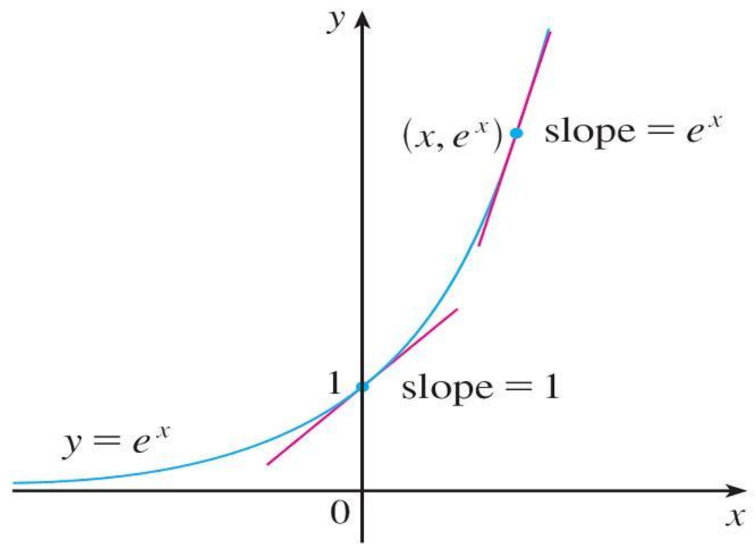


Figure 7

Exponential Functions

If we put $a = e$ and, therefore, $f'(0) = 1$ in Equation 4, it becomes the following important differentiation formula.

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Thus the exponential function $f(x) = e^x$ has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve $y = e^x$ is equal to the y-coordinate of the point (see Figure 7).

Example 8

If $f(x) = e^x - x$, find f' and f''. Compare the graphs of f and f'.

Solution:

Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx}(e^x - x)$$

$$= \frac{d}{dx}(e^x) - \frac{d}{dx}(x)$$

$$= e^x - 1$$

Example 8 – *Solution*

cont'd

We defined the second derivative as the derivative of f', so

$$f''(x) = \frac{d}{dx}(e^x - 1)$$

$$= \frac{d}{dx}(e^x) - \frac{d}{dx}(1)$$

$$= e^x$$

Example 8 – *Solution*

cont'd

The function f and its derivative f' are graphed in Figure 8.

Notice that f has a horizontal tangent when $x = 0$; this corresponds to the fact that $f'(0) = 0$. Notice also that, for $x > 0$, $f'(x)$ is positive and f is increasing.

When $x < 0$, $f'(x)$ is negative and f is decreasing.

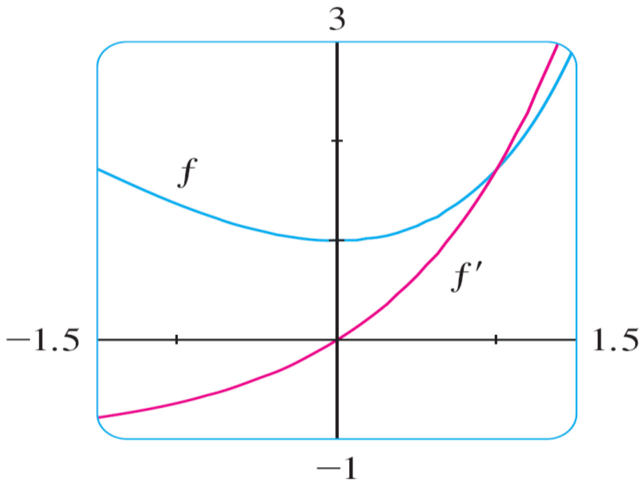


Figure 8

3 Differentiation Rules

3.2 The Product and Quotient Rules

The Product Rule

The Product Rule

By analogy with the Sum and Difference Rules, one might be tempted to guess, that the derivative of a product is the product of the derivatives.

We can see, however, that this guess is wrong by looking at a particular example.

Let $f(x) = x$ and $g(x) = x^2$. Then the Power Rule gives $f'(x) = 1$ and $g'(x) = 2x$.

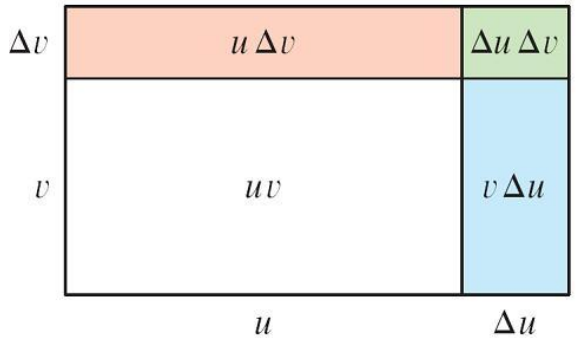
But $(fg)(x) = x^3$, so $(fg)'(x) = 3x^2$. Thus $(fg)' \neq f'g'$.

The Product Rule

The correct formula was discovered by Leibniz and is called the Product Rule.

Before stating the Product Rule, let's see how we might discover it.

We start by assuming that $u = f(x)$ and $v = g(x)$ are both positive differentiable functions. Then we can interpret the product uv as an area of a rectangle (see Figure 1).



The geometry of the Product Rule

Figure 1

The Product Rule

If x changes by an amount Δx, then the corresponding changes in u and v are

$$\Delta u = f(x + \Delta x) - f(x) \qquad \Delta v = g(x + \Delta x) - g(x)$$

and the new value of the product, $(u + \Delta u)(v + \Delta v)$, can be interpreted as the area of the large rectangle in Figure 1 (provided that Δu and Δv happen to be positive).

The change in the area of the rectangle is

$$\boxed{1} \qquad \Delta(uv) = (u + \Delta u)(v + \Delta v) - uv = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v$$

$$= \text{the sum of the three shaded areas}$$

The Product Rule

If we divide by Δx, we get

$$\frac{\Delta(uv)}{\Delta x} = u\,\frac{\Delta v}{\Delta x} + v\,\frac{\Delta u}{\Delta x} + \Delta u\,\frac{\Delta v}{\Delta x}$$

If we now let $\Delta x \to 0$, we get the derivative of uv:

$$\begin{aligned}
\frac{d}{dx}(uv) &= \lim_{\Delta x \to 0} \frac{\Delta(uv)}{\Delta x} = \lim_{\Delta x \to 0} \left(u\,\frac{\Delta v}{\Delta x} + v\,\frac{\Delta u}{\Delta x} + \Delta u\,\frac{\Delta v}{\Delta x} \right) \\
&= u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \left(\lim_{\Delta x \to 0} \Delta u \right)\left(\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \right) \\
&= u\,\frac{dv}{dx} + v\,\frac{du}{dx} + 0 \cdot \frac{dv}{dx}
\end{aligned}$$

The Product Rule

$$\boxed{2} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

(Notice that $\Delta u \to 0$ as $\Delta x \to 0$ since f is differentiable and therefore continuous.)

Although we started by assuming (for the geometric interpretation) that all the quantities are positive, we notice that Equation 1 is always true. (The algebra is valid whether u, v, Δu, Δv and are positive or negative.)

The Product Rule

So we have proved Equation 2, known as the Product Rule, for all differentiable functions *u* and *v*.

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

In words, the Product Rule says that *the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.*

Example 1

(a) If $f(x) = xe^x$, find $f'(x)$.

(b) Find the nth derivative, $f^{(n)}(x)$.

Solution:

(a) By the Product Rule, we have

$$f'(x) = \frac{d}{dx}(xe^x)$$

$$= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$$

$$= xe^x + e^x \cdot 1 = (x + 1)e^x$$

Example 1 – *Solution*

cont'd

(b) Using the Product Rule a second time, we get

$$f''(x) = \frac{d}{dx}[(x + 1)e^x]$$

$$= (x + 1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x + 1)$$

$$= (x + 1)e^x + e^x \cdot 1$$

$$= (x + 2)e^x$$

Example 1 – *Solution*

cont'd

Further applications of the Product Rule give

$$f'''(x) = (x + 3)e^x \qquad f^{(4)}(x) = (x + 4)e^x$$

In fact, each successive differentiation adds another term e^x, so

$$f^{(n)}(x) = (x + n)e^x$$

The Quotient Rule

The Quotient Rule

We find a rule for differentiating the quotient of two differentiable functions $u = f(x)$ and $v = g(x)$ in much the same way that we found the Product Rule.

If x, u, and v change by amounts Δx, Δu, and Δv, then the corresponding change in the quotient u/v is

$$\Delta\left(\frac{u}{v}\right) = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{(u + \Delta u)v - u(v + \Delta v)}{v(v + \Delta v)}$$

$$= \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}$$

The Quotient Rule

so

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{\Delta x \to 0} \frac{\Delta(u/v)}{\Delta x} = \lim_{\Delta x \to 0} \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}$$

As $\Delta x \to 0$, $\Delta v \to 0$ also, because $v = g(x)$ is differentiable and therefore continuous.

Thus, using the Limit Laws, we get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \displaystyle\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} - u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}}{v \displaystyle\lim_{\Delta x \to 0} (v + \Delta v)} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

The Quotient Rule

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

In words, the Quotient Rule says that the *derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

Example 4

Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Then

$$y' = \frac{(x^3 + 6)\dfrac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\dfrac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

The Quotient Rule

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f + g)' = f' + g' \qquad (f - g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

3

Derivatives

3.3

Derivatives of Trigonometric Functions

Derivatives of Trigonometric Functions

In particular, it is important to remember that when we talk about the function f defined for all real numbers x by

$$f(x) = \sin x$$

it is understood that $\sin x$ means the sine of the angle whose *radian* measure is x. A similar convention holds for the other trigonometric functions cos, tan, csc, sec, and cot.

All of the trigonometric functions are continuous at every number in their domains.

Derivatives of Trigonometric Functions

If we sketch the graph of the function $f(x) = \sin x$ and use the interpretation of $f'(x)$ as the slope of the tangent to the sine curve in order to sketch the graph of f', then it looks as if the graph of f' may be the same as the cosine curve. (See Figure 1).

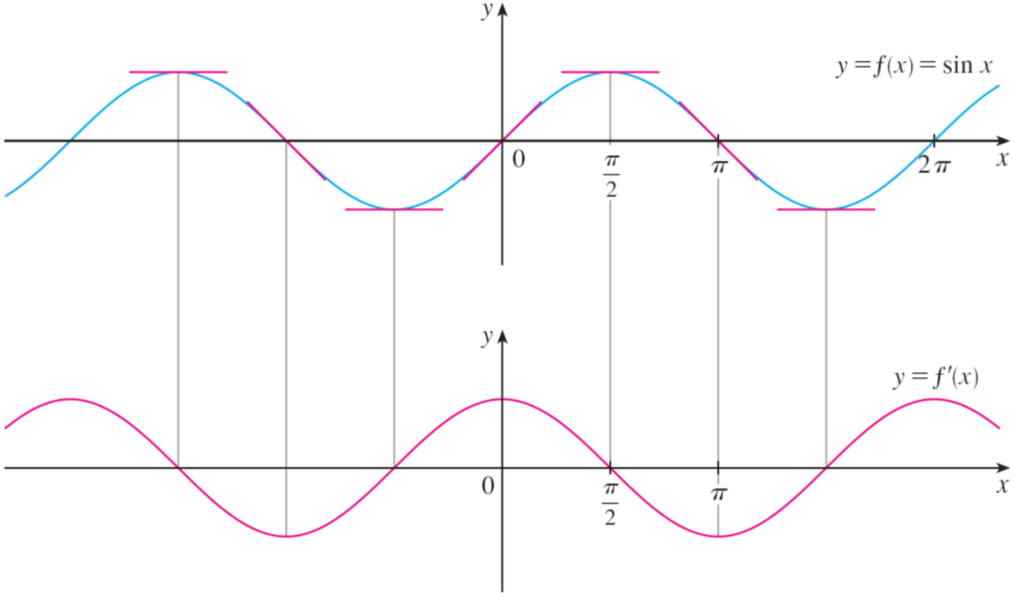


Figure 1

Derivatives of Trigonometric Functions

Let's try to confirm our guess that if $f(x) = \sin x$, then $f'(x) = \cos x$. From the definition of a derivative, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right]$$

Derivatives of Trigonometric Functions

$$= \lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right]$$

$$\boxed{1} \qquad = \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

Two of these four limits are easy to evaluate. Since we regard *x* as a constant when computing a limit as $h \to 0$, we have

$$\lim_{h \to 0} \sin x = \sin x \qquad \text{and} \qquad \lim_{h \to 0} \cos x = \cos x$$

Derivatives of Trigonometric Functions

The limit of (sin *h*)/*h* is not so obvious. We made the guess, on the basis of numerical and graphical evidence, that

2 $$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$

Derivatives of Trigonometric Functions

We now use a geometric argument to prove Equation 2. Assume first that θ lies between 0 and $\pi/2$. Figure 2(a) shows a sector of a circle with center *O*, central angle θ, and radius 1.

BC is drawn perpendicular to *OA*.
By the definition of radian measure, we have arc $AB = \theta$.
Also $|BC| = |OB| \sin \theta = \sin \theta$.

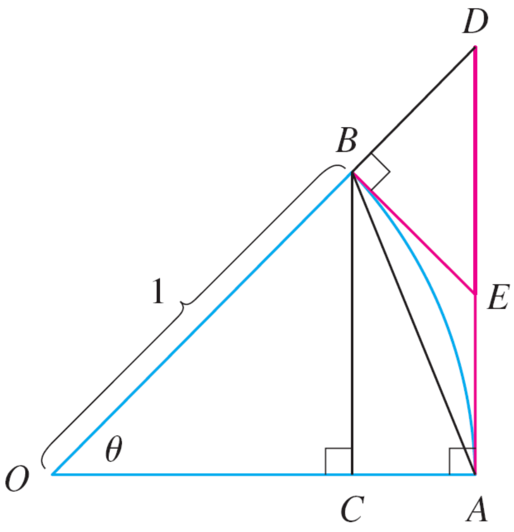


Figure 2(a)

Derivatives of Trigonometric Functions

From the diagram we see that

$$|BC| < |AB| < \text{arc } AB$$

Therefore $\quad \sin\theta < \theta \quad$ so $\quad \dfrac{\sin\theta}{\theta} < 1$

Let the tangent lines at *A* and *B* intersect at *E*. You can see from Figure 2(b) that the circumference of a circle is smaller than the length of a circumscribed polygon, and so arc $AB < |AE| + |EB|$.

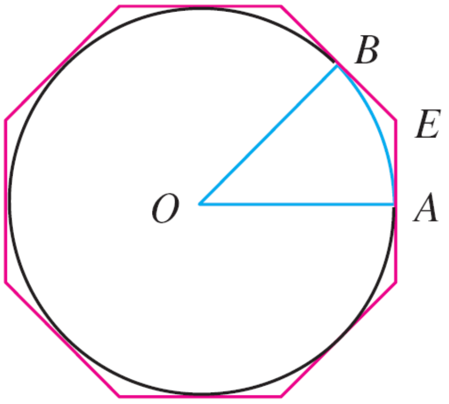


Figure 2(b)

Derivatives of Trigonometric Functions

Thus

$$\begin{aligned}\theta = \text{arc } AB &< |AE| + |EB| \\ &< |AE| + |ED| \\ &= |AD| = |OA| \tan \theta \\ &= \tan \theta\end{aligned}$$

Therefore we have

$$\theta < \frac{\sin \theta}{\cos \theta}$$

so

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

Derivatives of Trigonometric Functions

We know that $\lim_{\theta \to 0} 1 = 1$ and $\lim_{\theta \to 0} \cos \theta = 1$, so by the Squeeze Theorem, we have

$$\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1$$

But the function $(\sin \theta)/\theta$ is an even function, so its right and left limits must be equal. Hence, we have

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

so we have proved Equation 2.

Derivatives of Trigonometric Functions

We can deduce the value of the remaining limit in $\boxed{1}$ as follows:

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = \lim_{\theta \to 0} \left(\frac{\cos \theta - 1}{\theta} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} \right) = \lim_{\theta \to 0} \frac{\cos^2 \theta - 1}{\theta (\cos \theta + 1)}$$

$$= \lim_{\theta \to 0} \frac{-\sin^2 \theta}{\theta (\cos \theta + 1)} = -\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\cos \theta + 1} \right)$$

$$= -\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin \theta}{\cos \theta + 1}$$

$$= -1 \cdot \left(\frac{0}{1 + 1} \right) = 0$$

(by Equation 2)

Derivatives of Trigonometric Functions

$\boxed{3}$

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$$

If we now put the limits $\boxed{2}$ and $\boxed{3}$ in $\boxed{1}$, we get

$$f'(x) = \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x$$

Derivatives of Trigonometric Functions

So we have proved the formula for the derivative of the sine function:

4

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Example 1

Differentiate $y = x^2 \sin x$.

Solution:

Using the Product Rule and Formula 4, we have

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$$

$$= x^2 \cos x + 2x \sin x$$

Derivatives of Trigonometric Functions

Using the same methods as in the proof of Formula 4, one can prove that

5

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

The tangent function can also be differentiated by using the definition of a derivative, but it is easier to use the Quotient Rule together with Formulas 4 and 5:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

Derivatives of Trigonometric Functions

$$= \frac{\cos x \dfrac{d}{dx}(\sin x) - \sin x \dfrac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Derivatives of Trigonometric Functions

6 $$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

The derivatives of the remaining trigonometric functions, csc, sec, and cot, can also be found easily using the Quotient Rule.

Derivatives of Trigonometric Functions

We collect all the differentiation formulas for trigonometric functions in the following table. Remember that they are valid only when x is measured in radians.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Derivatives of Trigonometric Functions

Trigonometric functions are often used in modeling real-world phenomena. In particular, vibrations, waves, elastic motions, and other quantities that vary in a periodic manner can be described using trigonometric functions. In the following example we discuss an instance of simple harmonic motion.

Example 3

An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. (See Figure 5 and note that the downward direction is positive.)

Its position at time t is

$$s = f(t) = 4 \cos t$$

Find the velocity and acceleration at time t and use them to analyze the motion of the object.

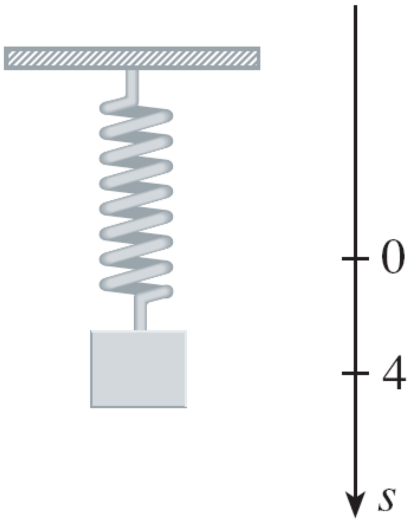


Figure 5

Example 3 – *Solution*

The velocity and acceleration are

$$v = \frac{ds}{dt}$$

$$= \frac{d}{dt}(4\cos t)$$

$$= 4\frac{d}{dt}(\cos t)$$

$$= -4\sin t$$

Example 3 – *Solution*

cont'd

$$
\begin{aligned}
a &= \frac{dv}{dt} \\
&= \frac{d}{dt}(-4 \sin t) \\
&= -4 \frac{d}{dt}(\sin t) \\
&= -4 \cos t
\end{aligned}
$$

The object oscillates from the lowest point ($s = 4$ cm) to the highest point ($s = -4$ cm). The period of the oscillation is 2π, the period of $\cos t$.

Example 3 – *Solution*

cont'd

The speed is $|v| = 4\,|\sin t|$, which is greatest when $|\sin t| = 1$, that is, when $\cos t = 0$.

So the object moves fastest as it passes through its equilibrium position ($s = 0$). Its speed is 0 when $\sin t = 0$, that is, at the high and low points.

The acceleration $a = -4 \cos t = 0$ when $s = 0$. It has greatest magnitude at the high and low points. See the graphs in Figure 6.

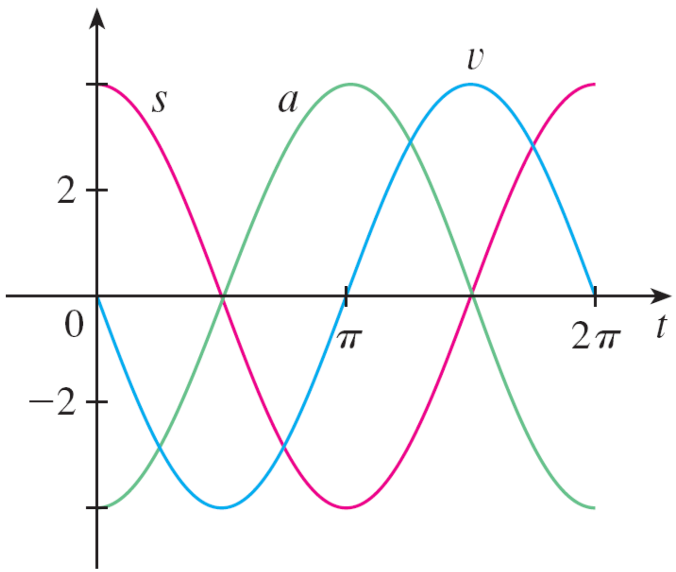


Figure 6

3 Derivatives

3.4 The Chain Rule

The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F'(x)$.

Observe that F is a composite function. In fact, if we let $y = f(u) = \sqrt{u}$ and let $u = g(x) = x^2 + 1$, then we can write $y = F(x) = f(g(x))$, that is, $F = f \circ g$.

We know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $F = f \circ g$ in terms of the derivatives of f and g.

The Chain Rule

It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of *f* and *g*. This fact is one of the most important of the differentiation rules and is called the *Chain Rule.*

It seems plausible if we interpret derivatives as rates of change. Regard du/dx as the rate of change of *u* with respect to *x*, dy/du as the rate of change of *y* with respect to *u*, and dy/dx as the rate of change of *y* with respect to *x*. If *u* changes twice as fast as *x* and *y* changes three times as fast as *u*, then it seems reasonable that *y* changes six times as fast as *x*, and so we expect that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The Chain Rule

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The Chain Rule

The Chain Rule can be written either in the prime notation

2 $$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or, if $y = f(u)$ and $u = g(x)$, in Leibniz notation:

3 $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Equation 3 is easy to remember because if dy/du and du/dx were quotients, then we could cancel du.

Remember, however, that du has not been defined and du/dx should not be thought of as an actual quotient.

Example 1

Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

Solution 1:

(Using Equation 2): We have expressed F as
$F(x) = (f \circ g)(x) = f(g(x))$ where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$.

Since

$$f'(u) = \tfrac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \quad \text{and} \quad g'(x) = 2x$$

we have $\quad F'(x) = f'(g(x)) \cdot g'(x)$

$$= \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}$$

Example 1 – *Solution 2*

cont'd

(Using Equation 3): If we let $u = x^2 + 1$ and $y = \sqrt{u}$, then

$$F'(x) = \frac{dy}{du}\frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}}(2x)$$

$$= \frac{1}{2\sqrt{x^2 + 1}}(2x)$$

$$= \frac{x}{\sqrt{x^2 + 1}}$$

The Chain Rule

When using Formula 3 we should bear in mind that dy/dx refers to the derivative of y when y is considered as a function of x (called the *derivative of y with respect to x*), whereas dy/du refers to the derivative of y when considered as a function of u (the derivative of y with respect to u). For instance, in Example 1, y can be considered as a function of x ($y = \sqrt{x^2 + 1}$) and also as a function of u ($y = \sqrt{u}$).

Note that

$$\frac{dy}{dx} = F'(x) = \frac{x}{\sqrt{x^2 + 1}} \quad \text{whereas} \quad \frac{dy}{du} = f'(u) = \frac{1}{2\sqrt{u}}$$

The Chain Rule

In general, if $y = \sin u$, where u is a differentiable function of x, then, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos u \frac{du}{dx}$$

Thus
$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

In a similar fashion, all of the formulas for differentiating trigonometric functions can be combined with the Chain Rule.

The Chain Rule

Let’s make explicit the special case of the Chain Rule where the outer function *f* is a power function.

If $y = [g(x)]^n$, then we can write $y = f(u) = u^n$ where $u = g(x)$. By using the Chain Rule and then the Power Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx} = n[g(x)]^{n-1}g'(x)$$

$\boxed{4}$ **The Power Rule Combined with the Chain Rule** If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively, $$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Example 3

Differentiate $y = (x^3 - 1)^{100}$.

Solution:

Taking $u = g(x) = x^3 - 1$ and $n = 100$ in $\boxed{4}$, we have

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 1)^{100}$$

$$= 100(x^3 - 1)^{99} \frac{d}{dx}(x^3 - 1)$$

$$= 100(x^3 - 1)^{99} \cdot 3x^2$$

$$= 300x^2(x^3 - 1)^{99}$$

The Chain Rule

We can use the Chain Rule to differentiate an exponential function with any base $a > 0$. Recall that $a = e^{\ln a}$. So

$$a^x = (e^{\ln a})^x = e^{(\ln a)x}$$

and the Chain Rule gives

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{(\ln a)x}) = e^{(\ln a)x}\frac{d}{dx}(\ln a)x$$

$$= e^{(\ln a)x} \cdot \ln a = a^x \ln a$$

because $\ln a$ is a constant. So we have the formula

5

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$

The Chain Rule

In particular, if $a = 2$, we get

$$\boxed{6} \qquad \frac{d}{dx}(2^x) = 2^x \ln 2$$

We had given the estimate

$$\frac{d}{dx}(2^x) \approx (0.69)2^x$$

This is consistent with the exact formula (6) because
$\ln 2 \approx 0.693147$.

The reason for the name "Chain Rule" becomes clear when we make a longer chain by adding another link.

The Chain Rule

Suppose that $y = f(u)$, $u = g(x)$, and $x = h(t)$, where f, g, and h are differentiable functions.

Then, to compute the derivative of y with respect to t, we use the Chain Rule twice:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt}$$

How to Prove the Chain Rule

How to Prove the Chain Rule

Recall that if $y = f(x)$ and x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

According to the definition of a derivative, we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a)$$

So if we denote by ε the difference between the difference quotient and the derivative, we obtain

$$\lim_{\Delta x \to 0} \varepsilon = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} - f'(a) \right) = f'(a) - f'(a) = 0$$

How to Prove the Chain Rule

But

$$\varepsilon = \frac{\Delta y}{\Delta x} - f'(a) \qquad \Rightarrow \qquad \Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x$$

If we define ε to be 0 when $\Delta x = 0$, then ε becomes a continuous function of Δx. Thus, for a differentiable function f, we can write

7 $\quad \Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \quad \text{where} \quad \varepsilon \to 0 \text{ as } \Delta x \to 0$

and ε is a continuous function of Δx. This property of differentiable functions is what enables us to prove the Chain Rule.

3 Derivatives

3.5 Implicit Differentiation

Implicit Differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3 + 1} \quad \text{or} \quad y = x \sin x$$

or, in general, $y = f(x)$.

Some functions, however, are defined implicitly by a relation between x and y such as

$$\boxed{1} \qquad x^2 + y^2 = 25$$

or

$$\boxed{2} \qquad x^3 + y^3 = 6xy$$

Implicit Differentiation

In some cases it is possible to solve such an equation for *y* as an explicit function (or several functions) of *x*.

For instance, if we solve Equation 1 for *y*, we get $y = \pm\sqrt{25 - x^2}$, so two of the functions determined by the implicit Equation 1 are $f(x) = \sqrt{25 - x^2}$ and $g(x) = -\sqrt{25 - x^2}$.

Implicit Differentiation

The graphs of *f* and *g* are the upper and lower semicircles of the circle $x^2 + y^2 = 25$. (See Figure 1.)

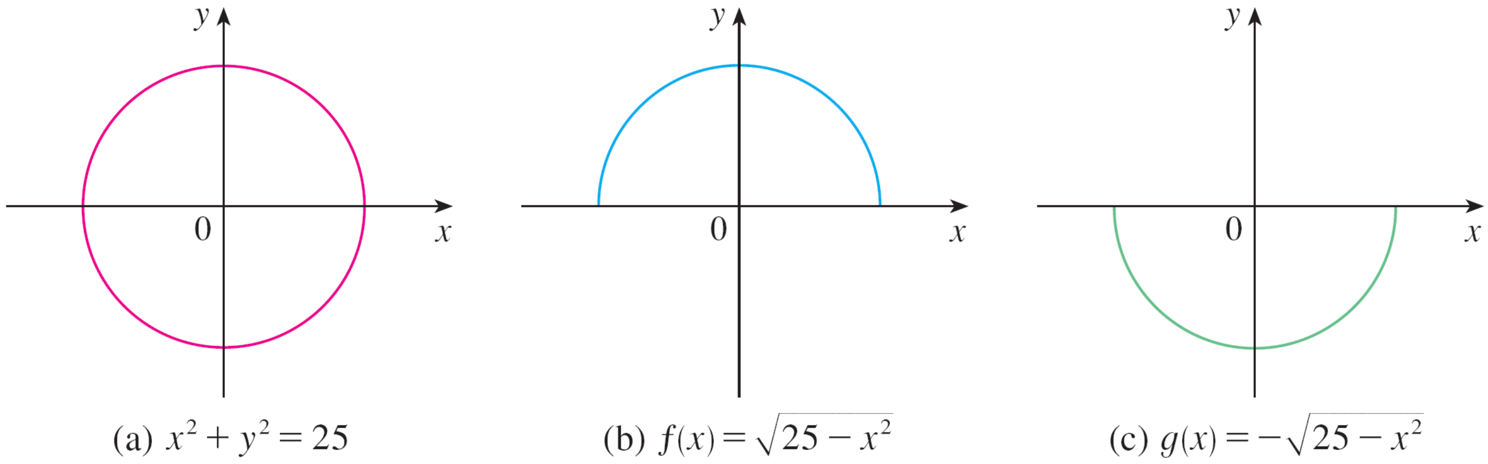
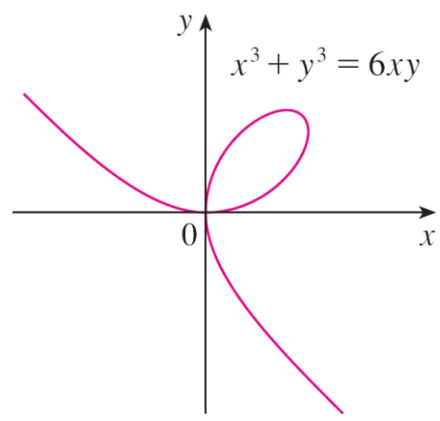


Figure 1

Implicit Differentiation

It's not easy to solve Equation 2 for *y* explicitly as a function of *x* by hand. (A computer algebra system has no trouble, but the expressions it obtains are very complicated.)

Nonetheless, $\boxed{2}$ is the equation of a curve called the **folium of Descartes** shown in Figure 2 and it implicitly defines *y* as several functions of *x*.

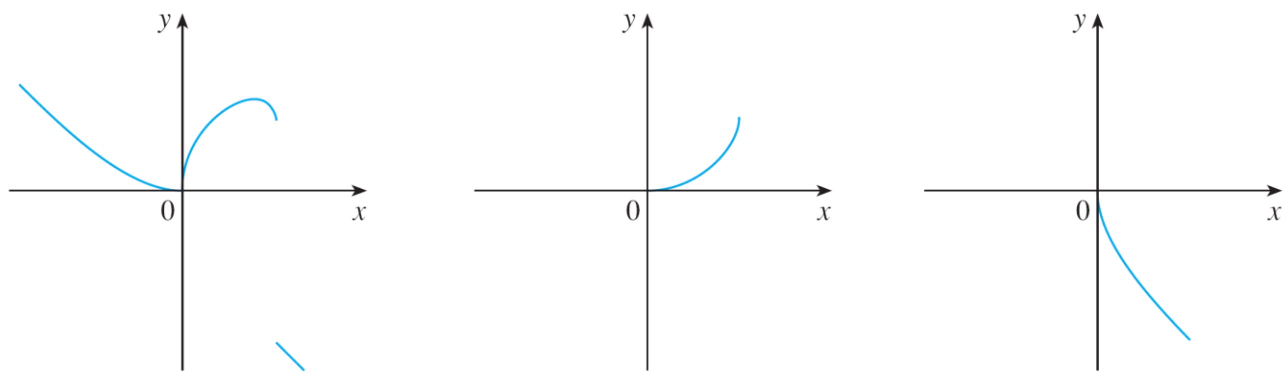


The folium of Descartes

Figure 2

Implicit Differentiation

The graphs of three such functions are shown in Figure 3.



Graphs of three functions defined by the folium of Descartes

Figure 3

When we say that is a function defined implicitly by Equation 2, we mean that the equation

$$x^3 + [f(x)^3] = 6xf(x)$$

is true for all values of in the domain of .

Implicit Differentiation

Fortunately, we don't need to solve an equation for y in terms of x in order to find the derivative of y. Instead we can use the method of **implicit differentiation**.

This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

Example 1

(a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

Solution 1:

(a) Differentiate both sides of the equation $x^2 + y^2 = 25$:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Example 1 – *Solution*

cont'd

Remembering that *y* is a function of *x* and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx}$$

$$= 2y\frac{dy}{dx}$$

Thus

$$2x + 2y\frac{dy}{dx} = 0$$

Now we solve this equation for *dy*/*dx*:

$$\frac{dy}{dx} = -\frac{x}{y}$$

Example 1 – *Solution*

cont'd

(b) At the point (3, 4) we have $x = 3$ and $y = 4$, so

$$\frac{dy}{dx} = -\frac{3}{4}$$

An equation of the tangent to the circle at (3, 4) is therefore

$$y - 4 = -\tfrac{3}{4}(x - 3) \qquad \text{or} \qquad 3x + 4y = 25$$

Solution 2:

(b) Solving the equation $x^2 + y^2 = 25$, we get $y = \pm\sqrt{25 - x^2}$. The point (3, 4) lies on the upper semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$.

Example 1 – *Solution*

cont'd

Differentiating *f* using the Chain Rule, we have

$$f'(x) = \tfrac{1}{2}(25 - x^2)^{-1/2}\frac{d}{dx}(25 - x^2)$$

$$= \tfrac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}}$$

So

$$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

and, as in Solution 1, an equation of the tangent is $3x + 4y = 25$.

3

Differentiation Rules

3.6 Derivatives of Logarithmic Functions

Derivatives of Logarithmic Functions

In this section we use implicit differentiation to find the derivatives of the logarithmic functions $y = \log_a x$ and, in particular, the natural logarithmic function $y = \ln x$. [It can be proved that logarithmic functions are differentiable; this is certainly plausible from their graphs (see Figure 12 in Section 1.6).]

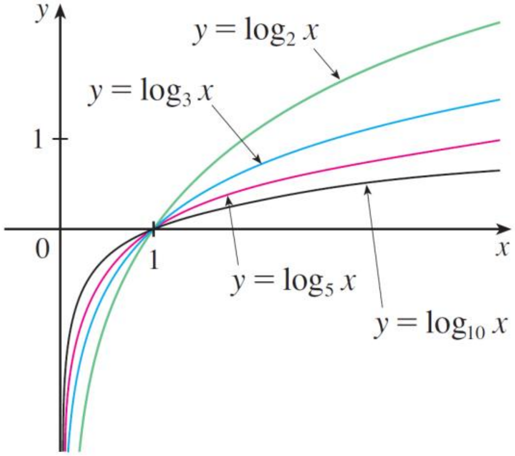


Figure 12

Derivatives of Logarithmic Functions

1 $$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

In general, if we combine Formula 2 with the Chain Rule, we get

3 $$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$ or $$\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

Example 2

Find $\dfrac{d}{dx}\ln(\sin x)$.

Solution:

Using $\boxed{3}$, we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x$$

$$= \cot x$$

$\boxed{4}$

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x}}$$

Logarithmic Differentiation

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms.

The method used in the next example is called **logarithmic differentiation**.

Example 15

Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}$.

Solution:

We take logarithms of both sides of the equation and use the properties of logarithms to simplify:

$$\ln y = \tfrac{3}{4} \ln x + \tfrac{1}{2} \ln(x^2 + 1) - 5 \ln(3x + 2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2 + 1} - 5 \cdot \frac{3}{3x + 2}$$

Example 15 – *Solution*

cont'd

Solving for *dy*/*dx*, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

Because we have an explicit expression for *y*, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

Logarithmic Differentiation

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the properties of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

The Power Rule If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

The Number *e* as a Limit

The Number e as a Limit

If $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$. We now use this fact to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x)$$

$$= \lim_{x \to 0} \ln(1+x)^{1/x}$$

The Number e as a Limit

Because $f'(1) = 1$, we have

$$\lim_{x \to 0} \ln(1 + x)^{1/x} = 1$$

Then, by the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}}$$

$$= \lim_{x \to 0} (1 + x)^{1/x}$$

5

$$\boxed{e = \lim_{x \to 0} (1 + x)^{1/x}}$$

The Number e as a Limit

Formula 5 is illustrated by the graph of the function $y = (1 + x)^{1/x}$ in Figure 4 and a table of values for small values of x. This illustrates the fact that, correct to seven decimal places,

$$e \approx 2.7182818$$

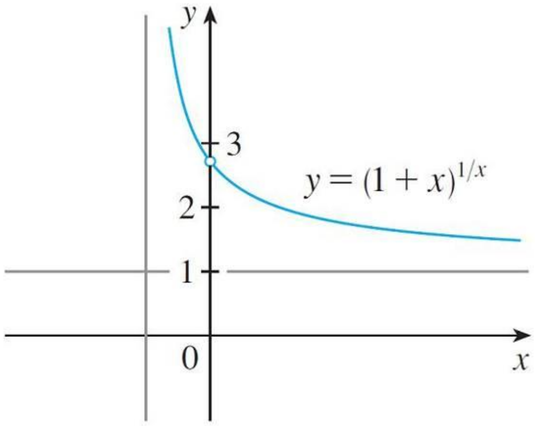


Figure 4

x	$(1 + x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

The Number *e* as a Limit

If we put $n = 1/x$ in Formula 5, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for *e* is

$$\boxed{6} \qquad \boxed{e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$

3

Derivatives

3.7 Rates of Change in the Natural and Social Sciences

Rates of Change in the Natural and Social Sciences

We know that if $y = f(x)$, then the derivative dy/dx can be interpreted as the rate of change of y with respect to x. In this section we examine some of the applications of this idea to physics, chemistry, biology, economics, and other sciences.

Let's recall from Section 2.6 the basic idea behind rates of change. If x changes from x_1 to x_2, then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

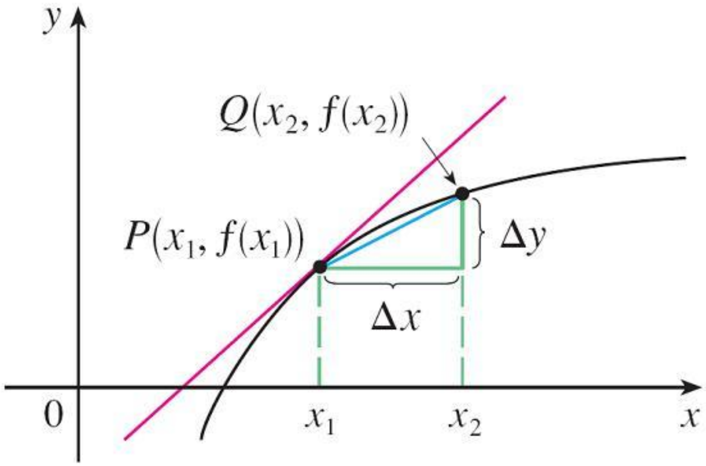
$$\Delta y = f(x_2) - f(x_1)$$

Rates of Change in the Natural and Social Sciences

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the **average rate of change of *y* with respect to *x*** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line *PQ* in Figure 1.



m_{PQ} = average rate of change
$m = f'(x_1)$ = instantaneous rate of change

Figure 1

Rates of Change in the Natural and Social Sciences

Its limit as $\Delta x \to 0$ is the derivative $f'(x_1)$, which can therefore be interpreted as the **instantaneous rate of change of *y* with respect to *x*** or the slope of the tangent line at $P(x_1, f(x_1))$.

Using Leibniz notation, we write the process in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Physics

Physics

If $s = f(t)$ is the position function of a particle that is moving in a straight line, then $\Delta s/\Delta t$ represents the average velocity over a time period Δt, and $v = ds/dt$ represents the instantaneous **velocity** (the rate of change of displacement with respect to time).

The instantaneous rate of change of velocity with respect to time is **acceleration**: $a(t) = v'(t) = s''(t)$.

Now that we know the differentiation formulas, we are able to solve problems involving the motion of objects more easily.

Example 1

The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.

(a) Find the velocity at time t.

(b) What is the velocity after 2 s? After 4 s?

(c) When is the particle at rest?

(d) When is the particle moving forward (that is, in the positive direction)?

Example 1

cont'd

(e) Draw a diagram to represent the motion of the particle.

(f) Find the total distance traveled by the particle during the first five seconds.

(g) Find the acceleration at time t and after 4 s.

(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.

(i) When is the particle speeding up? When is it slowing down?

Example 1 – *Solution*

(a) The velocity function is the derivative of the position function.

$$s = f(t) = t^3 - 6t^2 + 9t$$

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$$

Example 1 – *Solution*

cont'd

(b) The velocity after 2 s means the instantaneous velocity when $t = 2$, that is,

$$v(2) = \left.\frac{ds}{dt}\right|_{t=2} = 3(2)^2 - 12(2) + 9$$

$$= -3 \text{ m/s}$$

The velocity after 4 s is

$$v(4) = 3(4)^2 - 12(4) + 9$$

$$= 9 \text{ m/s}$$

Example 1 – *Solution*

cont'd

(c) The particle is at rest when $v(t) = 0$, that is,

$$\begin{aligned} 3t^2 - 12t + 9 &= 3(t^2 - 4t + 3) \\ &= 3(t - 1)(t - 3) \\ &= 0 \end{aligned}$$

and this is true when $t = 1$ or $t = 3$.

Thus the particle is at rest after 1 s and after 3 s.

Example 1 – *Solution*

cont'd

(d) The particle moves in the positive direction when $v(t) > 0$, that is,

$$3t^2 - 12t + 9 = 3(t - 1)(t - 3) > 0$$

This inequality is true when both factors are positive ($t > 3$) or when both factors are negative ($t < 1$).

Thus the particle moves in the positive direction in the time intervals $t < 1$ and $t > 3$.

It moves backward (in the negative direction) when $1 < t < 3$.

Example 1 – *Solution*

cont'd

(e) Using the information from part (d) we make a schematic sketch in Figure 2 of the motion of the particle back and forth along a line (the *s*-axis).

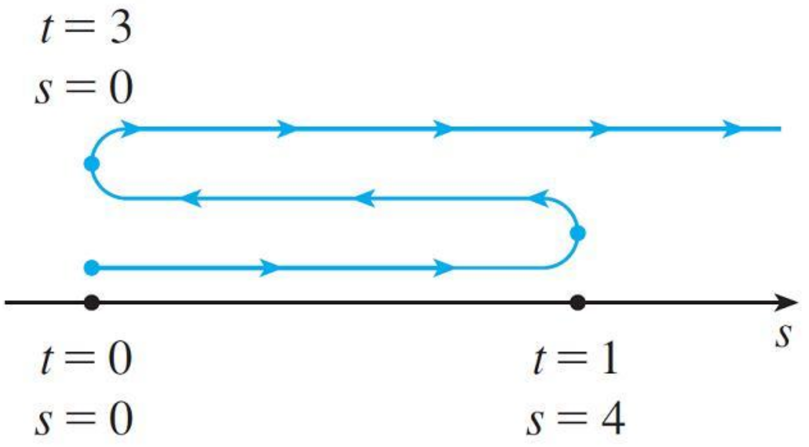


Figure 2

Example 1 – *Solution*

cont'd

(f) Because of what we learned in parts (d) and (e), we need to calculate the distances traveled during the time intervals [0, 1], [1, 3], and [3, 5] separately.

The distance traveled in the first second is

$$|f(1) - f(0)| = |4 - 0| = 4 \text{ m}$$

From $t = 1$ to $t = 3$ the distance traveled is

$$|f(3) - f(1)| = |0 - 4| = 4 \text{ m}$$

From $t = 3$ to $t = 5$ the distance traveled is

$$|f(5) - f(3)| = |20 - 0| = 20 \text{ m}$$

The total distance is 4 + 4 + 20 = 28 m.

Example 1 – *Solution*

cont'd

(g) The acceleration is the derivative of the velocity function:

$$a(t) = \frac{d^2s}{dt^2}$$

$$= \frac{dv}{dt}$$

$$= 6t - 12$$

$$a(4) = 6(4) - 12$$

$$= 12 \text{ m/s}^2$$

Example 1 – *Solution*

cont'd

(h) Figure 3 shows the graphs of *s*, *v*, and *a*.

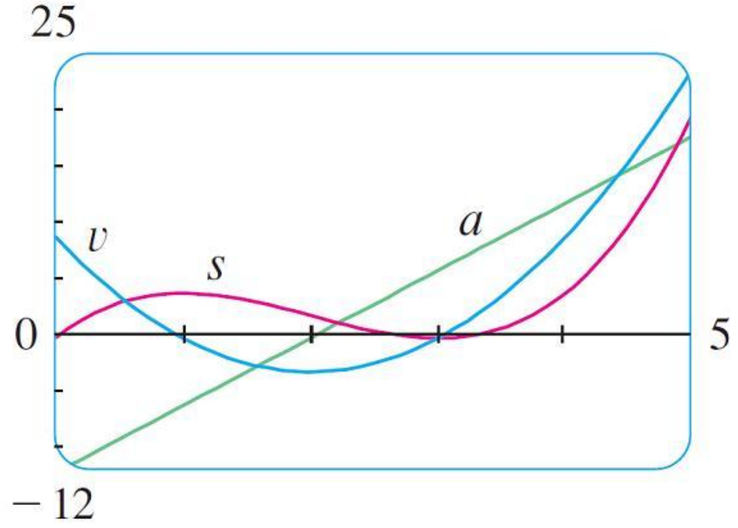


Figure 3

Example 1 – *Solution*

cont'd

(i) The particle speeds up when the velocity is positive and increasing (v and a are both positive) and also when the velocity is negative and decreasing (v and a are both negative).

In other words, the particle speeds up when the velocity and acceleration have the same sign. (The particle is pushed in the same direction it is moving.)

From Figure 3 we see that this happens when $1 < t < 2$ and when $t > 3$.

Example 1 – *Solution*

cont'd

The particle slows down when v and a have opposite signs, that is, when $0 \leq t < 1$ and when $2 < t < 3$.

Figure 4 summarizes the motion of the particle.

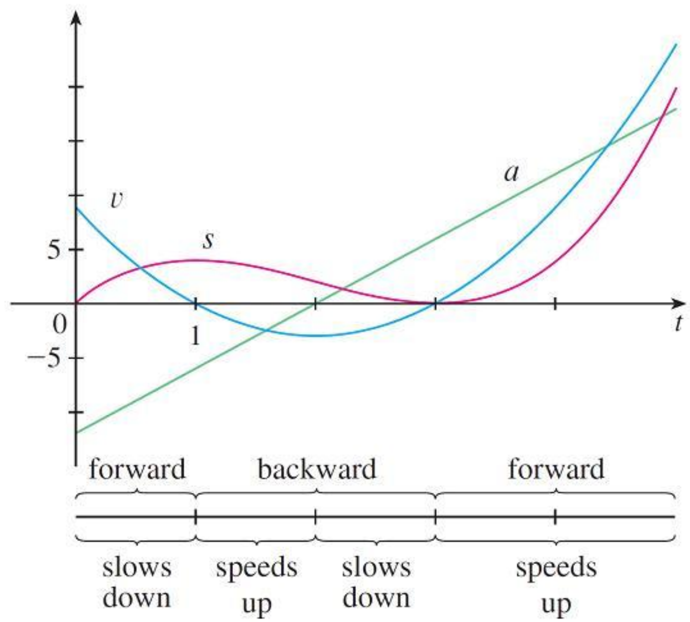


Figure 4

Example 2

If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ($\rho = m/l$) and measured in kilograms per meter.

Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point x is $m = f(x)$, as shown in Figure 5.

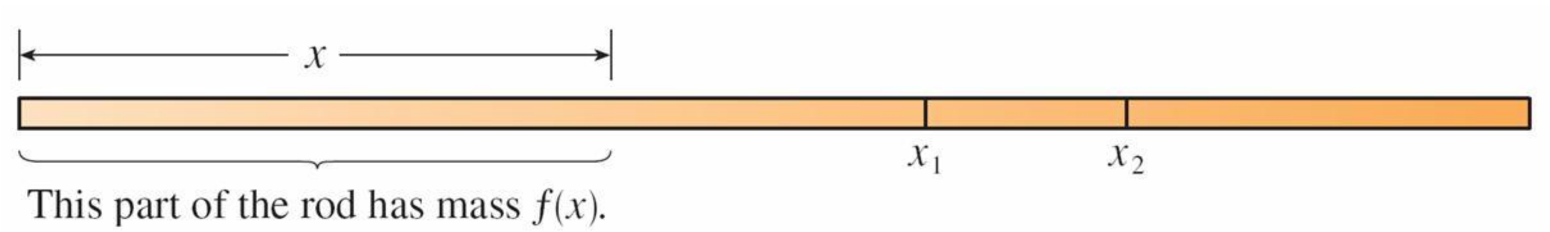


Figure 5

Example 2

cont'd

The mass of the part of the rod that lies between $x = x_1$ and $x = x_2$ is given by $\Delta m = f(x_2) - f(x_1)$, so the average density of that part of the rod is

$$\text{average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

If we now let $\Delta x \to 0$ (that is, $x_2 \to x_1$), we are computing the average density over smaller and smaller intervals.

The **linear density** ρ at x_1 is the limit of these average densities as $\Delta x \to 0$; that is, the linear density is the rate of change of mass with respect to length.

Example 2

cont'd

Symbolically,

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$$

Thus the linear density of the rod is the derivative of mass with respect to length.

For instance, if $m = f(x) = \sqrt{x}$, where x is measured in meters and m in kilograms, then the average density of the part of the rod given by $1 \le x \le 1.2$ is

$$\frac{\Delta m}{\Delta x} = \frac{f(1.2) - f(1)}{1.2 - 1} = \frac{\sqrt{1.2} - 1}{0.2}$$

Example 2

cont'd

$$\approx 0.48 \text{ kgm}$$

while the density right at $x = 1$ is

$$\rho = \left.\frac{dm}{dx}\right|_{x=1} = \left.\frac{1}{2\sqrt{x}}\right|_{x=1}$$

$$= 0.50 \text{ kg/m}$$

Example 3

A current exists whenever electric charges move. Figure 6 shows part of a wire and electrons moving through a plane surface, shaded red.

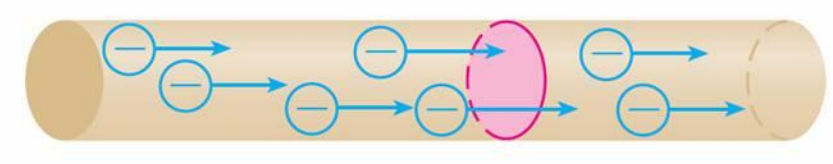


Figure 6

If ΔQ is the net charge that passes through this surface during a time period Δt, then the average current during this time interval is defined as

$$\text{average current} = \frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1}$$

Example 3

cont'd

If we take the limit of this average current over smaller and smaller time intervals, we get what is called the **current *I*** at a given time t_1:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

Thus the current is the rate at which charge flows through a surface. It is measured in units of charge per unit time (often coulombs per second, called amperes).

Chemistry

Example 4

A chemical reaction results in the formation of one or more substances (called *products*) from one or more starting materials (called *reactants*). For instance, the "equation"

$$2H_2 + O_2 \rightarrow 2H_2O$$

indicates that two molecules of hydrogen and one molecule of oxygen form two molecules of water.

Let's consider the reaction

$$A + B \rightarrow C$$

where A and B are the reactants and C is the product.

Example 4

cont'd

The **concentration** of a reactant A is the number of moles (1 mole = 6.022×10^{23} molecules) per liter and is denoted by [A].

The concentration varies during a reaction, so [A], [B], and [C] are all functions of time (t).

The average rate of reaction of the product C over a time interval $t_1 \leq t \leq t_2$ is

$$\frac{\Delta[\mathrm{C}]}{\Delta t} = \frac{[\mathrm{C}](t_2) - [\mathrm{C}](t_1)}{t_2 - t_1}$$

Example 4

cont'd

But chemists are more interested in the **instantaneous rate of reaction**, which is obtained by taking the limit of the average rate of reaction as the time interval Δt approaches 0:

$$\text{rate of reaction} = \lim_{\Delta t \to 0} \frac{\Delta[\mathrm{C}]}{\Delta t} = \frac{d[\mathrm{C}]}{dt}$$

Since the concentration of the product increases as the reaction proceeds, the derivative $d[\mathrm{C}]/dt$ will be positive, and so the rate of reaction of C is positive.

Example 4

cont'd

The concentrations of the reactants, however, decrease during the reaction, so, to make the rates of reaction of A and B positive numbers, we put minus signs in front of the derivatives $d[A]/dt$ and $d[B]/dt$.

Since [A] and [B] each decrease at the same rate that [C] increases, we have

$$\text{rate of reaction} = \frac{d[\mathrm{C}]}{dt} = -\frac{d[\mathrm{A}]}{dt} = -\frac{d[\mathrm{B}]}{dt}$$

Example 4

cont'd

More generally, it turns out that for a reaction of the form

$$aA + bB \rightarrow cC + dD$$

we have

$$-\frac{1}{a}\frac{d[A]}{dt} = -\frac{1}{b}\frac{d[B]}{dt} = \frac{1}{c}\frac{d[C]}{dt} = \frac{1}{d}\frac{d[D]}{dt}$$

The rate of reaction can be determined from data and graphical methods. In some cases there are explicit formulas for the concentrations as functions of time, which enable us to compute the rate of reaction.

Example 5

One of the quantities of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume V depends on its pressure P.

We can consider the rate of change of volume with respect to pressure—namely, the derivative dV/dP. As P increases, V decreases, so $dV/dP < 0$.

The **compressibility** is defined by introducing a minus sign and dividing this derivative by the volume V:

$$\text{isothermal compressibility} = \beta = -\frac{1}{V}\frac{dV}{dP}$$

Example 5

cont'd

Thus β measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

For instance, the volume V (in cubic meters) of a sample of air at 25°C was found to be related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}$$

Example 5

cont'd

The rate of change of V with respect to P when P = 50 kPa is

$$\left.\frac{dV}{dP}\right|_{P=50} = -\left.\frac{5.3}{P^2}\right|_{P=50}$$

$$= -\frac{5.3}{2500}$$

$$= -0.00212 \text{ m}^3/\text{kPa}$$

Example 5

cont'd

The compressibility at that pressure is

$$\beta = -\frac{1}{V}\frac{dV}{dP}\bigg|_{P=50}$$

$$= \frac{0.00212}{\frac{5.3}{50}}$$

$$= 0.02\ (\mathrm{m}^3/\mathrm{kPa})/\mathrm{m}^3$$

Biology

Example 6

Let $n = f(t)$ be the number of individuals in an animal or plant population at time t.

The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \leq t \leq t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The **instantaneous rate of growth** is obtained from this average rate of growth by letting the time period Δt approach 0:

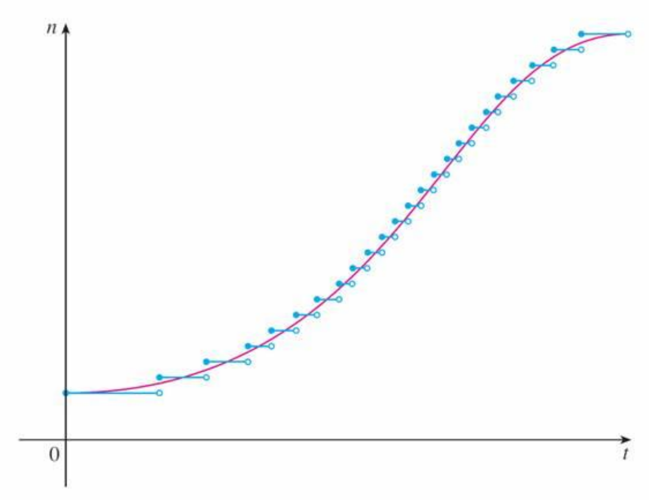
$$\text{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Example 6

cont'd

Strictly speaking, this is not quite accurate because the actual graph of a population function $n = f(t)$ would be a step function that is discontinuous whenever a birth or death occurs and therefore not differentiable.

However, for a large animal or plant population, we can replace the graph by a smooth approximating curve as in Figure 7.



A smooth curve approximating a growth function

Figure 7

Example 6

cont'd

To be more specific, consider a population of bacteria in a homogeneous nutrient medium.

Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour.

If the initial population is n_0 and the time t is measured in hours, then

$$f(1) = 2f(0) = 2n_0$$

$$f(2) = 2f(1) = 2^2 n_0$$

Example 6

cont'd

$$f(3) = 2f(2) = 2^3 n_0$$

and, in general,

$$f(t) = 2^t n_0$$

The population function is $n = n_0 2^t$.

We have shown that

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Example 6

cont'd

So the rate of growth of the bacteria population at time *t* is

$$\frac{dn}{dt} = \frac{d}{dt}(n_0 2^t)$$

$$= n_0 2^t \ln 2$$

For example, suppose that we start with an initial population of $n_0 = 100$ bacteria. Then the rate of growth after 4 hours is

$$\left.\frac{dn}{dt}\right|_{t=4} = 100 \cdot 2^4 \ln 2$$

Example 6

cont'd

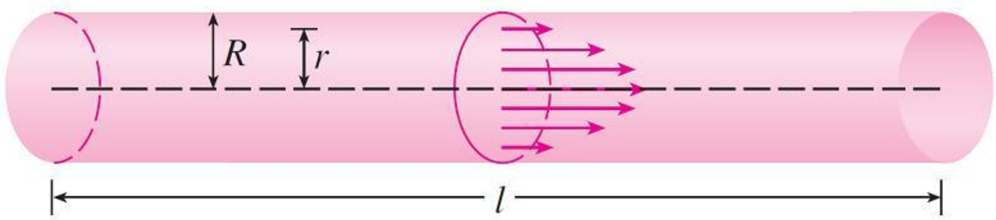
$$= 1600 \ln 2$$

$$\approx 1109$$

This means that, after 4 hours, the bacteria population is growing at a rate of about 1109 bacteria per hour.

Example 7

When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius *R* and length *l* as illustrated in Figure 8.



Blood flow in an artery

Figure 8

Because of friction at the walls of the tube, the velocity *v* of the blood is greatest along the central axis of the tube and decreases as the distance *r* from the axis increases until *v* becomes 0 at the wall.

Example 7

cont'd

The relationship between *v* and *r* is given by the **law of laminar flow** discovered by the French physician Jean-Louis-Marie Poiseuille in 1840.

This law states that

$$v = \frac{P}{4\eta l}(R^2 - r^2) \tag{1}$$

where η is the viscosity of the blood and *P* is the pressure difference between the ends of the tube.

If *P* and *l* are constant, then *v* is a function of *r* with domain [0, *R*].

Example 7

cont'd

The average rate of change of the velocity as we move from $r = r_1$ outward to $r = r_2$ is given by

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}$$

and if we let $\Delta r \to 0$, we obtain the **velocity gradient**, that is, the instantaneous rate of change of velocity with respect to r:

$$\text{velocity gradient} = \lim_{\Delta r \to 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

Example 7

cont'd

Using Equation 1, we obtain

$$\frac{dv}{dr} = \frac{P}{4\eta l}(0 - 2r)$$

$$= -\frac{Pr}{2\eta l}$$

For one of the smaller human arteries we can take $\eta = 0.027$, $R = 0.008$ cm, $l = 2$ cm, and $P = 4000$ dynes/cm^2, which gives

$$v = \frac{4000}{4(0.027)2}(0.000064 - r^2)$$

Example 7

cont'd

$$\approx 1.85 \times 10^4(6.4 \times 10^{-5} - r^2)$$

At $r = 0.02$ cm the blood is flowing at a speed of

$$v(0.002) \approx 1.85 \times 10^4(64 \times 10^{-6} - 4 \times 10^{-6})$$

$$= 1.11 \text{ cm/s}$$

and the velocity gradient at that point is

$$\left.\frac{dv}{dr}\right|_{r=0.002} = -\frac{4000(0.002)}{2(0.027)2} \approx -74 \text{ (cm/s)/cm}$$

Example 7

cont'd

To get a feeling for what this statement means, let's change our units from centimeters to micrometers (1 cm = 10,000 μm). Then the radius of the artery is 80 μm.

The velocity at the central axis is 11,850 μm/s, which decreases to 11,110 μm/s at a distance of r = 20 μm.

The fact that $dv/dr = -74$ (μm/s)/μm means that, when r = 20 μm, the velocity is decreasing at a rate of about 74 μm/s for each micrometer that we proceed away from the center.

Economics

Example 8

Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity.

The function C is called a **cost function**. If the number of items produced is increased from x_1 to x_2, then the additional cost is $\Delta C = C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1}$$

$$= \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

Example 8

cont'd

The limit of this quantity as $\Delta x \to 0$, that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the **marginal cost** by economists:

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

[Since x often takes on only integer values, it may not make literal sense to let Δx approach 0, but we can always replace $C(x)$ by a smooth approximating function as in Example 6.]

Example 8

cont'd

Taking $\Delta x = 1$ and n large (so that Δx is small compared to n), we have

$$C'(n) \approx C(n + 1) - C(n)$$

Thus the marginal cost of producing n units is approximately equal to the cost of producing one more unit [the $(n + 1)$st unit].

Example 8

cont'd

It is often appropriate to represent a total cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3$$

where *a* represents the overhead cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labor, and so on. (The cost of raw materials may be proportional to *x*, but labor costs might depend partly on higher powers of *x* because of overtime costs and inefficiencies involved in large-scale operations.)

Example 8

cont'd

For instance, suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10{,}000 + 5x + 0.01x^2$$

Then the marginal cost function is

$$C'(x) = 5 + 0.02x$$

The marginal cost at the production level of 500 items is

$$C'(500) = 5 + 0.02(500)$$

$$= \$15/\text{item}$$

Example 8

cont'd

This gives the rate at which costs are increasing with respect to the production level when $x = 500$ and predicts the cost of the 501st item.

The actual cost of producing the 501st item is

$$C(501) - C(500) = [10{,}000 + 5(501) + 0.01(501)^2]$$

$$- [10{,}000 + 5(500) + 0.01(500)^2]$$

$$= \$15.01$$

Notice that $C'(500) \approx C(501) - C(500)$.

Other Sciences

Other Sciences

Rates of change occur in all the sciences. A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks.

An engineer wants to know the rate at which water flows into or out of a reservoir.

An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases.

A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height.

A Single Idea, Many Interpretations

A Single Idea, Many Interpretations

Velocity, density, current, power, and temperature gradient in physics; rate of reaction and compressibility in chemistry; rate of growth and blood velocity gradient in biology; marginal cost and marginal profit in economics; rate of heat flow in geology; rate of improvement of performance in psychology; rate of spread of a rumor in sociology—these are all special cases of a single mathematical concept, the derivative.

This is an illustration of the fact that part of the power of mathematics lies in its abstractness.

A Single Idea, Many Interpretations

A single abstract mathematical concept (such as the derivative) can have different interpretations in each of the sciences.

When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences.

This is much more efficient than developing properties of special concepts in each separate science.

3

Derivatives

3.8 Exponential Growth and Decay

Exponential Growth and Decay

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For instance, if $y = f(t)$ is the number of individuals in a population of animals or bacteria at time t, then it seems reasonable to expect that the rate of growth $f'(t)$ is proportional to the population $f(t)$; that is, $f'(t) = kf(t)$ for some constant k.

Indeed, under ideal conditions (unlimited environment, adequate nutrition, immunity to disease) the mathematical model given by the equation $f'(t) = kf(t)$ predicts what actually happens fairly accurately.

Exponential Growth and Decay

Another example occurs in nuclear physics where the mass of a radioactive substance decays at a rate proportional to the mass.

In chemistry, the rate of a unimolecular first-order reaction is proportional to the concentration of the substance.

In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

Exponential Growth and Decay

In general, if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

1 $$\boxed{\frac{dy}{dt} = ky}$$

where k is a constant.

Equation 1 is sometimes called the **law of natural growth** (if $k > 0$) or the **law of natural decay** (if $k < 0$). It is called a **differential equation** because it involves an unknown function y and its derivative dy/dt.

Exponential Growth and Decay

It's not hard to think of a solution of Equation 1. This equation asks us to find a function whose derivative is a constant multiple of itself.

Any exponential function of the form $y(t) = Ce^{kt}$, where C is a constant, satisfies

$$y'(t) = C(ke^{kt}) = k(Ce^{kt}) = ky(t)$$

Exponential Growth and Decay

We will see later that *any* function that satisfies $dy/dt = ky$ must be of the form $y = Ce^{kt}$. To see the significance of the constant C, we observe that

$$y(0) = Ce^{k \cdot 0} = C$$

Therefore C is the initial value of the function.

2 Theorem The only solutions of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}$$

Population Growth

Population Growth

What is the significance of the proportionality constant k? In the context of population growth, where $P(t)$ is the size of a population at time t, we can write

$$\boxed{3} \qquad \frac{dP}{dt} = kP \qquad \text{or} \qquad \frac{1}{P}\frac{dP}{dt} = k$$

The quantity

$$\frac{1}{P}\frac{dP}{dt}$$

is the growth rate divided by the population size; it is called the **relative growth rate**.

Population Growth

According to $\boxed{3}$, instead of saying "the growth rate is proportional to population size" we could say "the relative growth rate is constant."

Then $\boxed{2}$ says that a population with constant relative growth rate must grow exponentially.

Notice that the relative growth rate k appears as the coefficient of t in the exponential function Ce^{kt}.

Population Growth

For instance, if

$$\frac{dP}{dt} = 0.02P$$

and t is measured in years, then the relative growth rate is $k = 0.02$ and the population grows at a relative rate of 2% per year.

If the population at time 0 is P_0, then the expression for the population is

$$P(t) = P_0 e^{0.02t}$$

Example 1

Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

Solution:

We measure the time t in years and let $t = 0$ in the year 1950.

Example 1 – *Solution*

cont'd

We measure the population $P(t)$ in millions of people. Then $P(0) = 2560$ and $P(10) = 3040$.

Since we are assuming that $dP/dt = kP$, Theorem 2 gives

$$P(t) = P(0)e^{kt} = 2560e^{kt}$$

$$P(10) = 2560e^{10k} = 3040$$

$$k = \frac{1}{10} \ln \frac{3040}{2560} \approx 0.017185$$

Example 1 – *Solution*

cont'd

The relative growth rate is about 1.7% per year and the model is

$$P(t) = 2560e^{0.017185t}$$

We estimate that the world population in 1993 was

$$P(43) = 2560e^{0.017185(43)} \approx 5360 \text{ million}$$

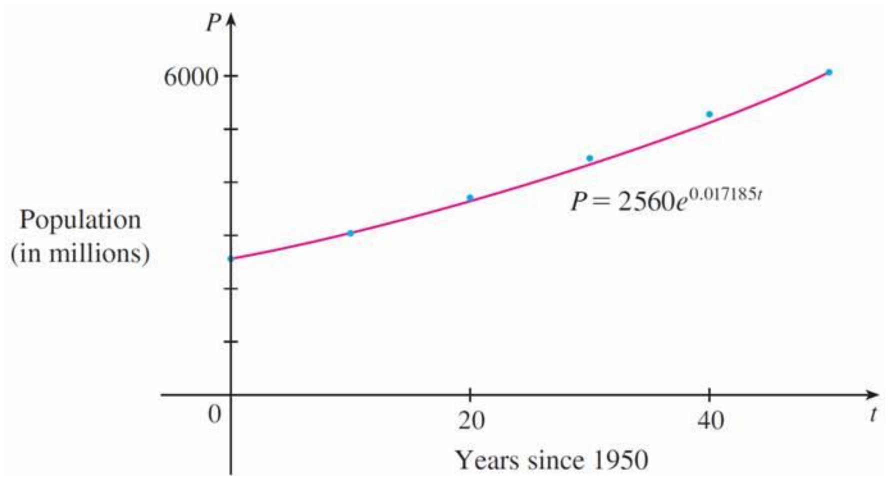
The model predicts that the population in 2020 will be

$$P(70) = 2560e^{0.017185(70)} \approx 8524 \text{ million}$$

Example 1 – *Solution*

cont'd

The graph in Figure 1 shows that the model is fairly accurate to the end of the 20th century (the dots represent the actual population), so the estimate for 1993 is quite reliable. But the prediction for 2020 is riskier.



A model for world population growth in the second half of the 20th century

Figure 1

Radioactive Decay

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. If $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t, then the relative decay rate

$$-\frac{1}{m}\frac{dm}{dt}$$

has been found experimentally to be constant. (Since dm/dt is negative, the relative decay rate is positive.) It follows that

$$\frac{dm}{dt} = km$$

where k is a negative constant.

Radioactive Decay

In other words, radioactive substances decay at a rate proportional to the remaining mass. This means that we can use $\boxed{2}$ to show that the mass decays exponentially:

$$m(t) = m_0 e^{kt}$$

Physicists express the rate of decay in terms of **half-life**, the time required for half of any given quantity to decay.

Example 2

The half-life of radium-226 is 1590 years.

(a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of the sample that remains after t years.

(b) Find the mass after 1000 years correct to the nearest milligram.

(c) When will the mass be reduced to 30 mg?

Solution:

(a) Let $m(t)$ be the mass of radium-226 (in milligrams) that remains after t years.

Example 2 – *Solution*

cont'd

Then $dm/dt = km$ and $y(0) = 100$, so $\boxed{2}$ gives

$$m(t) = m(0)e^{kt} = 100e^{kt}$$

In order to determine the value of k, we use the fact that $y(1590) = \frac{1}{2}(100)$. Thus

$$100e^{1590k} = 50 \qquad \text{so} \qquad e^{1590k} = \frac{1}{2}$$

and

$$1590k = \ln \tfrac{1}{2} = -\ln 2$$

Example 2 – *Solution*

cont'd

$$k = -\frac{\ln 2}{1590}$$

Therefore

$$m(t) = 100e^{-(\ln 2)t/1590}$$

We could use the fact that $e^{\ln 2} = 2$ to write the expression for $m(t)$ in the alternative form

$$m(t) = 100 \times 2^{-t/1590}$$

Example 2 – *Solution*

cont'd

(b) The mass after 1000 years is

$$m(1000) = 100e^{-(\ln 2)1000/1590} \approx 65 \text{ mg}$$

(c) We want to find the value of t such that $m(t) = 30$, that is,

$$100e^{-(\ln 2)t/1590} = 30 \quad \text{or} \quad e^{-(\ln 2)t/1590} = 0.3$$

We solve this equation for t by taking the natural logarithm of both sides:

$$-\frac{\ln 2}{1590}t = \ln 0.3$$

Example 2 – *Solution*

cont'd

Thus

$$t = -1590 \frac{\ln 0.3}{\ln 2}$$

$$\approx 2762 \text{ years}$$

Radioactive Decay

As a check on our work in Example 2, we use a graphing device to draw the graph of $m(t)$ in Figure 2 together with the horizontal line $m = 30$. These curves intersect when $t \approx 2800$, and this agrees with the answer to part (c).

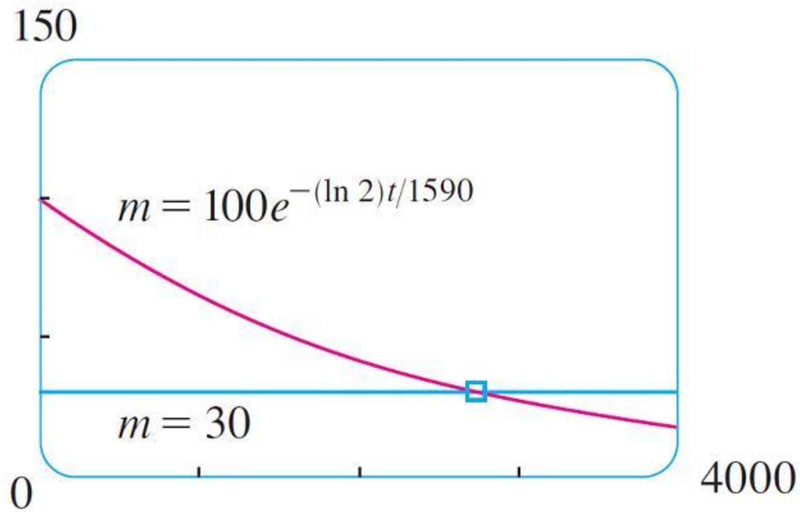


Figure 2

Newton’s Law of Cooling

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. (This law also applies to warming.)

If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant.

Newton's Law of Cooling

This equation is not quite the same as Equation 1, so we make the change of variable $y(t) = T(t) - T_s$. Because T_s is constant, we have $y'(t) = T'(t)$ and so the equation becomes

$$\frac{dy}{dt} = ky$$

We can then use $\boxed{2}$ to find an expression for y, from which we can find T.

Example 3

A bottle of soda pop at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F. After half an hour the soda pop has cooled to 61°F.

(a) What is the temperature of the soda pop after another half hour?

(b) How long does it take for the soda pop to cool to 50°F?

Solution:

(a) Let $T(t)$ be the temperature of the soda after t minutes.

Example 3 – *Solution*

cont'd

The surrounding temperature is $T_s = 44°F$, so Newton's Law of Cooling states that

$$\frac{dT}{dt} = k(T - 44)$$

If we let $y = T - 44$, then $y(0) = T(0) - 44 = 72 - 44 = 28$, so y satisfies

$$\frac{dy}{dt} = ky \qquad y(0) = 28$$

and by $\boxed{2}$ we have

$$y(t) = y(0)e^{kt} = 28e^{kt}$$

Example 3 – *Solution*

cont'd

We are given that $T(30) = 61$, so $y(30) = 61 – 44 = 17$ and

$$28e^{30k} = 17 \qquad e^{30k} = \frac{17}{28}$$

Taking logarithms, we have

$$k = \frac{\ln\left(\frac{17}{28}\right)}{30}$$

$$\approx -0.01663$$

Example 3 – *Solution*

cont'd

Thus

$$y(t) = 28e^{-0.01663t}$$

$$T(t) = 44 + 28e^{-0.01663t}$$

$$T(60) = 44 + 28e^{-0.01663(60)}$$

$$\approx 54.3$$

So after another half hour the pop has cooled to about 54° F.

Example 3 – *Solution*

cont'd

(b) We have $T(t) = 50$ when

$$44 + 28e^{-0.01663t} = 50$$

$$e^{-0.01663t} = \tfrac{6}{28}$$

$$t = \frac{\ln\left(\frac{6}{28}\right)}{-0.01663}$$

$$\approx 92.6$$

The pop cools to 50°F after about 1 hour 33 minutes.

Newton's Law of Cooling

Notice that in Example 3, we have

$$\lim_{t \to \infty} T(t) = \lim_{t \to \infty} (44 + 28e^{-0.01663t}) = 44 + 28 \cdot 0 = 44$$

which is to be expected. The graph of the temperature function is shown in Figure 3.

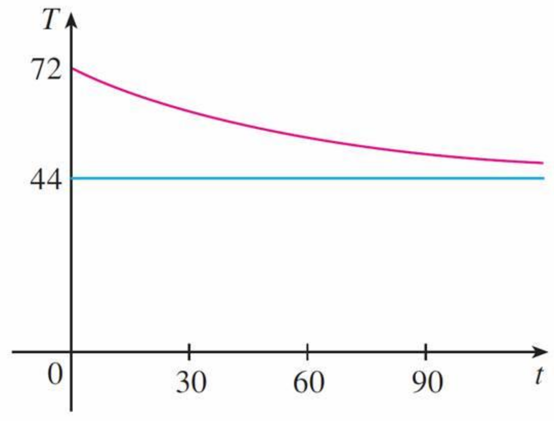


Figure 3

Continuously Compounded Interest

Example 4

If \$1000 is invested at 6% interest, compounded annually, then after 1 year the investment is worth \$1000(1.06) = \$1060, after 2 years it's worth \$[1000(1.06)]1.06 = \$1123.60, and after t years it's worth \$1000$(1.06)^t$.

In general, if an amount A_0 is invested at an interest rate r (r = 0.06 in this example), then after t years it's worth $A_0(1 + r)^t$.

Usually, however, interest is compounded more frequently, say, n times a year.

Example 4

cont'd

Then in each compounding period the interest rate is r/n and there are nt compounding periods in t years, so the value of the investment is

$$A_0\left(1 + \frac{r}{n}\right)^{nt}$$

For instance, after 3 years at 6% interest a \$1000 investment will be worth

$\$1000(1.06)^3 = \$1191.02$ with annual compounding

$\$1000(1.03)^6 = \$1194.05$ with semiannual compounding

Example 4

cont'd

$\$1000(1.015)^{12} = \$1195.62$ with quarterly compounding

$\$1000(1.005)^{36} = \$1196.68$ with monthly compounding

$\$1000\left(1 + \frac{0.06}{365}\right)^{365 \cdot 3} = \$1197.20$ with daily compounding

You can see that the interest paid increases as the number of compounding periods (n) increases. If we let $n \to \infty$, then we will be compounding the interest **continuously** and the value of the investment will be

$$A(t) = \lim_{n \to \infty} A_0\left(1 + \frac{r}{n}\right)^{nt}$$

Example 4

cont'd

$$= \lim_{n \to \infty} A_0 \left[\left(1 + \frac{r}{n} \right)^{n/r} \right]^{rt}$$

$$= A_0 \left[\lim_{n \to \infty} \left(1 + \frac{r}{n} \right)^{n/r} \right]^{rt}$$

$$= A_0 \left[\lim_{m \to \infty} \left(1 + \frac{1}{m} \right)^{m} \right]^{rt}$$

(where $m = n/r$)

But the limit in this expression is equal to the number *e*.

Example 4

cont'd

So with continuous compounding of interest at interest rate r, the amount after t years is

$$A(t) = A_0 e^{rt}$$

If we differentiate this equation, we get

$$\frac{dA}{dt} = rA_0 e^{rt} = rA(t)$$

which says that, with continuous compounding of interest, the rate of increase of an investment is proportional to its size.

Example 4

cont'd

Returning to the example of \$1000 invested for 3 years at 6% interest, we see that with continuous compounding of interest the value of the investment will be

$$A(3) = \$1000e^{(0.06)3} = \$1197.22$$

Notice how close this is to the amount we calculated for daily compounding, \$1197.20. But the amount is easier to compute if we use continuous compounding.

3 Derivatives

3.9 Related Rates

Related Rates

If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other.

But it is much easier to measure directly the rate of increase of the volume than the rate of increase of the radius.

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured).

The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Example 1

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Solution:

We start by identifying two things:

the *given information*:

the rate of increase of the volume of air is 100 cm^3/s

and the *unknown*:

the rate of increase of the radius when the diameter is 50 cm

Example 1 – *Solution*

cont'd

In order to express these quantities mathematically, we introduce some suggestive *notation*:

Let V be the volume of the balloon and let r be its radius.

The key thing to remember is that rates of change are derivatives. In this problem, the volume and the radius are both functions of the time t.

The rate of increase of the volume with respect to time is the derivative dV/dt, and the rate of increase of the radius is dr/dt.

Example 1 – *Solution*

cont'd

We can therefore restate the given and the unknown as follows:

Given: $\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$

Unknown: $\frac{dr}{dt}$ when $r = 25$ cm

In order to connect dV/dt and dr/dt, we first relate V and r by the formula for the volume of a sphere:

$$V = \tfrac{4}{3}\pi r^3$$

Example 1 – *Solution*

cont'd

In order to use the given information, we differentiate each side of this equation with respect to *t*. To differentiate the right side, we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now we solve for the unknown quantity:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

Example 1 – *Solution*

cont'd

If we put r = 25 and dV/dt = 100 in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}100$$

$$= \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s.

3

Derivatives

3.9

Linear Approximations and Differentials

Linear Approximations and Differentials

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line.

This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f.

Linear Approximations and Differentials

So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. (See Figure 1.)

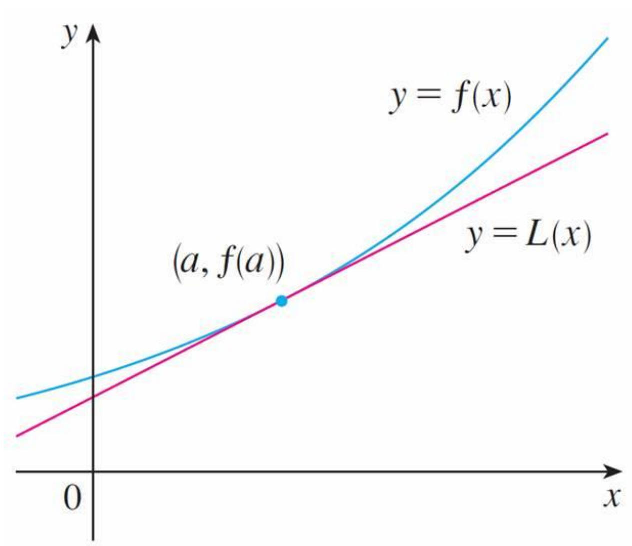


Figure 1

Linear Approximations and Differentials

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$\boxed{1} \qquad f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a.

Linear Approximations and Differentials

The linear function whose graph is this tangent line, that is,

$$\boxed{2} \qquad L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of *f* at *a*.

Example 1

Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

Solution:

The derivative of $f(x) = (x + 3)^{1/2}$ is

$$f'(x) = \tfrac{1}{2}(x+3)^{-1/2}$$

$$= \frac{1}{2\sqrt{x+3}}$$

and so we have $f(1) = 2$ and $f'(1) = \tfrac{1}{4}$.

Example 1 – *Solution*

cont'd

Putting these values into Equation 2, we see that the linearization is

$$L(x) = f(1) + f'(1)(x - 1)$$

$$= 2 + \tfrac{1}{4}(x - 1)$$

$$= \frac{7}{4} + \frac{x}{4}$$

The corresponding linear approximation $\boxed{1}$ is

$$\sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4} \qquad \text{(when } x \text{ is near 1)}$$

Example 1 – *Solution*

cont'd

In particular, we have

$$\sqrt{3.98} \approx \tfrac{7}{4} + \tfrac{0.98}{4}$$

$$= 1.995$$

and

$$\sqrt{4.05} \approx \tfrac{7}{4} + \tfrac{1.05}{4}$$

$$= 2.0125$$

Example 1 – *Solution*

cont'd

The linear approximation is illustrated in Figure 2.

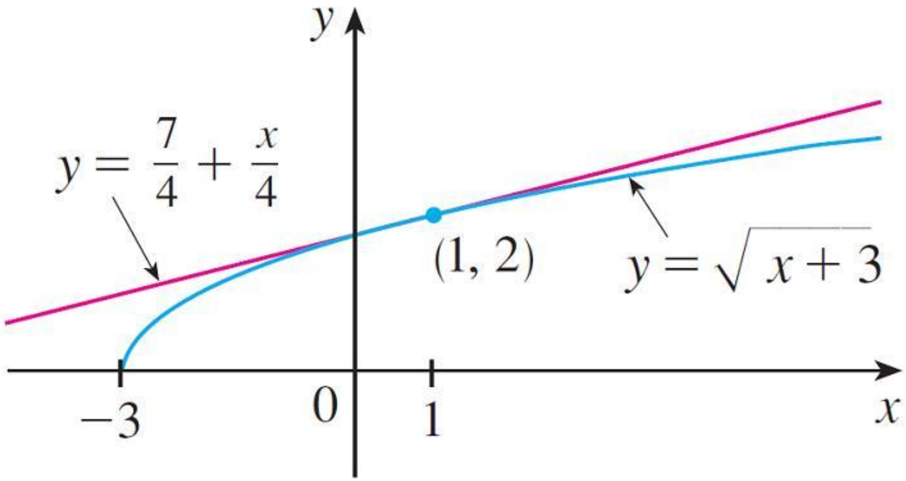


Figure 2

Example 1 – *Solution*

cont'd

We see that, indeed, the tangent line approximation is a good approximation to the given function when x is near 1.

We also see that our approximations are overestimates because the tangent line lies above the curve.

Of course, a calculator could give us approximations for $\sqrt{3.98}$ and $\sqrt{4.05}$, but the linear approximation gives an approximation *over an entire interval.*

Linear Approximations and Differentials

In the following table we compare the estimates from the linear approximation in Example 1 with the true values.

	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176 . . .
$\sqrt{3.98}$	0.98	1.995	1.99499373 . . .
$\sqrt{4}$	1	2	2.00000000 . . .
$\sqrt{4.05}$	1.05	2.0125	2.01246117 . . .
$\sqrt{4.1}$	1.1	2.025	2.02484567 . . .
$\sqrt{5}$	2	2.25	2.23606797 . . .
$\sqrt{6}$	3	2.5	2.44948974 . . .

Linear Approximations and Differentials

Notice from this table, and also from Figure 2, that the tangent line approximation gives good estimates when *x* is close to 1 but the accuracy of the approximation deteriorates when *x* is farther away from 1.

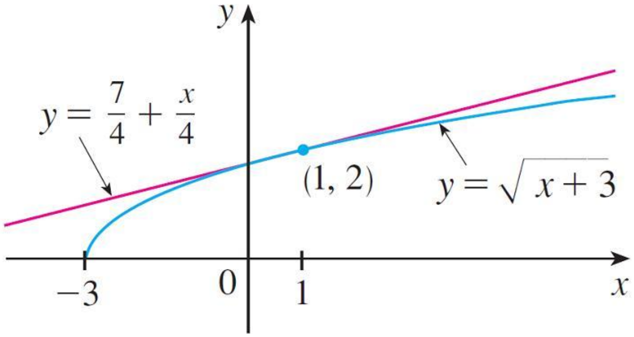


Figure 2

Linear Approximations and Differentials

The next example shows that by using a graphing calculator or computer we can determine an interval throughout which a linear approximation provides a specified accuracy.

Example 2

For what values of *x* is the linear approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

accurate to within 0.5? What about accuracy to within 0.1?

Solution:

Accuracy to within 0.5 means that the functions should differ by less than 0.5:

$$\left| \sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4} \right) \right| < 0.5$$

Example 2 – *Solution*

cont'd

Equivalently, we could write

$$\sqrt{x + 3} - 0.5 < \frac{7}{4} + \frac{x}{4} < \sqrt{x + 3} + 0.5$$

This says that the linear approximation should lie between the curves obtained by shifting the curve $y = \sqrt{x + 3}$ upward and downward by an amount 0.5.

Example 2 – *Solution*

cont'd

Figure 3 shows the tangent line $y = (7 + x)/4$ intersecting the upper curve $y = \sqrt{x+3} + 0.5$ at P and Q.

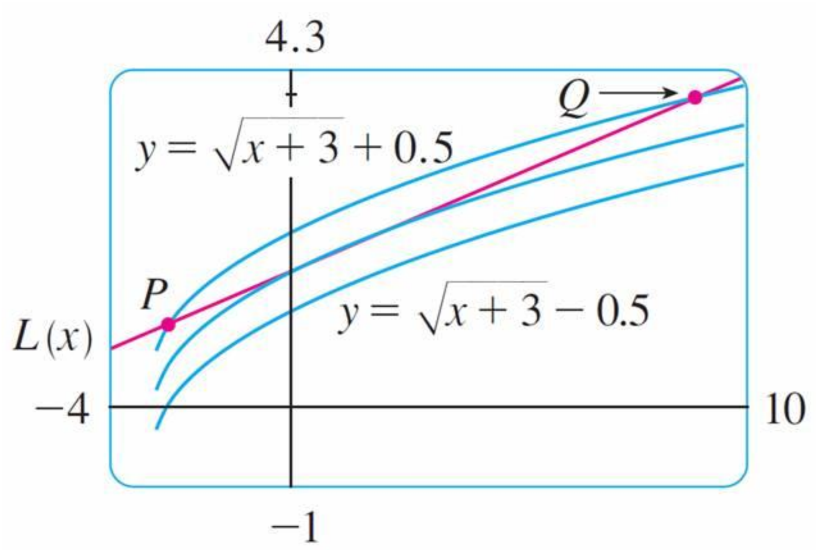


Figure 3

Example 2 – *Solution*

cont'd

Zooming in and using the cursor, we estimate that the x-coordinate of P is about –2.66 and the x-coordinate of Q is about 8.66.

Thus we see from the graph that the approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

is accurate to within 0.5 when $-2.6 < x < 8.6$. (We have rounded to be safe.)

Example 2 – *Solution*

cont'd

Similarly, from Figure 4 we see that the approximation is accurate to within 0.1 when $-1.1 < x < 3.9$.

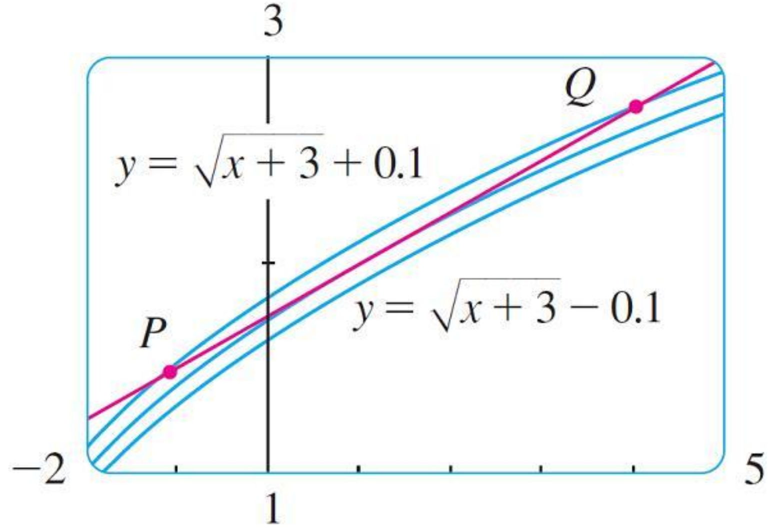


Figure 4

Applications to Physics

Applications to Physics

Linear approximations are often used in physics. In analyzing the consequences of an equation, a physicist sometimes needs to simplify a function by replacing it with its linear approximation.

For instance, in deriving a formula for the period of a pendulum, physics textbooks obtain the expression $a_T = -g \sin \theta$ for tangential acceleration and then replace θ by θ with the remark that $\sin \theta$ is very close to θ if θ is not too large.

Applications to Physics

You can verify that the linearization of the function $f(x) = \sin x$ at $a = 0$ is $L(x) = x$ and so the linear approximation at 0 is

$$\sin x \approx x$$

So, in effect, the derivation of the formula for the period of a pendulum uses the tangent line approximation for the sine function.

Applications to Physics

Another example occurs in the theory of optics, where light rays that arrive at shallow angles relative to the optical axis are called *paraxial rays.*

In paraxial (or Gaussian) optics, both $\sin\theta$ and $\cos\theta$ are replaced by their linearizations. In other words, the linear approximations

$$\sin\theta \approx \theta \quad \text{and} \quad \cos\theta \approx 1$$

are used because θ is close to 0.

Differentials

Differentials

The ideas behind linear approximations are sometimes formulated in the terminology and notation of *differentials.*

If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number.

The **differential** dy is then defined in terms of dx by the equation

$$dy = f'(x)\,dx \tag{3}$$

Differentials

So *dy* is a dependent variable; it depends on the values of *x* and *dx*.

If *dx* is given a specific value and *x* is taken to be some specific number in the domain of *f*, then the numerical value of *dy* is determined.

Differentials

The geometric meaning of differentials is shown in Figure 5.

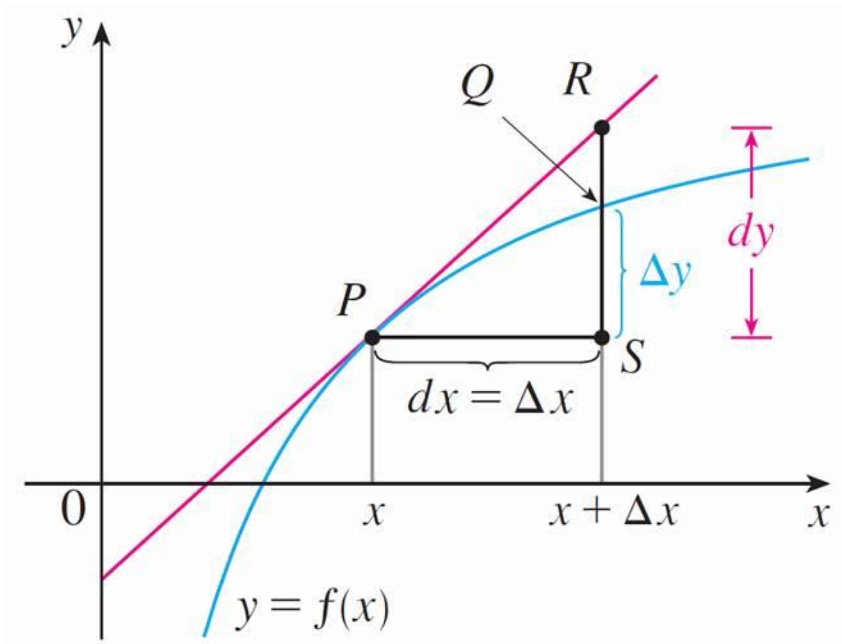


Figure 5

Differentials

Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x)$$

The slope of the tangent line PR is the derivative $f'(x)$. Thus the directed distance from S to R is $f'(x)\, dx = dy$.

Therefore dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.

Example 3

Compare the values of Δy and dy if
$y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

Solution:

(a) We have

$$f(2) = 2^3 + 2^2 - 2(2) + 1$$

$$= 9$$

$$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1$$

$$= 9.717625$$

Example 3 – *Solution*

cont'd

$$\Delta y = f(2.05) - f(2)$$

$$= 0.717625$$

In general,

$$dy = f'(x)\,dx$$

$$= (3x^2 + 2x - 2)\,dx$$

When $x = 2$ and $dx = \Delta x = 0.05$, this becomes

$$dy = [3(2)^2 + 2(2) - 2]0.05$$

$$= 0.7$$

Example 3 – *Solution*

cont'd

(b) $f(2.01) = (2.01)^3 + (2.01)^2 - 2(2.01) + 1$

$= 9.140701$

$\Delta y = f(2.01) - f(2)$

$= 0.140701$

When $dx = \Delta x = 0.01$,

$dy = [3(2)^2 + 2(2) - 2]0.01$

$= 0.14$

Differentials

Our final example illustrates the use of differentials in estimating the errors that occur because of approximate measurements.

Example 4

The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

Solution:

If the radius of the sphere is r, then its volume is $V = \frac{4}{3}\pi r^3$. If the error in the measured value of r is denoted by $dr = \Delta r$, then the corresponding error in the calculated value of V is ΔV, which can be approximated by the differential

$$dV = 4\pi r^2\, dr$$

Example 4 – *Solution*

cont'd

When $r = 21$ and $dr = 0.05$, this becomes

$$dV = 4\pi(21)^2 0.05$$

$$\approx 277$$

The maximum error in the calculated volume is about 277 cm^3.

Differentials

Note:

Although the possible error in Example 4 may appear to be rather large, a better picture of the error is given by the **relative error**, which is computed by dividing the error by the total volume:

$$\frac{\Delta V}{V} \approx \frac{dV}{V}$$

$$= \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3}$$

$$= 3\,\frac{dr}{r}$$

Differentials

Thus the relative error in the volume is about three times the relative error in the radius.

In Example 4 the relative error in the radius is approximately $dr/r = 0.05/21 \approx 0.0024$ and it produces a relative error of about 0.007 in the volume.

The errors could also be expressed as **percentage errors** of 0.24% in the radius and 0.7% in the volume.

3 Differentiation Rules

3.11 Hyperbolic Functions

Hyperbolic Functions

Certain even and odd combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names.

In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle.

Hyperbolic Functions

For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine**, **hyperbolic cosine**, and so on.

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

Hyperbolic Functions

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities.

We list some of them here and leave most of the proofs to the exercises.

Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

Example 1

Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and

(b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

Solution:

(a) $\cosh^2 x - \sinh^2 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{e^x - e^{-x}}{2}\right)^2$

$= \dfrac{e^{2x} + 2 + e^{-2x}}{4} - \dfrac{e^{2x} - 2 + e^{-2x}}{4}$

$= \dfrac{4}{4}$

$= 1$

Example 1 – *Solution*

cont'd

(b) We start with the identity proved in part (a):

$$\cosh^2 x - \sinh^2 x = 1$$

If we divide both sides by $\cosh^2 x$, we get

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

or

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

Hyperbolic Functions

The derivatives of the hyperbolic functions are easily computed. For example,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{e^x + e^{-x}}{2}$$

$$= \cosh x$$

Hyperbolic Functions

We list the differentiation formulas for the hyperbolic functions as Table 1.

1 Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Example 2

Any of these differentiation rules can be combined with the Chain Rule. For instance,

$$\frac{d}{dx}\left(\cosh \sqrt{x}\right) = \sinh \sqrt{x} \cdot \frac{d}{dx}\sqrt{x}$$

$$= \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

Inverse Hyperbolic Functions

Inverse Hyperbolic Functions

The sinh and tanh are one-to-one functions and so they have inverse functions denoted by $\sinh^{-1}$ and $\tanh^{-1}$. The cosh is not one-to-one, but when restricted to the domain $[0, \infty)$ it becomes one-to-one.

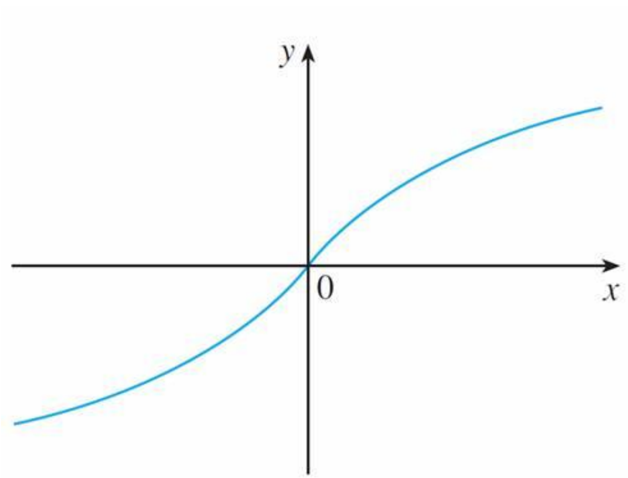
The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

2

$$
\begin{aligned}
y = \sinh^{-1}x &\iff \sinh y = x \\
y = \cosh^{-1}x &\iff \cosh y = x \quad \text{and} \quad y \geqslant 0 \\
y = \tanh^{-1}x &\iff \tanh y = x
\end{aligned}
$$

Inverse Hyperbolic Functions

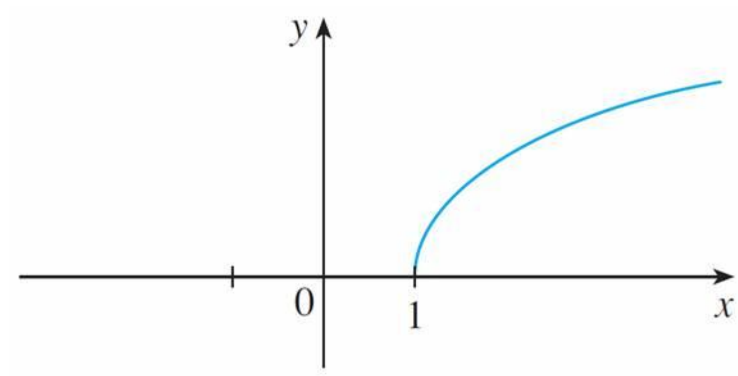
We can sketch the graphs of $\sinh^{-1}$, $\cosh^{-1}$, and $\tanh^{-1}$ in Figures 8, 9, and 10.



$y = \sinh^{-1} x$

domain = $\mathbb{R}$ range = $\mathbb{R}$

Figure 8

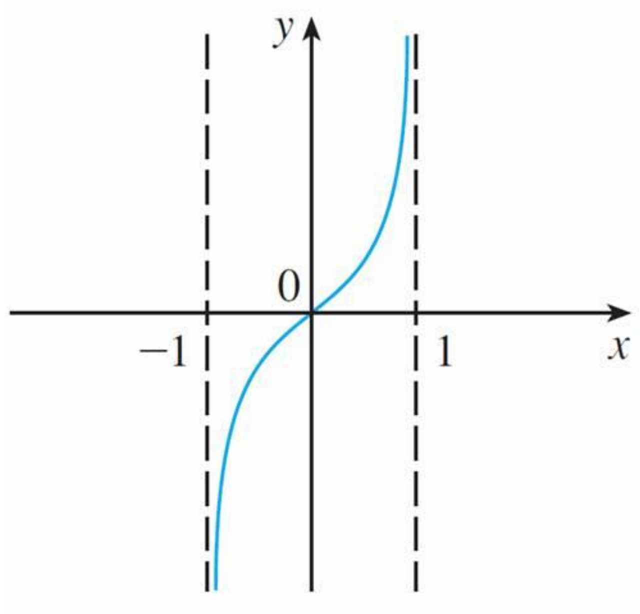


$y = \cosh^{-1} x$

domain = $[1, \infty)$ range = $[0, \infty)$

Figure 9

Inverse Hyperbolic Functions



$y = \tanh^{-1} x$

domain = (–1, 1)   range = $\mathbb{R}$

Figure 10

Inverse Hyperbolic Functions

Since the hyperbolic functions are defined in terms of exponential functions, it's not surprising to learn that the inverse hyperbolic functions can be expressed in terms of logarithms.

In particular, we have:

$$\boxed{3} \qquad \sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad x \in \mathbb{R}$$

$$\boxed{4} \qquad \cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geqslant 1$$

$$\boxed{5} \qquad \tanh^{-1}x = \tfrac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \qquad -1 < x < 1$$

Example 3

Show that $\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

Solution:

Let $y = \sinh^{-1}x$. Then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

so $$e^y - 2x - e^{-y} = 0$$

or, multiplying by e^y,

$$e^{2y} - 2xe^y - 1 = 0$$

Example 3 – *Solution*

cont'd

This is really a quadratic equation in e^y:

$$(e^y)^2 - 2x(e^y) - 1 = 0$$

Solving by the quadratic formula, we get

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$= x \pm \sqrt{x^2 + 1}$$

Note that $e^y > 0$, but $x - \sqrt{x^2 + 1} < 0 \left(\text{because } x < \sqrt{x^2 + 1}\right)$.

Example 3 – *Solution*

cont'd

Thus the minus sign is inadmissible and we have

$$e^y = x + \sqrt{x^2 + 1}$$

Therefore

$$y = \ln(e^y)$$

$$= \ln\left(x + \sqrt{x^2 + 1}\right)$$

Inverse Hyperbolic Functions

6 Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}$$

The inverse hyperbolic functions are all differentiable because the hyperbolic functions are differentiable.

Example 4

Prove that $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{1 + x^2}}$.

Solution:

Let $y = \sinh^{-1}x$. Then $\sinh y = x$. If we differentiate this equation implicitly with respect to x, we get

$$\cosh y \frac{dy}{dx} = 1$$

Example 4 – *Solution*

cont'd

Since $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y \geq 0$, we have $\cosh y = \sqrt{1 + \sinh^2 y}$, so

$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

$$= \frac{1}{\sqrt{1 + \sinh^2 y}}$$

$$= \frac{1}{\sqrt{1 + x^2}}$$