5 Integrals

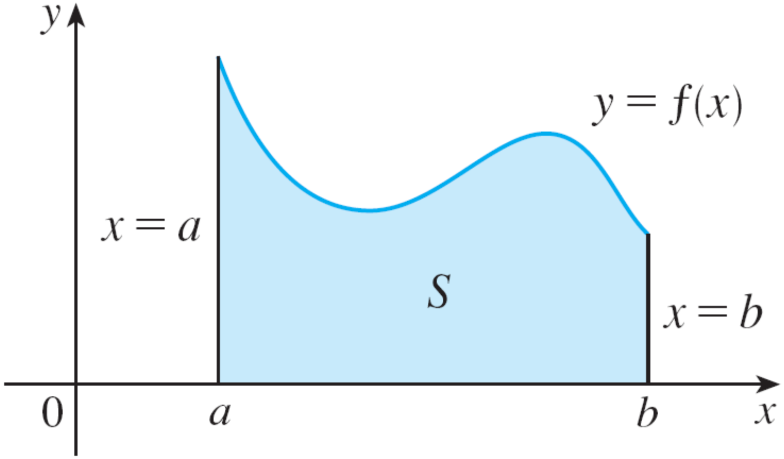
5.1 Areas and Distances

The Area Problem

The Area Problems

We begin by attempting to solve the *area problem*: Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

This means that S, illustrated in Figure 1, is bounded by the graph of a continuous function f [where $f(x) \geq 0$], the vertical lines $x = a$ and $x = b$, and the x-axis.



$S = \{(x, y) \mid a \leq x \leq b,\ 0 \leq y \leq f(x)\}$

Figure 1

The Area Problems

For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height.

The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.

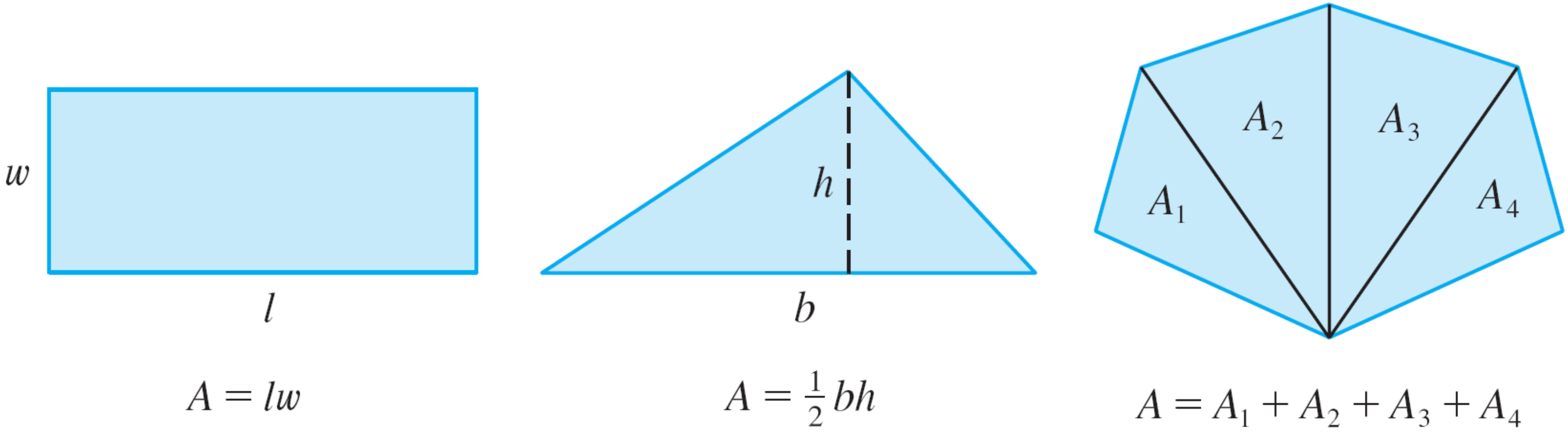


Figure 2

The Area Problems

However, it isn’t so easy to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations.

We pursue a similar idea for areas. We first approximate the region S by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles.

Example 1

Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure 3).

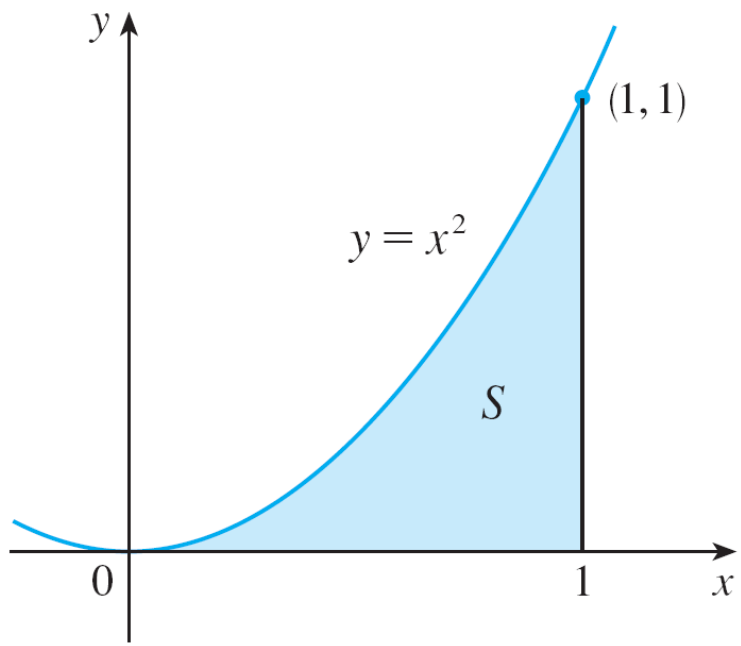


Figure 3

Example 1 – *Solution*

We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that.

Suppose we divide S into four strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$ as in Figure 4(a).

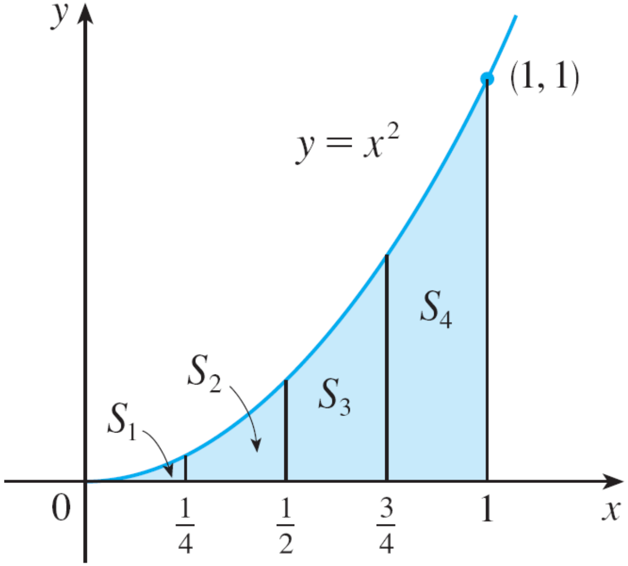


Figure 4(a)

Example 1 – *Solution*

cont'd

We can approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip [see Figure 4(b)].

In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the *right* endpoints of the subintervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$, and $\left[\frac{3}{4}, 1\right]$.

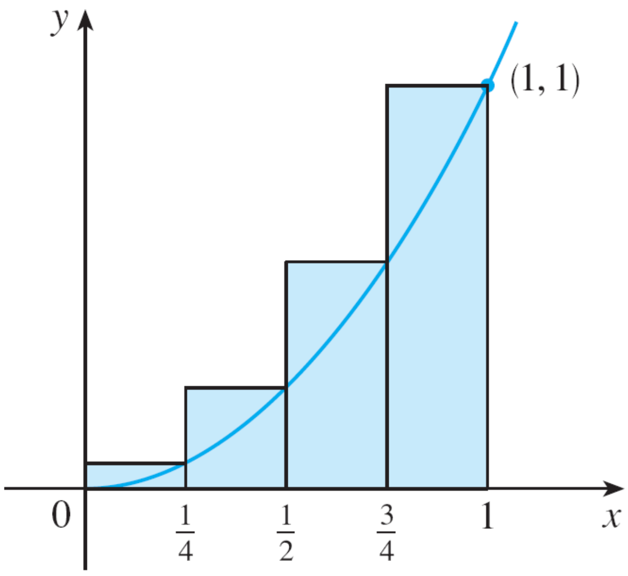


Figure 4(b)

Each rectangle has width $\frac{1}{4}$ and the heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, and 1^2.

Example 1 – *Solution*

cont'd

If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot 1^2$$

$$= \frac{15}{32}$$

$$= 0.46875$$

From Figure 4(b) we see that the area A of S is less than R_4, so

$$A < 0.46875$$

Example 1 – *Solution*

cont'd

Instead of using the rectangles in Figure 4(b) we could use the smaller rectangles in Figure 5 whose heights are the values of *f* at the *left* endpoints of the subintervals. (The leftmost rectangle has collapsed because its height is 0.)

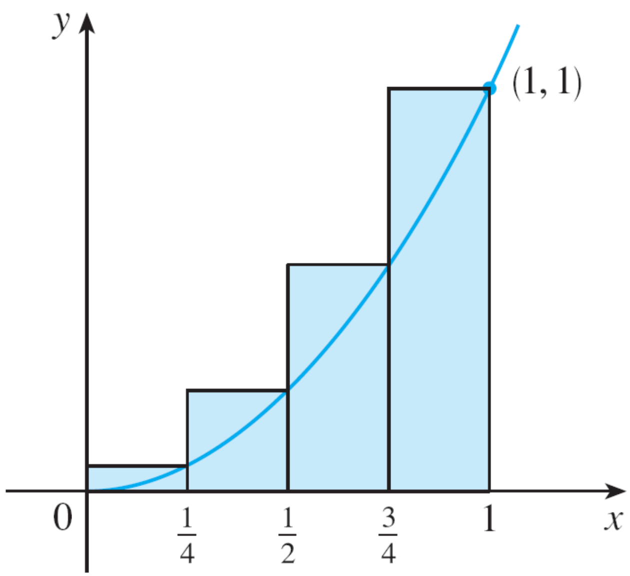


Figure 4(b)

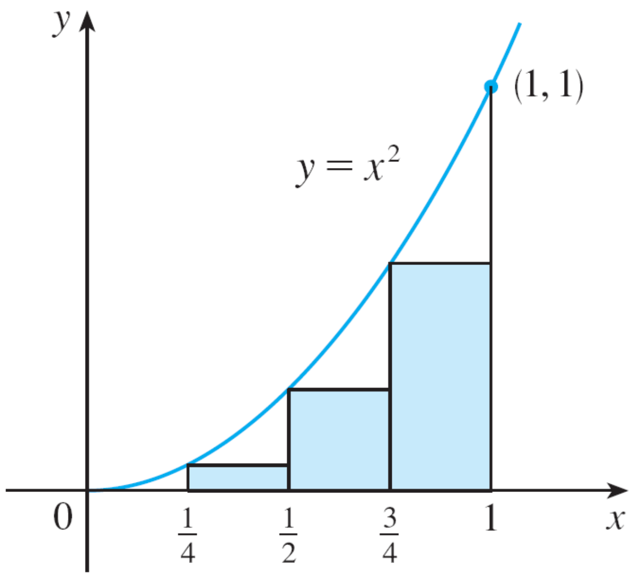


Figure 5

Example 1 – *Solution*

cont'd

The sum of the areas of these approximating rectangles is

$$L_4 = \tfrac{1}{4} \cdot 0^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4} \cdot \left(\tfrac{3}{4}\right)^2$$

$$= \frac{7}{32}$$

$$= 0.21875$$

We see that the area of *S* is larger than L_4, so we have lower and upper estimates for *A*:

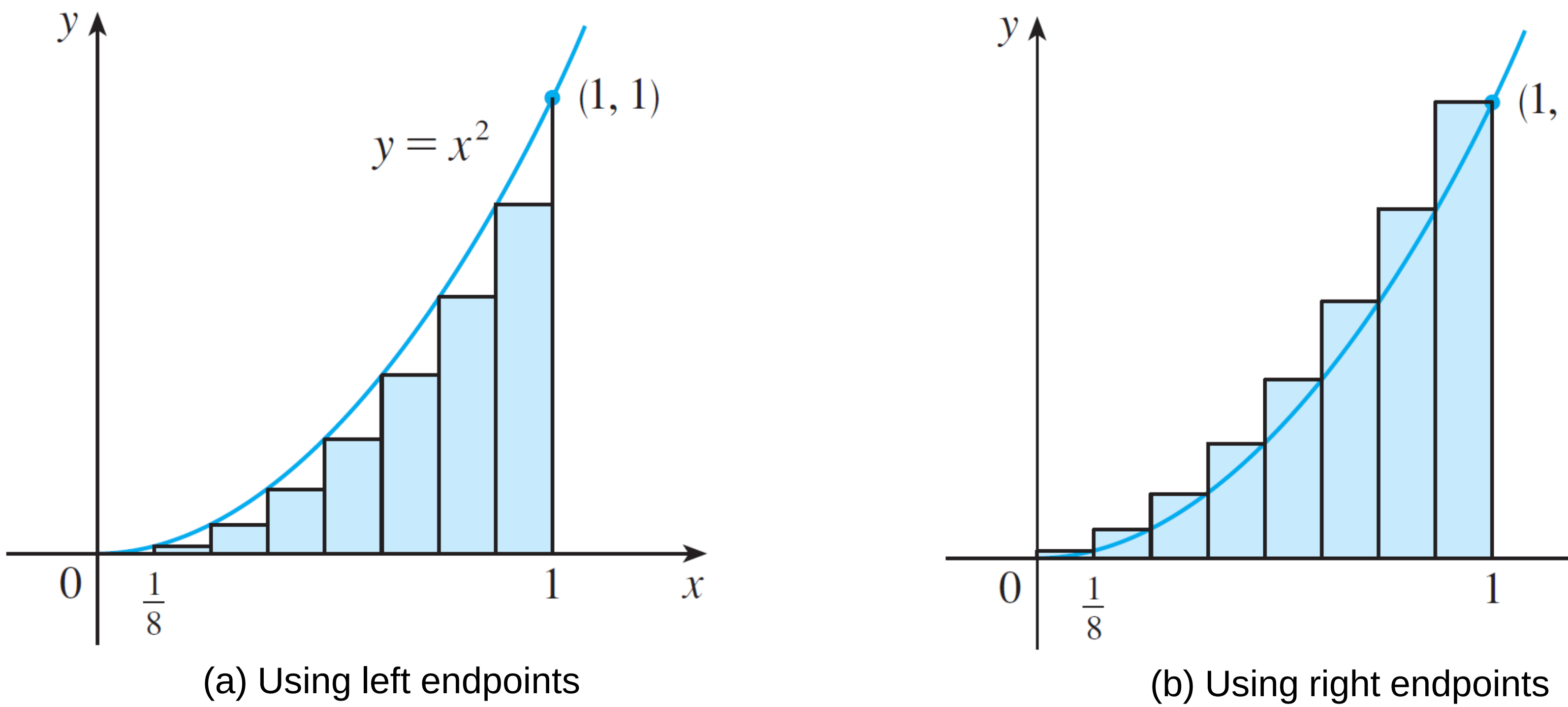
$$0.21875 < A < 0.46875$$

We can repeat this procedure with a larger number of strips.

Example 1 – *Solution*

cont'd

Figure 6 shows what happens when we divide the region *S* into eight strips of equal width.



Approximating *S* with eight rectangles

Figure 6

Example 1 – *Solution*

cont'd

By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A:

$$0.2734375 < A < 0.3984375$$

So one possible answer to the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375.

We could obtain better estimates by increasing the number of strips.

Example 1 – *Solution*

cont'd

The table at the right shows the results of similar calculations (with a computer) using n rectangles whose heights are found with left endpoints (L_n) or right endpoints (R_n).

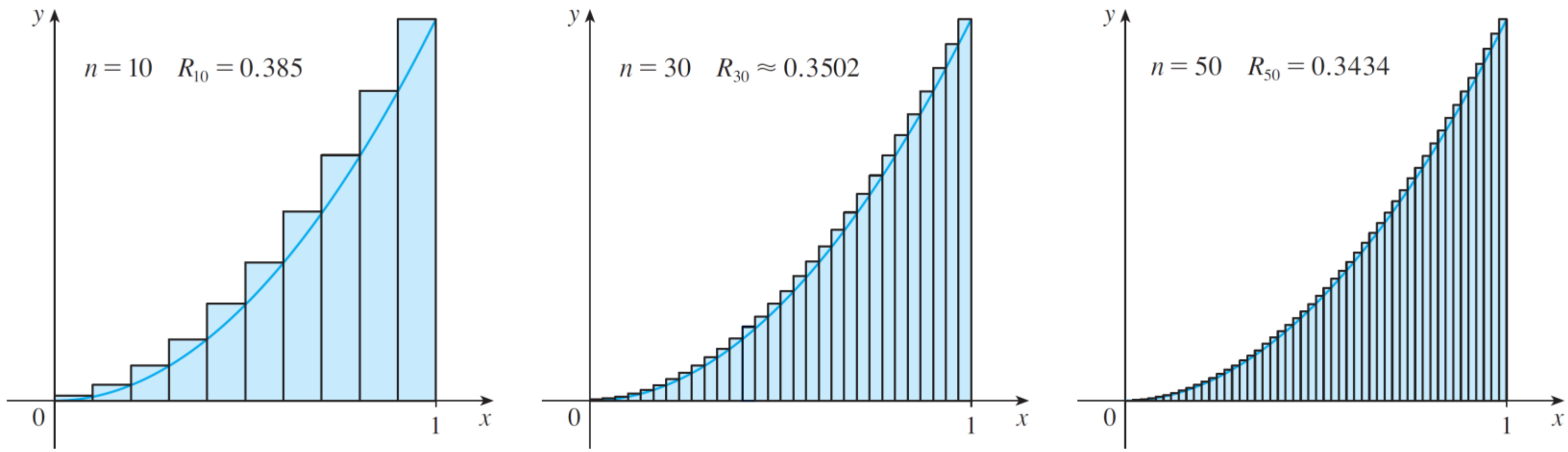
n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

In particular, we see by using 50 strips that the area lies between 0.3234 and 0.3434. With 1000 strips we narrow it down even more: A lies between 0.3328335 and 0.3338335.

A good estimate is obtained by averaging these numbers: $A \approx 0.3333335$.

The Area Problems

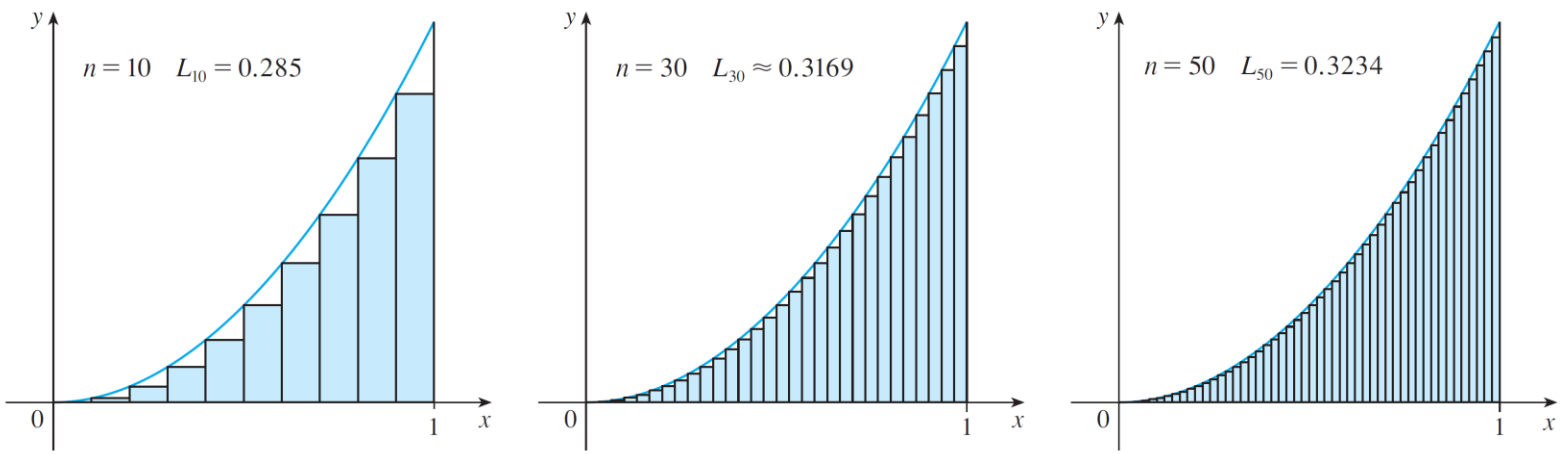
From Figures 8 and 9 it appears that, as n increases, both L_n and R_n become better and better approximations to the area of S.



Right endpoints produce upper sums because $f(x) = x^2$ is increasing

Figure 8

The Area Problems



Left endpoints produce upper sums because $f(x) = x^2$ is increasing

Figure 9

Therefore we *define* the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \frac{1}{3}$$

The Area Problems

We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width as in Figure 10.

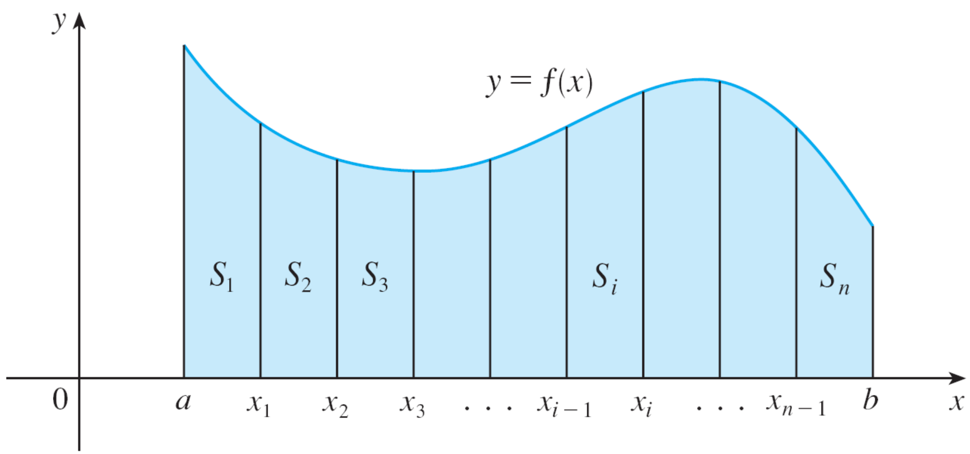


Figure 10

The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b - a}{n}$$

The Area Problems

These strips divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$.

The right endpoints of the subintervals are

$$x_1 = a + \Delta x,$$

$$x_2 = a + 2\,\Delta x,$$

$$x_3 = a + 3\,\Delta x,$$

$$\vdots$$

The Area Problems

Let's approximate the *i*th strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of *f* at the right endpoint (see Figure 11).

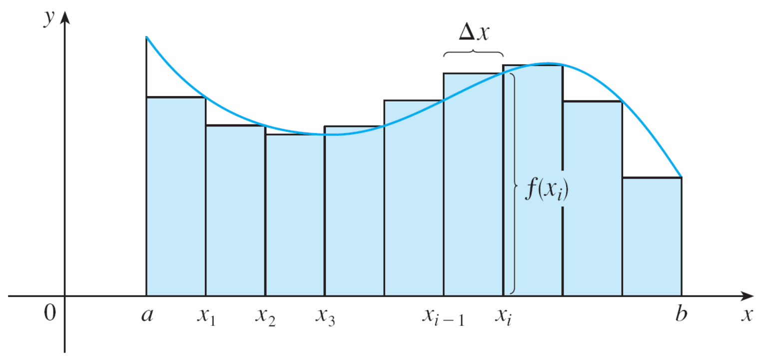


Figure 11

Then the area of the *i*th rectangle is $f(x_i)\,\Delta x$. What we think of intuitively as the area of *S* is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \ldots + f(x_n)\,\Delta x$$

The Area Problems

Figure 12 shows this approximation for n = 2, 4, 8, and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$.

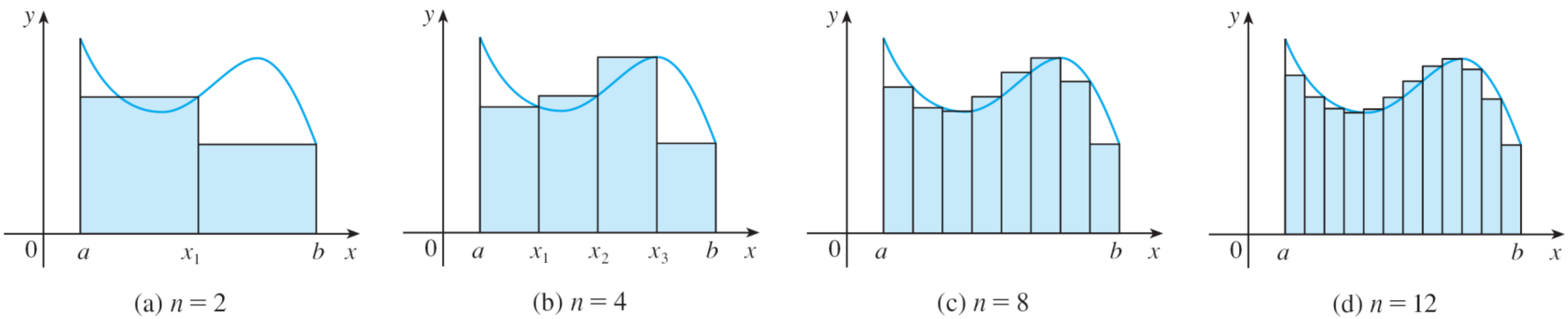


Figure 12

The Area Problems

Therefore we define the area *A* of the region *S* in the following way.

2 Definition The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

It can be proved that the limit in Definition 2 always exists, since we are assuming that *f* is continuous. It can also be shown that we get the same value if we use left endpoints:

$$\boxed{3} \qquad A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x]$$

The Area Problems

In fact, instead of using left endpoints or right endpoints, we could take the height of the *i*th rectangle to be the value of f at *any* number x_i* in the *i*th subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1*, x_2*, \ldots, x_n*$ the **sample points**.

Figure 13 shows approximating rectangles when the sample points are not chosen to be endpoints.

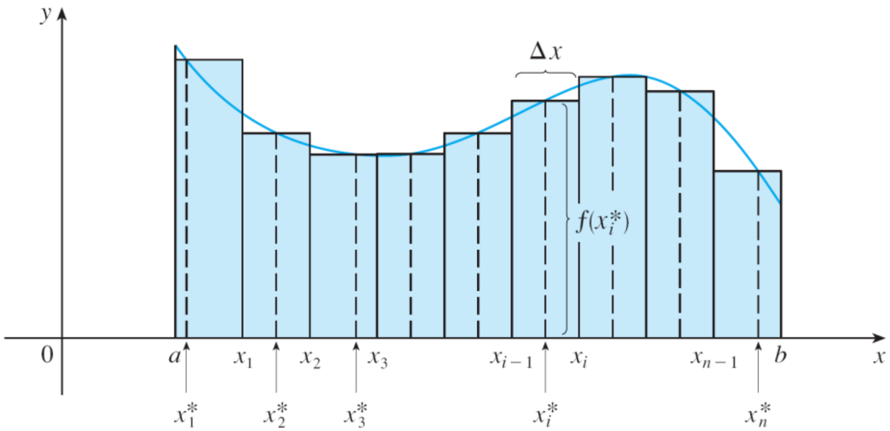


Figure 13

So a more general expression for the area of S is

$$\boxed{4} \qquad A = \lim_{n \to \infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

The Area Problems

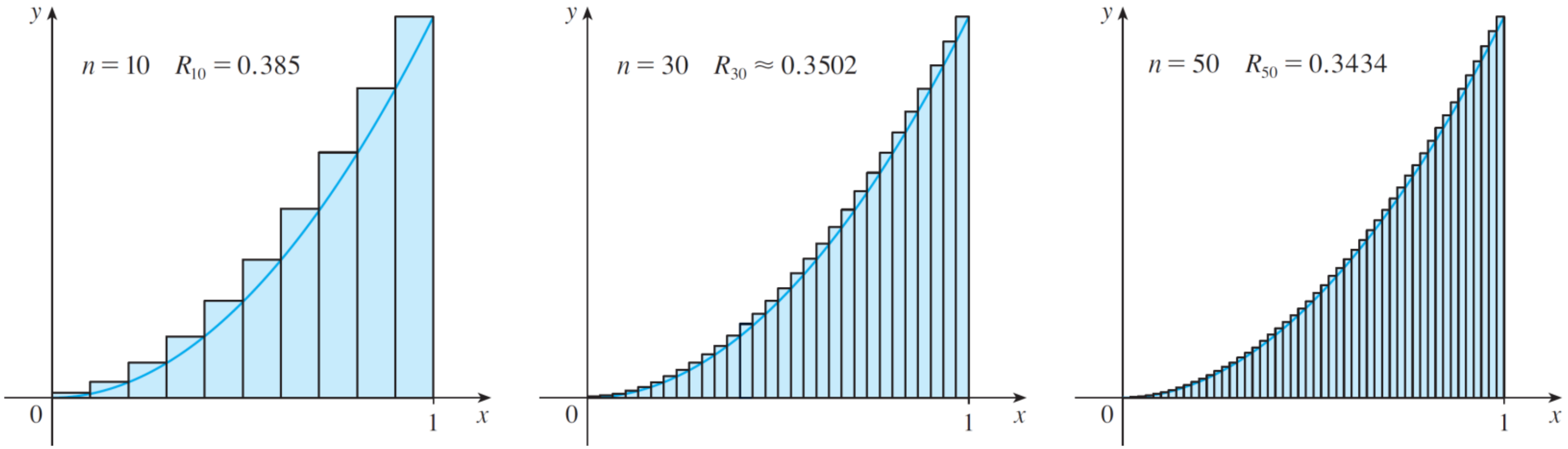
Note:
It can be shown that an equivalent definition of area is the following: *A is the unique number that is smaller than all the upper sums and bigger than all the lower sums*.

We saw in Example 1, for instance, that the area $\left(A = \frac{1}{3}\right)$ is trapped between all the left approximating sums L_n and all the right approximating sums R_n.

The function in those examples, $f(x) = x^2$, happens to be increasing on [0, 1] and so the lower sums arise from left endpoints and the upper sums from right endpoints.

The Area Problems

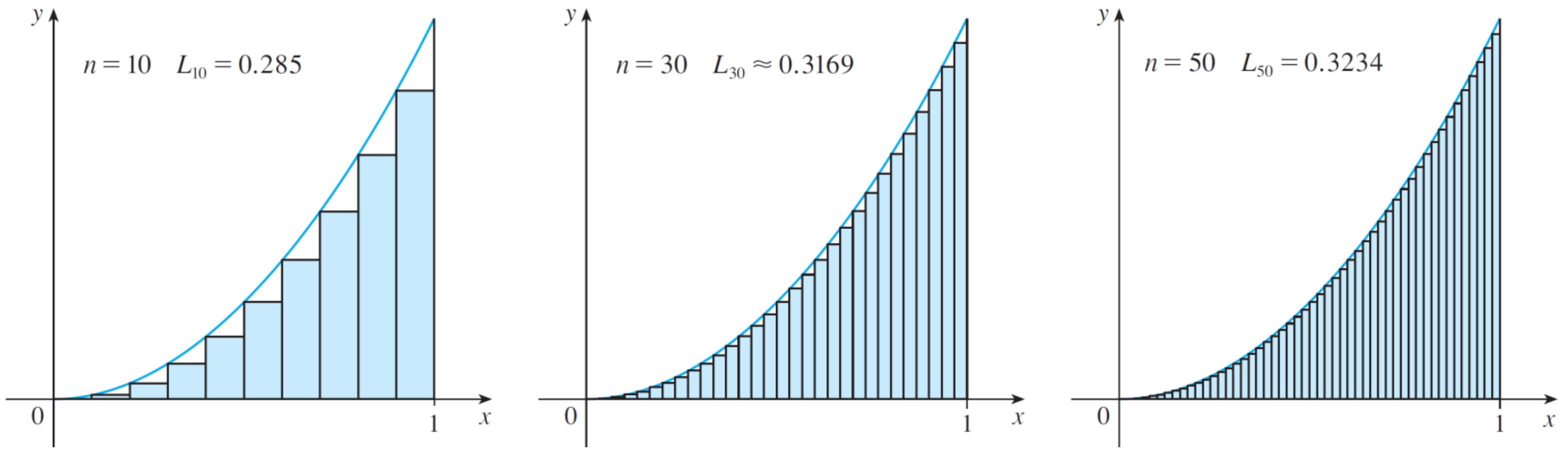
See Figures 8 and 9.



Right endpoints produce upper sums because $f(x) = x^2$ is increasing

Figure 8

The Area Problems

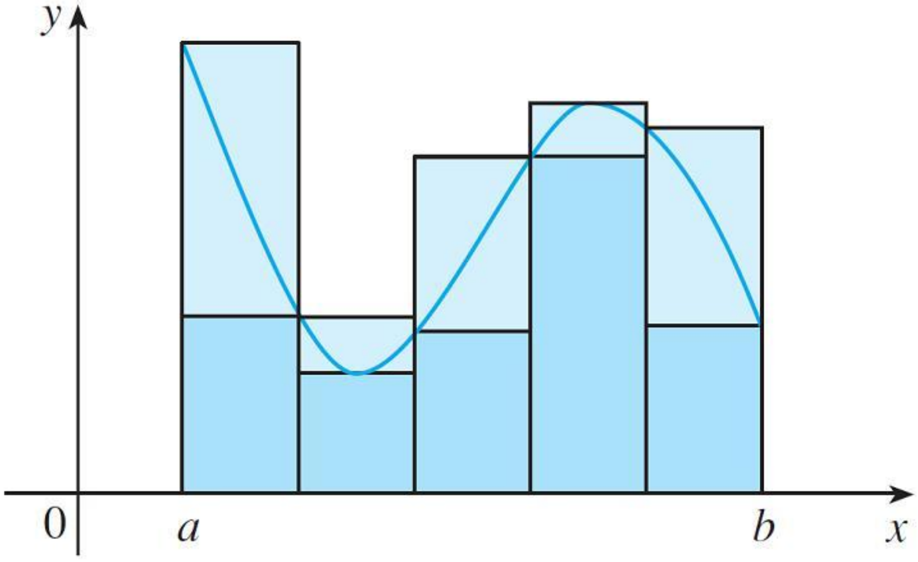


Left endpoints produce upper sums because $f(x) = x^2$ is increasing

Figure 9

The Area Problems

In general, we form **lower** (and **upper**) **sums** by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (and maximum) value of *f* on the *i*th subinterval. (See Figure 14)



Lower sums (short rectangles) and upper sums (tall rectangles)

Figure 14

The Area Problems

We often use **sigma notation** to write sums with many terms more compactly. For instance,

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

So the expressions for area in Equations 2, 3, and 4 can be written as follows:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

The Area Problems

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

We can also rewrite Formula 1 in the following way:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

The Distance Problem

The Distance Problems

Now let's consider the *distance problem:* Find the distance traveled by an object during a certain time period if the velocity of the object is known at all times.

If the velocity remains constant, then the distance problem is easy to solve by means of the formula

$$\text{distance} = \text{velocity} \times \text{time}$$

But if the velocity varies, it's not so easy to find the distance traveled.

Example 4

Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

In order to have the time and the velocity in consistent units, let's convert the velocity readings to feet per second (1 mi/h = 5280/3600 ft/s):

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	46	41

Example 4

cont'd

During the first five seconds the velocity doesn't change very much, so we can estimate the distance traveled during that time by assuming that the velocity is constant.

If we take the velocity during that time interval to be the initial velocity (25 ft/s), then we obtain the approximate distance traveled during the first five seconds:

$$25 \text{ ft/s} \times 5 \text{ s} = 125 \text{ ft}$$

Example 4

cont'd

Similarly, during the second time interval the velocity is approximately constant and we take it to be the velocity when $t = 5$ s.

So our estimate for the distance traveled from $t = 5$ s to $t = 10$ s is

$$31 \text{ ft/s} \times 5 \text{ s} = 155 \text{ ft}$$

If we add similar estimates for the other time intervals, we obtain an estimate for the total distance traveled:

$$(25 \times 5) + (31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (46 \times 5) = 1135 \text{ ft}$$

Example 4

cont'd

We could just as well have used the velocity at the *end* of each time period instead of the velocity at the beginning as our assumed constant velocity.

Then our estimate becomes

$$(31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (46 \times 5) + (41 \times 5) = 1215 \text{ ft}$$

If we had wanted a more accurate estimate, we could have taken velocity readings every two seconds, or even every second.

The Distance Problems

In general, suppose an object moves with velocity $v = f(t)$, where $a \leq t \leq b$ and $f(t) \geq 0$ (so the object always moves in the positive direction).

We take velocity readings at times t_0 (= a), t_1, t_2,. . ., t_n (= b) so that the velocity is approximately constant on each subinterval.

If these times are equally spaced, then the time between consecutive readings is $\Delta t = (b - a)/n$. During the first time interval the velocity is approximately $f(t_0)$ and so the distance traveled is approximately $f(t_0)\,\Delta t$.

The Distance Problems

Similarly, the distance traveled during the second time interval is about $f(t_1)\,\Delta t$ and the total distance traveled during the time interval [*a*, *b*] is approximately

$$f(t_0)\,\Delta t + f(t_1)\,\Delta t + \cdots + f(t_{n-1})\,\Delta t = \sum_{i=1}^{n} f(t_{i-1})\,\Delta t$$

If we use the velocity at right endpoints instead of left endpoints, our estimate for the total distance becomes

$$f(t_1)\,\Delta t + f(t_2)\,\Delta t + \cdots + f(t_n)\,\Delta t = \sum_{i=1}^{n} f(t_i)\,\Delta t$$

The Distance Problems

The more frequently we measure the velocity, the more accurate our estimates become, so it seems plausible that the *exact* distance *d* traveled is the *limit* of such expressions:

$$\boxed{5} \qquad d = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_{i-1})\,\Delta t = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i)\,\Delta t$$

5

Integrals

5.2 The Definite Integral

The Definite Integral

We have seen that a limit of the form

$$\boxed{1} \quad \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

arises when we compute an area.

We also have seen that it arises when we try to find the distance traveled by an object.

It turns out that this same type of limit occurs in a wide variety of situations even when *f* is not necessarily a positive function.

The Definite Integral

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 \ (= a), x_1, x_2, \ldots, x_n \ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of *f* from *a* to *b*** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Note 1: The symbol $\int$ was introduced by Leibniz and is called an **integral sign**.

It is an elongated *S* and was chosen because an integral is a limit of sums.

The Definite Integral

In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**.

For now, the symbol dx has no meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol.

The dx simply indicates that the independent variable is x. The procedure of calculating an integral is called **integration**.

The Definite Integral

Note 2: The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on *x.* In fact, we could use any letter in place of *x* without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

Note 3: The sum

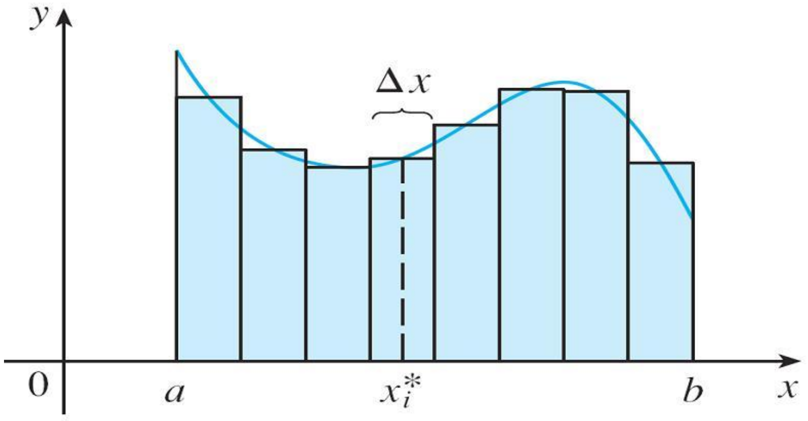
$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866).

The Definite Integral

So Definition 2 says that the definite integral of an integrable function can be approximated to within any desired degree of accuracy by a Riemann sum.

We know that if *f* happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see Figure 1).

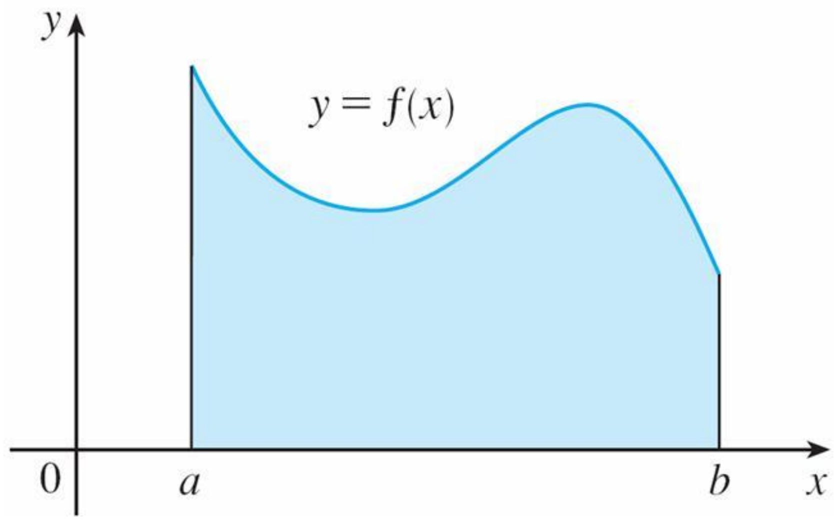


If $f(x) \geq 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

Figure 1

The Definite Integral

We see that the definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area under the curve $y = f(x)$ from a to b. (See Figure 2.)

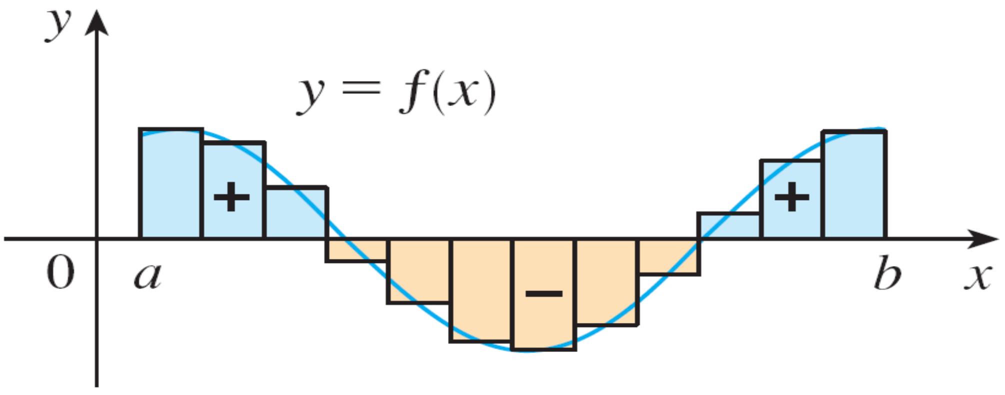


If $f(x) \geq 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

Figure 2

The Definite Integral

If f takes on both positive and negative values, as in Figure 3, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the *negatives* of the areas of the rectangles that lie below the x-axis (the areas of the blue rectangles *minus* the areas of the gold rectangles).



$\Sigma f(x_i^*)\,\Delta x$ is an approximation to the net area.

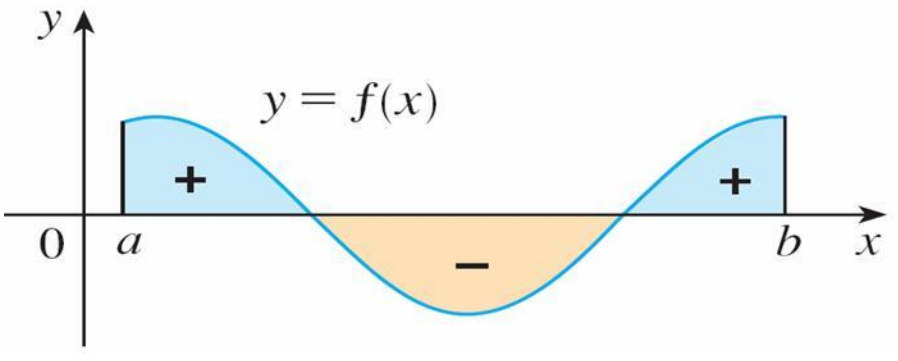
Figure 3

The Definite Integral

When we take the limit of such Riemann sums, we get the situation illustrated in Figure 4. A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.



$\int_a^b f(x)\,dx$ is the net area.

Figure 4

The Definite Integral

Note 4: Although we have defined $\int_a^b f(x)\,dx$ by dividing [a, b] into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width.

If the subinterval widths are $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, we have to ensure that all these widths approach 0 in the limiting process. This happens if the largest width, $\max \Delta x_i$, approaches 0. So in this case the definition of a definite integral becomes

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

The Definite Integral

Note 5: We have defined the definite integral for an integrable function, but not all functions are integrable. The following theorem shows that the most commonly occurring functions are in fact integrable. It is proved in more advanced courses.

> **3 Theorem** If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

If *f* is integrable on [*a*, *b*], then the limit in Definition 2 exists and gives the same value no matter how we choose the sample points x_i^*.

The Definite Integral

To simplify the calculation of the integral we often take the sample points to be right endpoints. Then $x_i^* = x_i$ and the definition of an integral simplifies as follows.

4 Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i\,\Delta x$$

Example 1

Express

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\,\Delta x$$

as an integral on the interval $[0, \pi]$.

Solution:

Comparing the given limit with the limit in Theorem 4, we see that they will be identical if we choose $f(x) = x^3 + x \sin x$. We are given that $a = 0$ and $b = \pi$.

Therefore, by Theorem 4, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\,\Delta x = \int_0^{\pi}(x^3 + x \sin x)\,dx$$

The Definite Integral

When Leibniz chose the notation for an integral, he chose the ingredients as reminders of the limiting process.

In general, when we write

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx$$

we replace $\lim \Sigma$ by $\int$, x_i^* by x, and Δx by dx.

Evaluating Integrals

Evaluating Integrals

When we use a limit to evaluate a definite integral, we need to know how to work with sums. The following three equations give formulas for sums of powers of positive integers.

5 $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

6 $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

7 $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Evaluating Integrals

The remaining formulas are simple rules for working with sigma notation:

$$\boxed{8} \qquad \sum_{i=1}^{n} c = nc$$

$$\boxed{9} \qquad \sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$$

$$\boxed{10} \qquad \sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\boxed{11} \qquad \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

Example 2 –*Evaluating an integral as a limit of Riemann sums*

(a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

Solution:

(a) With $n = 6$ the interval width is

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

and the right endpoints are $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$, $x_4 = 2.0$, $x_5 = 2.5$, and $x_6 = 3.0$.

Example 2 – *Solution*

cont'd

So the Riemann sum is

$$R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x$$

$$= f(0.5)\,\Delta x + f(1.0)\,\Delta x + f(1.5)\,\Delta x + f(2.0)\,\Delta x + f(2.5)\,\Delta x + f(3.0)\,\Delta x$$

$$= \tfrac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9)$$

$$= -3.9375$$

Example 2 – *Solution*

cont'd

Notice that f is not a positive function and so the Riemann sum does not represent a sum of areas of rectangles. But it does represent the sum of the areas of the blue rectangles (above the x-axis) minus the sum of the areas of the gold rectangles (below the x-axis) in Figure 5.

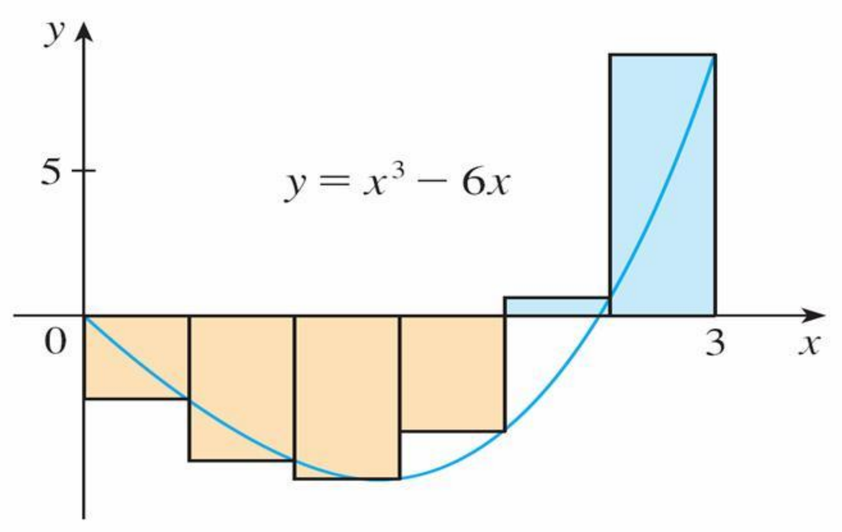


Figure 5

Example 2 – *Solution*

cont'd

(b) With n subintervals we have

$$\Delta x = \frac{b - a}{n} = \frac{3}{n}$$

Thus $x_0 = 0$, $x_1 = 3/n$, $x_2 = 6/n$, $x_3 = 9/n$, and, in general, $x_i = 3i/n$.

Since we are using right endpoints, we can use Theorem 4:

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

Example 2 – *Solution*

cont'd

$$= \lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n} \right)^3 - 6\left(\frac{3i}{n} \right) \right]$$ (Equation 9 with $c = 3/n$)

$$= \lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i \right]$$

$$= \lim_{n \to \infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i \right]$$ (Equations 11 and 9)

$$= \lim_{n \to \infty} \left\{ \frac{81}{n^4} \left[\frac{n(n+1)}{2} \right]^2 - \frac{54}{n^2} \frac{n(n+1)}{2} \right\}$$ (Equations 7 and 5)

Example 2 – *Solution*

cont'd

$$= \lim_{n \to \infty} \left[\frac{81}{4} \left(1 + \frac{1}{n}\right)^2 - 27\left(1 + \frac{1}{n}\right) \right]$$

$$= \frac{81}{4} - 27$$

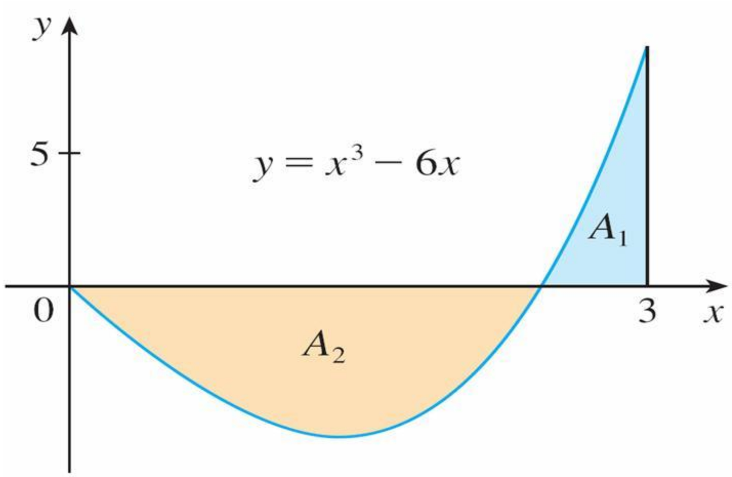
$$= -\frac{27}{4}$$

$$= -6.75$$

Example 2 – *Solution*

cont'd

This integral can't be interpreted as an area because f takes on both positive and negative values. But it can be interpreted as the difference of areas $A_1 - A_2$, where A_1 and A_2 are shown in Figure 6.



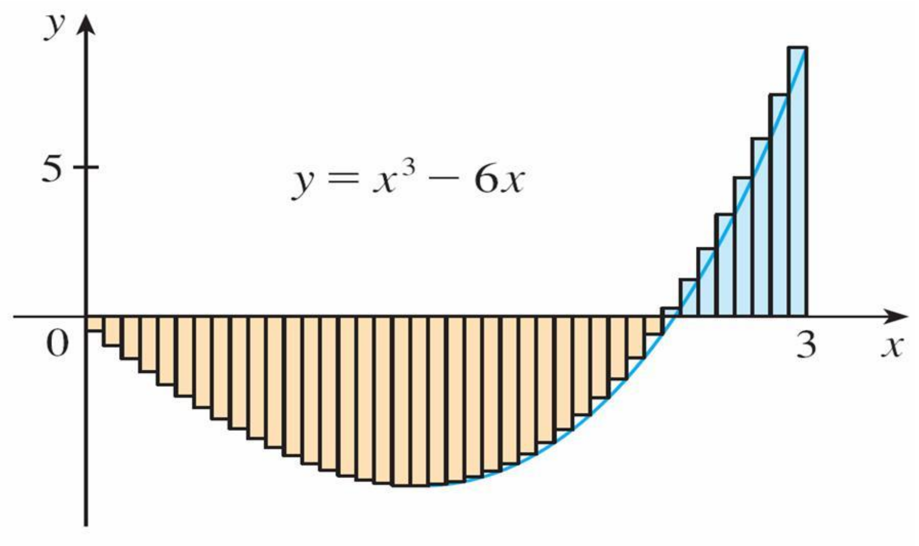
$$\int_0^3 (x^3 - 6x)\,dx = A_1 - A_2 = -6.75$$

Figure 6

Example 2 – *Solution*

cont'd

Figure 7 illustrates the calculation by showing the positive and negative terms in the right Riemann sum R_n for $n = 40$.



$R_{40} \approx -6.3998$

Figure 7

Example 2 – *Solution*

cont'd

The values in the table show the Riemann sums approaching the exact value of the integral, –6.75, as $n \to \infty$.

n	R_n
40	−6.3998
100	−6.6130
500	−6.7229
1000	−6.7365
5000	−6.7473

The Midpoint Rule

The Midpoint Rule

We often choose the sample point x_i^* to be the right endpoint of the *i*th subinterval because it is convenient for computing the limit.

But if the purpose is to find an *approximation* to an integral, it is usually better to choose x_i^* to be the midpoint of the interval, which we denote by $\bar{x}_i$.

The Midpoint Rule

Any Riemann sum is an approximation to an integral, but if we use midpoints we get the following approximation.

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\,\Delta x = \Delta x\,[f(\overline{x}_1) + \cdots + f(\overline{x}_n)]$$

where $$\Delta x = \frac{b-a}{n}$$

and $$\overline{x}_i = \tfrac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

Example 5

Use the Midpoint Rule with n = 5 to approximate $\int_1^2 \frac{1}{x}\,dx.$

Solution:

The endpoints of the five subintervals are 1, 1.2, 1.4, 1.6, 1.8, and 2.0, so the midpoints are 1.1, 1.3, 1.5, 1.7, and 1.9.

The width of the subintervals is Δx = (2 – 1)/5 = $\frac{1}{5}$, so the Midpoint Rule gives

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x\,[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$

Example 5 – *Solution*

cont'd

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$\approx \ 0.691908$$

Since $f(x) = 1/x > 0$ for $1 \le x \le 2$, the integral represents an area, and the approximation given by the Midpoint Rule is the sum of the areas of the rectangles shown in Figure 11.

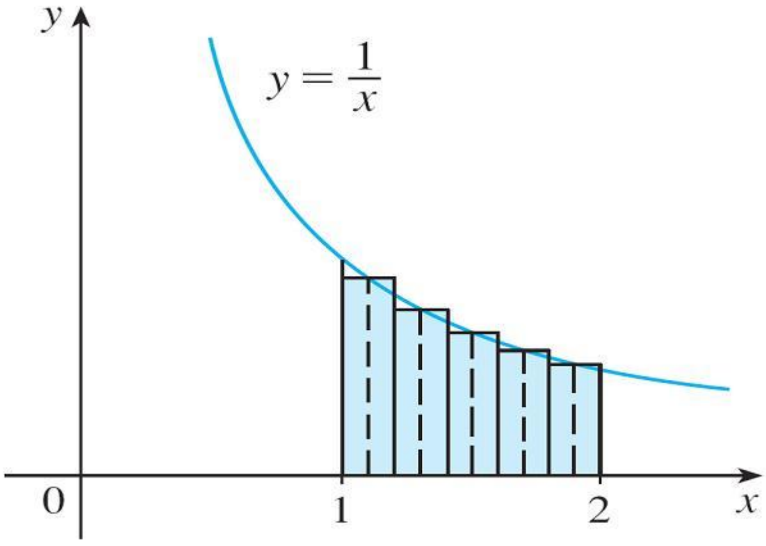


Figure 11

Properties of the Definite Integral

Properties of the Definite Integral

When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$.

But the definition as a limit of Riemann sums makes sense even if $a > b$.

Notice that if we reverse a and b, then Δx changes from $(b - a)/n$ to $(a - b)/n$. Therefore

$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$$

Properties of the Definite Integral

If $a = b$, then $\Delta x = 0$ and so

$$\int_a^a f(x)\,dx = 0$$

We now develop some basic properties of integrals that will help us to evaluate integrals in a simple manner. We assume that *f* and *g* are continuous functions.

Properties of the Integral

1. $\int_a^b c\,dx = c(b - a)$, where c is any constant

2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

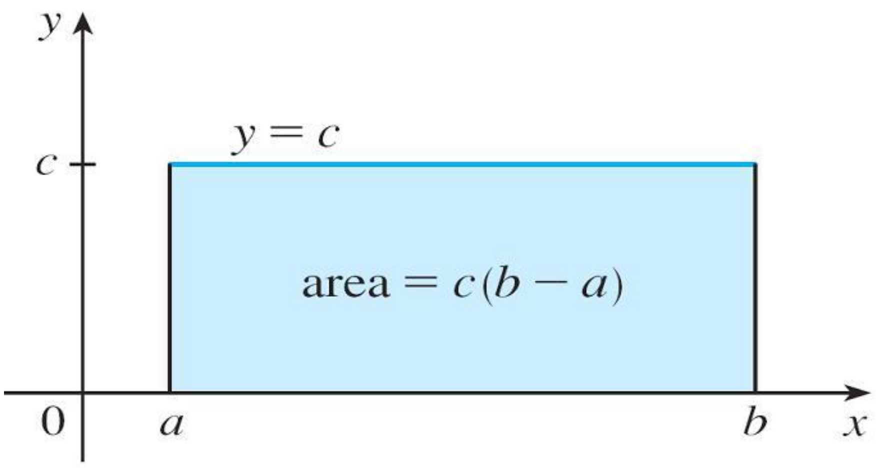
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Properties of the Definite Integral

Property 1 says that the integral of a constant function $f(x) = c$ is the constant times the length of the interval.

If $c > 0$ and $a < b$, this is to be expected because $c(b - a)$ is the area of the shaded rectangle in Figure 13.



$$\int_a^b c\,dx = c(b - a)$$

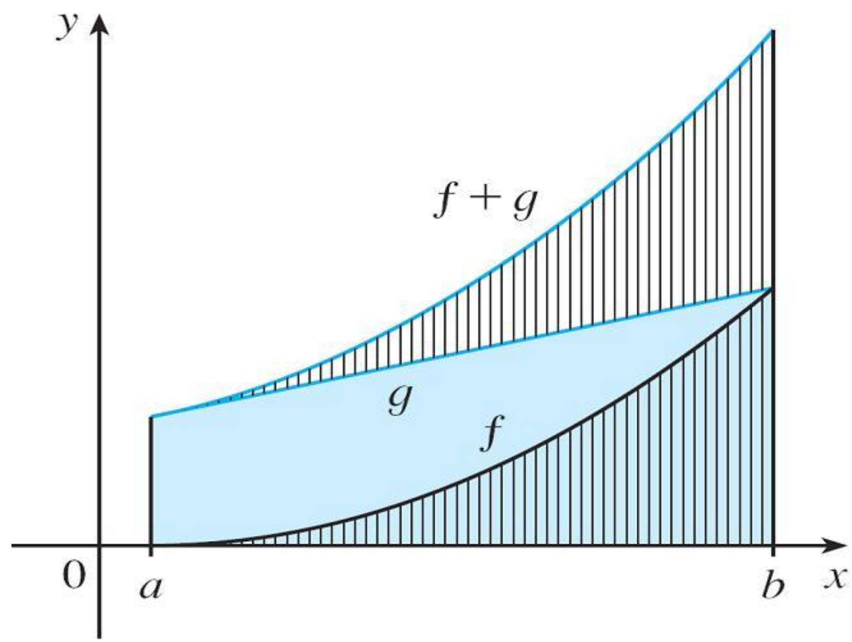
Figure 13

Properties of the Definite Integral

Property 2 says that the integral of a sum is the sum of the integrals.

For positive functions it says that the area under *f* + *g* is the area under *f* plus the area under *g*.

Figure 14 helps us understand why this is true: In view of how graphical addition works, the corresponding vertical line segments have equal height.



$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Figure 14

Properties of the Definite Integral

In general, Property 2 follows from Theorem 4 and the fact that the limit of a sum is the sum of the limits:

$$\begin{aligned}
\int_a^b [f(x) + g(x)]\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) + g(x_i)]\,\Delta x \\
&= \lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i)\,\Delta x + \sum_{i=1}^{n} g(x_i)\,\Delta x \right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\,\Delta x \\
&= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx
\end{aligned}$$

Properties of the Definite Integral

Property 3 can be proved in a similar manner and says that the integral of a constant times a function is the constant times the integral of the function.

In other words, a constant (but *only* a constant) can be taken in front of an integral sign.

Property 4 is proved by writing $f - g = f + (-g)$ and using Properties 2 and 3 with $c = -1$.

Example 6

Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\,dx$.

Solution:

Using Properties 2 and 3 of integrals, we have

$$\int_0^1 (4 + 3x^2)\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx$$

$$= \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

Example 6 – *Solution*

cont'd

We know from Property 1 that

$$\int_0^1 4\,dx = 4(1 - 0)$$

and we have found that $\int_0^1 x^2\,dx = \frac{1}{3}$.

So

$$\int_0^1 (4 + 3x^2)\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

$$= 4 + 3 \cdot \tfrac{1}{3}$$

$$= 5$$

Properties of the Definite Integral

The next property tells us how to combine integrals of the same function over adjacent intervals:

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Properties of the Definite Integral

This is not easy to prove in general, but for the case where $f(x) \geq 0$ and $a < c < b$ Property 5 can be seen from the geometric interpretation in Figure 15: The area under $y = f(x)$ from a to c plus the area from c to b is equal to the total area from a to b.

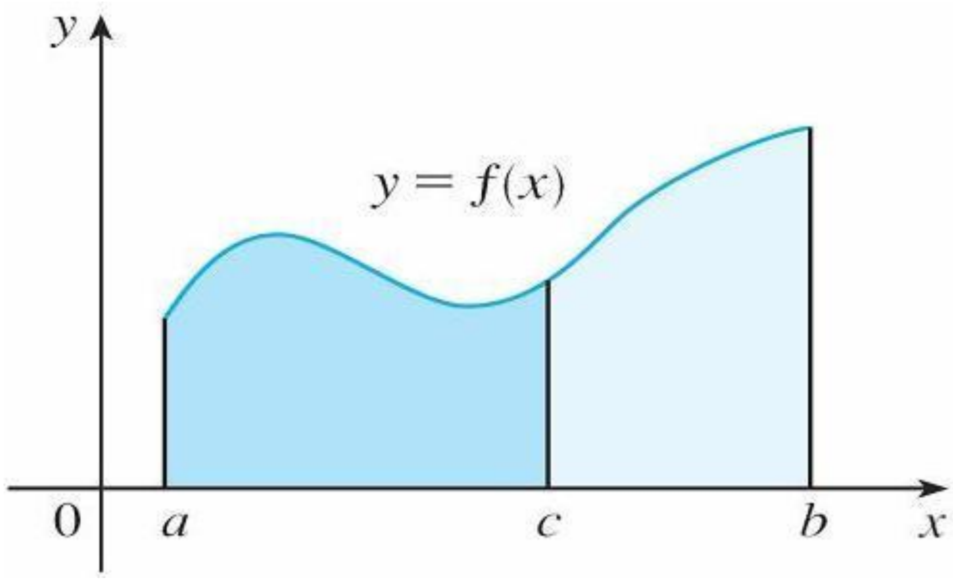


Figure 15

Properties of the Definite Integral

Properties 1–5 are true whether $a < b$, $a = b$, or $a > b$. The following properties, in which we compare sizes of functions and sizes of integrals, are true only if $a \leq b$.

Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$

Properties of the Definite Integral

If $f(x) \geq 0$, then $\int_a^b f(x)\,dx$ represents the area under the graph of f, so the geometric interpretation of Property 6 is simply that areas are positive. (It also follows directly from the definition because all the quantities involved are positive.)

Property 7 says that a bigger function has a bigger integral.

It follows from Properties 6 and 4 because $f - g \geq 0$.

Properties of the Definite Integral

Property 8 is illustrated by Figure 16 for the case where $f(x) \geq 0$.

If f is continuous we could take m and M to be the absolute minimum and maximum values of f on the interval [a, b].

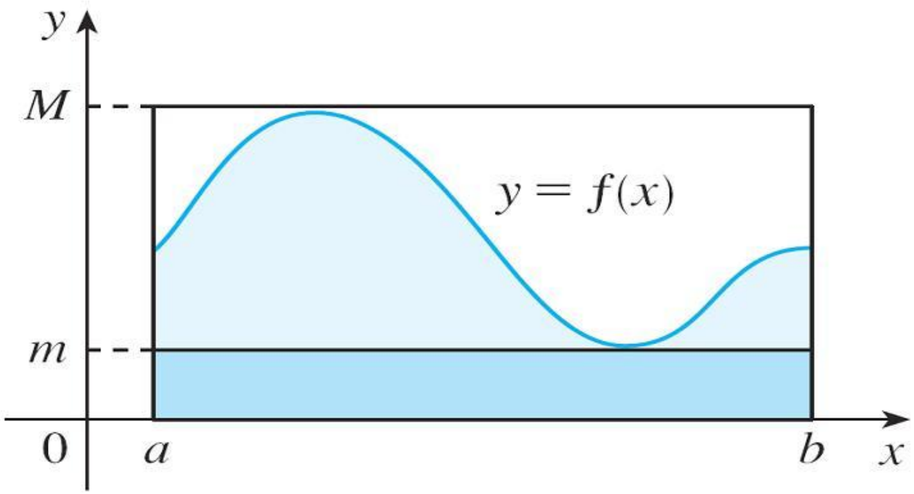


Figure 16

Properties of the Definite Integral

In this case Property 8 says that the area under the graph of f is greater than the area of the rectangle with height m and less than the area of the rectangle with height M.

Property 8 is useful when all we want is a rough estimate of the size of an integral without going to the bother of using the Midpoint Rule.

5 Integrals

5.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus.

It gives the precise inverse relationship between the derivative and the integral.

The Fundamental Theorem of Calculus

The first part of the Fundamental Theorem deals with functions defined by an equation of the form

$$\boxed{1} \qquad g(x) = \int_a^x f(t)\, dt$$

where *f* is a continuous function on [*a*, *b*] and *x* varies between *a* and *b*. Observe that *g* depends only on *x*, which appears as the variable upper limit in the integral.

If *x* is a fixed number, then the integral $\int_a^x f(t)\, dt$ is a definite number.

If we then let *x* vary, the number $\int_a^x f(t)\, dt$ also varies and defines a function of *x* denoted by *g*(*x*).

The Fundamental Theorem of Calculus

If f happens to be a positive function, then $g(x)$ can be interpreted as the area under the graph of f from a to x, where x can vary from a to b. (Think of g as the "area so far" function; see Figure 1.)

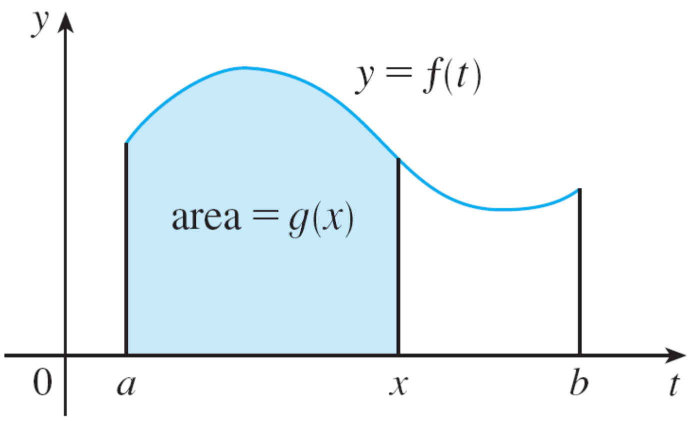


Figure 1

Example 1

If f is the function whose graph is shown in Figure 2 and $g(x) = \int_0^x f(t)\,dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$. Then sketch a rough graph of g.

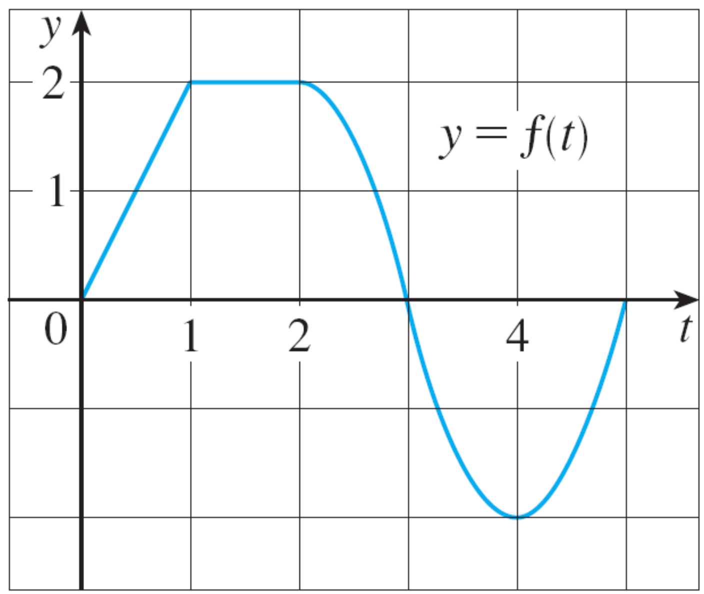
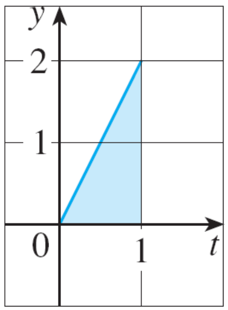


Figure 2

Example 1 – *Solution*

First we notice that $g(0) = \int_0^0 f(t)\,dt = 0$.

From Figure 3 we see that $g(1)$ is the area of a triangle:



$g(1) = 1$

Figure 3

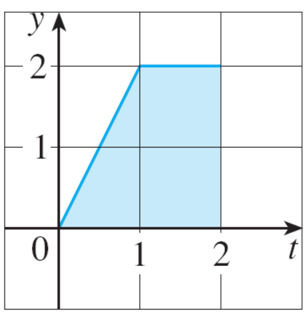
$$g(1) = \int_0^1 f(t)\,dt = \tfrac{1}{2}(1 \cdot 2) = 1$$

To find $g(2)$ we add to $g(1)$ the area of a rectangle:

$$g(2) = \int_0^2 f(t)\,dt$$

$$= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$$

$$= 1 + (1 \cdot 2) = 3$$



$g(2) = 3$

Figure 3

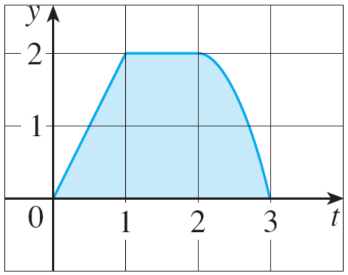
Example 1 – *Solution*

cont'd

We estimate that the area under f from 2 to 3 is about 1.3,

$$g(3) = g(2) + \int_2^3 f(t)\,dt$$

$$\approx 3 + 1.3 = 4.3$$



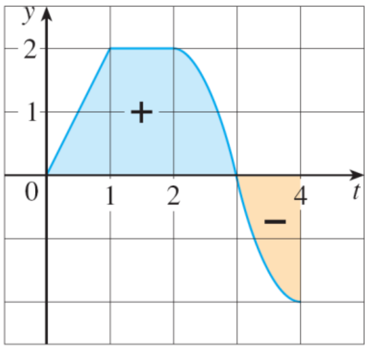
$g(3) \approx 4.3$

Figure 3

For $t > 3$, $f(t)$ is negative and so we start subtracting areas:

$$g(4) = g(3) + \int_3^4 f(t)\,dt$$

$$\approx 4.3 + (–1.3) = 3.0$$



$g(4) \approx 3$

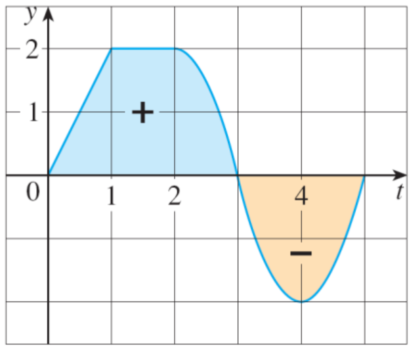
Figure 3

Example 1 – *Solution*

cont'd

$$g(5) = g(4) + \int_4^5 f(t)\,dt$$

$$\approx 3 + (–1.3) = 1.7$$

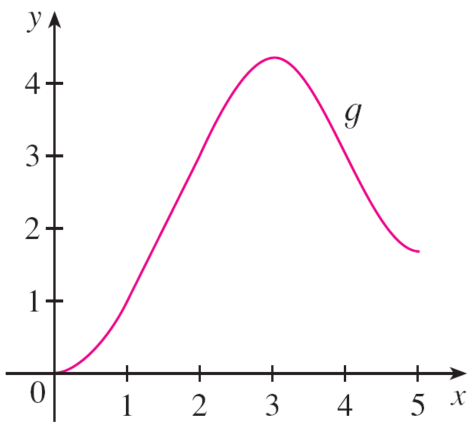


$g(5) \approx 1.7$

Figure 3

We use these values to sketch the graph of *g* in Figure 4.

Notice that, because $f(t)$ is positive for $t < 3$, we keep adding area for $t < 3$ and so *g* is increasing up to $x = 3$, where it attains a maximum value. For $x > 3$, *g* decreases because $f(t)$ is negative.



$g(x) = \int_a^x f(t)\,dt$

Figure 4

The Fundamental Theorem of Calculus

If we take, $f(t) = t$ and $a = 0$, then we have $g(x) = \int_0^x t\,dt = \frac{x^2}{2}$ Notice that $g'(x) = x$, that is, $g' = f$. In other words, if g is defined as the integral of f by Equation 1, then g turns out to be an antiderivative of f, at least in this case.

And if we sketch the derivative of the function g shown in Figure 4 by estimating slopes of tangents, we get a graph like that of f in Figure 2. So we suspect that $g' = f$ in Example 1 too.

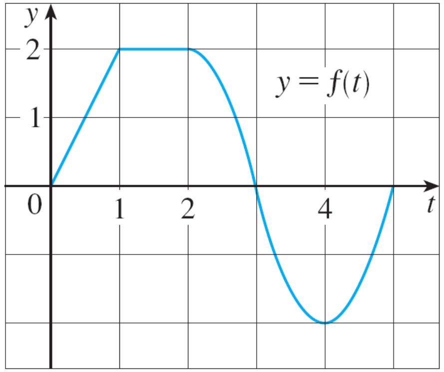


Figure 2

The Fundamental Theorem of Calculus

To see why this might be generally true we consider any continuous function *f* with $f(x) \geq 0$. Then $g(x) = \int_a^x f(t)\,dt$ can be interpreted as the area under the graph of *f* from *a* to *x*, as in Figure 1.

In order to compute $g'(x)$ from the definition of a derivative we first observe that, for $h > 0$, $g(x + h) - g(x)$ is obtained by subtracting areas, so it is the area under the graph of *f* from *x* to $x + h$ (the blue area in Figure 5).

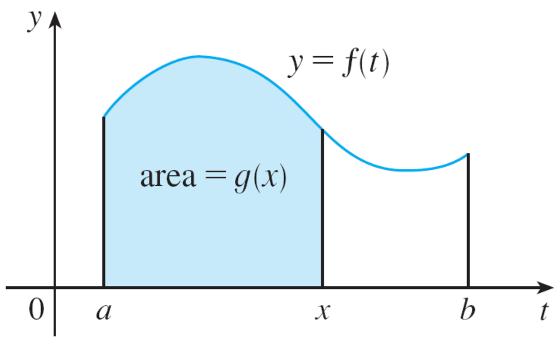


Figure 1

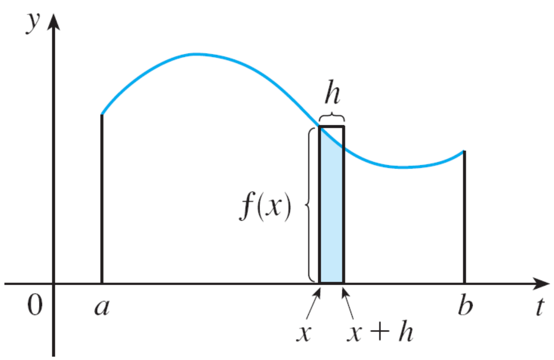


Figure 5

The Fundamental Theorem of Calculus

For small *h* you can see from the figure that this area is approximately equal to the area of the rectangle with height $f(x)$ and width h:

$$g(x + h) - g(x) \approx hf(x)$$

so

$$\frac{g(x+h) - g(x)}{h} \approx f(x)$$

Intuitively, we therefore expect that

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$$

The Fundamental Theorem of Calculus

The fact that this is true, even when *f* is not necessarily positive, is the first part of the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Using Leibniz notation for derivatives, we can write this theorem as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

when *f* is continuous.

The Fundamental Theorem of Calculus

Roughly speaking, this equation says that if we first integrate *f* and then differentiate the result, we get back to the original function *f*.

Example 2

Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\, dt.$

Solution:

Since $f(t) = \sqrt{1 + t^2}$ is continuous, Part 1 of the Fundamental Theorem of Calculus gives

$$g'(x) = \sqrt{1 + x^2}$$

Example 3

Although a formula of the form $g(x) = \int_a^x f(t)\,dt$ may seem like a strange way of defining a function, books on physics, chemistry, and statistics are full of such functions. For instance, the **Fresnel function**

$$S(x) = \int_0^x \sin(\pi t^2/2)\,dt$$

is named after the French physicist Augustin Fresnel (1788–1827), who is famous for his works in optics.

This function first appeared in Fresnel's theory of the diffraction of light waves, but more recently it has been applied to the design of highways.

Example 3

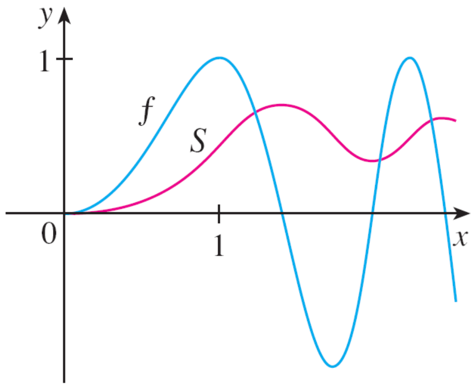
cont'd

Part 1 of the Fundamental Theorem tells us how to differentiate the Fresnel function:

$$S'(x) = \sin(\pi x^2/2)$$

This means that we can apply all the methods of differential calculus to analyze *S*.

Figure 7 shows the graphs of $f(x) = \sin(\pi x^2/2)$ and the Fresnel function $S(x) = \int_0^x f(t)\,dt$.



$f(x) = \sin(\pi x^2/2)$

$S(x) = \int_0^x \sin(\pi t^2/2)\,dt$

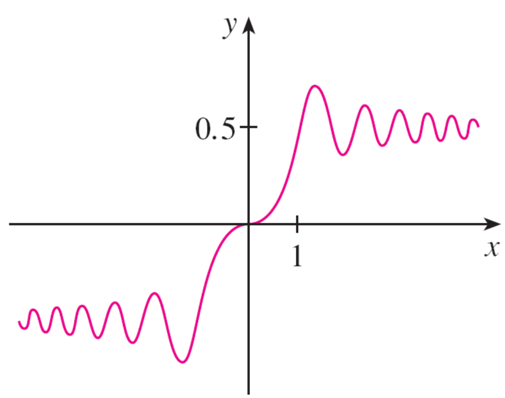
Figure 7

Example 3

cont'd

A computer was used to graph S by computing the value of this integral for many values of x.

It does indeed look as if $S(x)$ is the area under the graph of f from 0 to x [until $x \approx 1.4$ when $S(x)$ becomes a difference of areas]. Figure 8 shows a larger part of the graph of S.



The Fresnel function

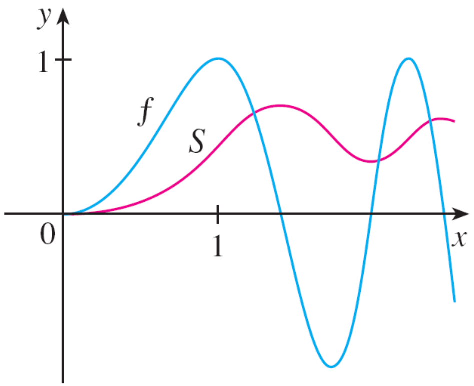
$$S(x) = \int_0^x \sin(\pi t^2/2)\,dt$$

Figure 8

Example 3

cont'd

If we now start with the graph of S in Figure 7 and think about what its derivative should look like, it seems reasonable that $S'(x) = f(x)$. [For instance, S is increasing when $f(x) > 0$ and decreasing when $f(x) < 0$.] So this gives a visual confirmation of Part 1 of the Fundamental Theorem of Calculus.



$$f(x) = \sin(\pi x^2/2)$$

$$S(x) = \int_0^x \sin(\pi t^2/2)\,dt$$

Figure 7

The Fundamental Theorem of Calculus

The second part of the Fundamental Theorem of Calculus, which follows easily from the first part, provides us with a much simpler method for the evaluation of integrals.

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Differentiation and Integration as Inverse Processes

Differentiation and Integration as Inverse Processes

We end this section by bringing together the two parts of the Fundamental Theorem.

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

We noted that Part 1 can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

which says that if *f* is integrated and then the result is differentiated, we arrive back at the original function *f*.

Differentiation and Integration as Inverse Processes

Since $F'(x) = f(x)$, Part 2 can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This version says that if we take a function F, first differentiate it, and then integrate the result, we arrive back at the original function F, but in the form $F(b) - F(a)$.

Taken together, the two parts of the Fundamental Theorem of Calculus say that differentiation and integration are inverse processes. Each undoes what the other does.

5

Integrals

5.4 Indefinite Integrals and the Net Change Theorem

Indefinite Integrals and the Net Change Theorem

In this section we introduce a notation for antiderivatives, review the formulas for antiderivatives, and use them to evaluate definite integrals.

We also reformulate FTC2 in a way that makes it easier to apply to science and engineering problems.

Indefinite Integrals

Indefinite Integrals

Both parts of the Fundamental Theorem establish connections between antiderivatives and definite integrals. Part 1 says that if *f* is continuous, then $\int_a^x f(t)\,dt$ is an antiderivative of *f*. Part 2 says that $\int_a^b f(x)\,dx$ can be found by evaluating $F(b) - F(a)$, where *F* is an antiderivative of *f*.

We need a convenient notation for antiderivatives that makes them easy to work with.

Because of the relation given by the Fundamental Theorem between antiderivatives and integrals, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of *f* and is called an **indefinite integral.**

Indefinite Integrals

Thus

$$\boxed{\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)}$$

For example, we can write

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

So we can regard an indefinite integral as representing an entire *family* of functions (one antiderivative for each value of the constant *C*).

Indefinite Integrals

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a *number*, whereas an indefinite integral $\int f(x)\,dx$ is a *function* (or family of functions).

The connection between them is given by Part 2 of the Fundamental Theorem:

If *f* is continuous on [*a*, *b*], then

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big]_a^b$$

The effectiveness of the Fundamental Theorem depends on having a supply of antiderivatives of functions.

Indefinite Integrals

Any formula can be verified by differentiating the function on the right side and obtaining the integrand.

For instance,

$$\int \sec^2 x \, dx = \tan x + C \qquad \text{because} \qquad \frac{d}{dx}(\tan x + C) = \sec^2 x$$

Indefinite Integrals

1 Table of Indefinite Integrals

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int k\,dx = kx + C$	
$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int a^x\,dx = \dfrac{a^x}{\ln a} + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \dfrac{1}{x^2+1}\,dx = \tan^{-1}x + C$	$\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$
$\int \sinh x\,dx = \cosh x + C$	$\int \cosh x\,dx = \sinh x + C$

The most general antiderivative *on a given interval* is obtained by adding a constant to a particular antiderivative.

Indefinite Integrals

We adopt the convention that when a formula for a general indefinite integral is given, it is valid only on an interval.

Thus we write

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

with the understanding that it is valid on the interval $(0, \infty)$ or on the interval $(-\infty, 0)$.

Indefinite Integrals

This is true despite the fact that the general antiderivative of the function $f(x) = 1/x^2$, $x \neq 0$, is

$$F(x) = \begin{cases} -\dfrac{1}{x} + C_1 & \text{if } x < 0 \\[2ex] -\dfrac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$$

Example 2

Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

Solution:

This indefinite integral isn't immediately apparent in Table 1, so we use trigonometric identities to rewrite the function before integrating:

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$= \int \csc\theta \cot\theta \, d\theta$$

$$= -\csc\theta + C$$

Example 5

Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt.$

Solution:

First we need to write the integrand in a simpler form by carrying out the division:

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int_1^9 (2 + t^{1/2} - t^{-2})\,dt$$

$$= 2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\Bigg]_1^9$$

$$= 2t + \tfrac{2}{3}t^{3/2} + \frac{1}{t}\Bigg]_1^9$$

Example 5 – *Solution*

cont'd

$$= \left(2 \cdot 9 + \tfrac{2}{3} \cdot 9^{3/2} + \tfrac{1}{9}\right) - \left(2 \cdot 1 + \tfrac{2}{3} \cdot 1^{3/2} + \tfrac{1}{1}\right)$$

$$= 18 + 18 + \tfrac{1}{9} - 2 - \tfrac{2}{3} - 1$$

$$= 32\tfrac{4}{9}$$

Applications

Applications

Part 2 of the Fundamental Theorem says that if *f* is continuous on [*a*, *b*], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where *F* is any antiderivative of *f*. This means that $F' = f$, so the equation can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Applications

We know that $F'(x)$ represents the rate of change of $y = F(x)$ with respect to x and $F(b) – F(a)$ is the change in y when x changes from a to b.

[Note that y could, for instance, increase, then decrease, then increase again.

Although y might change in both directions, $F(b) – F(a)$ represents the *net* change in y.]

Applications

So we can reformulate FTC2 in words as follows.

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This principle can be applied to all of the rates of change in the natural and social sciences. Here are a few instances of this idea:

- If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into the reservoir at time t.

Applications

So

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

is the change in the amount of water in the reservoir between time t_1 and time t_2.

- If [C](t) is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative $d[C]/dt$.

Applications

So

$$\int_{t_1}^{t_2} \frac{d[\mathrm{C}]}{dt}\,dt = [\mathrm{C}](t_2) - [\mathrm{C}](t_1)$$

is the change in the concentration of C from time t_1 to time t_2.

- If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$. So

$$\int_a^b \rho(x)\,dx = m(b) - m(a)$$

is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

Applications

- If the rate of growth of a population is dn/dt, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$

is the net change in population during the time period from t_1 to t_2.

(The population increases when births happen and decreases when deaths occur. The net change takes into account both births and deaths.)

Applications

- If $C(x)$ is the cost of producing x units of a commodity, then the marginal cost is the derivative $C'(x)$.

  So

$$\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1)$$

  is the increase in cost when production is increased from x_1 units to x_2 units.

Applications

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\boxed{2} \qquad \int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

is the net change of position, or *displacement*, of the particle during the time period from t_1 to t_2.

This was true for the case where the object moves in the positive direction, but now we have proved that it is always true.

Applications

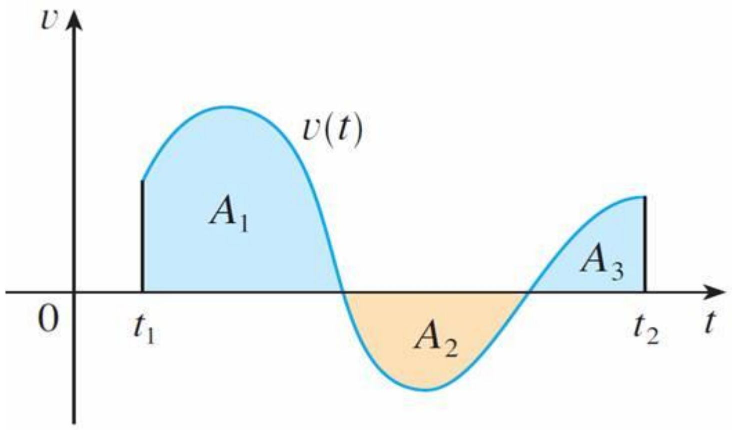
- If we want to calculate the distance the object travels during the time interval, we have to consider the intervals when $v(t) \geq 0$ (the particle moves to the right) and also the intervals when $v(t) \leq 0$ (the particle moves to the left).

  In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore

$$\boxed{3} \qquad \int_{t_1}^{t_2} |v(t)|\, dt = \text{total distance traveled}$$

Applications

Figure 3 shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.



$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

Figure 3

Applications

- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$$

is the change in velocity from time t_1 to time t_2.

Example 6

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \le t \le 4$.

(b) Find the distance traveled during this time period.

Example 6 – *Solution*

(a) By Equation 2, the displacement is

$$s(4) - s(1) = \int_1^4 v(t)\,dt$$

$$= \int_1^4 (t^2 - t - 6)\,dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4$$

$$= -\frac{9}{2}$$

Example 6 – *Solution*

cont'd

This means that the particle moved 4.5 m toward the left.

(b) Note that $v(t) = t^2 - t - 6 = (t - 3)(t + 2)$ and so $v(t) \leq 0$ on the interval [1, 3] and $v(t) \geq 0$ on [3, 4].

Thus, from Equation 3, the distance traveled is

$$\int_1^4 |v(t)|\,dt = \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt$$

$$= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt$$

Example 6 – *Solution*

cont'd

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4$$

$$= \frac{61}{6}$$

$$\approx 10.17 \text{ m}$$

5 Integrals

5.5 The Substitution Rule

The Substitution Rule

Because of the Fundamental Theorem, it’s important to be able to find antiderivatives.

But our antidifferentiation formulas don’t tell us how to evaluate integrals such as

$$\boxed{1} \qquad \int 2x\sqrt{1 + x^2}\, dx$$

To find this integral we use the problem-solving strategy of *introducing something extra.* Here the “something extra” is a new variable; we change from the variable x to a new variable u.

The Substitution Rule

Suppose that we let u be the quantity under the root sign in $\boxed{1}$, $u = 1 + x^2$. Then the differential of u is $du = 2x dx$.

Notice that if the dx in the notation for an integral were to be interpreted as a differential, then the differential $2x dx$ would occur in $\boxed{1}$ and so, formally, without justifying our calculation, we could write

$$\boxed{2} \qquad \int 2x\sqrt{1 + x^2}\,dx = \int \sqrt{1 + x^2}\,2x\,dx = \int \sqrt{u}\,du$$

$$= \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}(x^2 + 1)^{3/2} + C$$

The Substitution Rule

But now we can check that we have the correct answer by using the Chain Rule to differentiate the final function of Equation 2:

$$\frac{d}{dx}\left[\tfrac{2}{3}(x^2+1)^{3/2}+C\right]=\tfrac{2}{3}\cdot\tfrac{3}{2}(x^2+1)^{1/2}\cdot 2x=2x\sqrt{x^2+1}$$

In general, this method works whenever we have an integral that we can write in the form $\int f(g(x))\,g'(x)\,dx$.

The Substitution Rule

Observe that if $F' = f$, then

$$\boxed{3} \qquad \int F'(g(x))\, g'(x)\, dx = F(g(x)) + C$$

because, by the Chain Rule,

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

If we make the "change of variable" or "substitution" $u = g(x)$, then from Equation 3 we have

$$\int F'(g(x))g'(x)\, dx = F(g(x)) + C = F(u) + C = \int F'(u)\, du$$

or, writing $F' = f$, we get

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du$$

The Substitution Rule

Thus we have proved the following rule.

$\boxed{4}$ **The Substitution Rule** If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Notice that the Substitution Rule for integration was proved using the Chain Rule for differentiation.

Notice also that if $u = g(x)$, then $du = g'(x)\,dx$, so a way to remember the Substitution Rule is to think of dx and du in $\boxed{4}$ as differentials.

The Substitution Rule

Thus the Substitution Rule says: **It is permissible to operate with *dx* and *du* after integral signs as if they were differentials.**

Example 1

Find $\int x^3 \cos(x^4 + 2)\, dx.$

Solution:

We make the substitution $u = x^4 + 2$ because its differential is $du = 4x^3\, dx$, which, apart from the constant factor 4, occurs in the integral.

Thus, using $x^3\, dx = \frac{1}{4}\, du$ and the Substitution Rule, we have

$$\int x^3 \cos(x^4 + 2)\, dx = \int \cos u \cdot \frac{1}{4}\, du$$

$$= \frac{1}{4} \int \cos u\, du$$

Example 1 – *Solution*

cont'd

$$= \frac{1}{4}\sin u + C$$

$$= \frac{1}{4}\sin(x^4 + 2) + C$$

Notice that at the final stage we had to return to the original variable x.

Definite Integrals

Definite Integrals

When evaluating a *definite* integral by substitution, two methods are possible. One method is to evaluate the indefinite integral first and then use the Fundamental Theorem.

For example,

$$\int_0^4 \sqrt{2x+1}\,dx = \int \sqrt{2x+1}\,dx\Big]_0^4 = \tfrac{1}{3}(2x+1)^{3/2}\Big]_0^4$$

$$= \tfrac{1}{3}(9)^{3/2} - \tfrac{1}{3}(1)^{3/2} = \tfrac{1}{3}(27-1) = \tfrac{26}{3}$$

Another method, which is usually preferable, is to change the limits of integration when the variable is changed.

Definite Integrals

$\boxed{6}$ **The Substitution Rule for Definite Integrals** If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example 7

Evaluate $\int_0^4 \sqrt{2x + 1}\, dx$ using $\boxed{6}$.

Solution:

Let $u = 2x + 1$. Then $du = 2\, dx$, so $dx = \frac{1}{2}\, du$.

To find the new limits of integration we note that

when $x = 0$, $u = 2(0) + 1 = 1$

and

when $x = 4$, $u = 2(4) + 1 = 9$

Example 7 – *Solution*

cont'd

Therefore

$$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \tfrac{1}{2}\sqrt{u}\,du$$

$$= \tfrac{1}{2} \cdot \tfrac{2}{3} u^{3/2} \Big]_1^9$$

$$= \tfrac{1}{3}(9^{3/2} - 1^{3/2})$$

$$= \frac{26}{3}$$

Observe that when using $\boxed{6}$ we do *not* return to the variable *x* after integrating. We simply evaluate the expression in *u* between the appropriate values of *u*.

Symmetry

Symmetry

The following theorem uses the Substitution Rule for Definite Integrals $\boxed{6}$ to simplify the calculation of integrals of functions that possess symmetry properties.

$\boxed{7}$ **Integrals of Symmetric Functions** Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

Symmetry

Theorem 7 is illustrated by Figure 3.

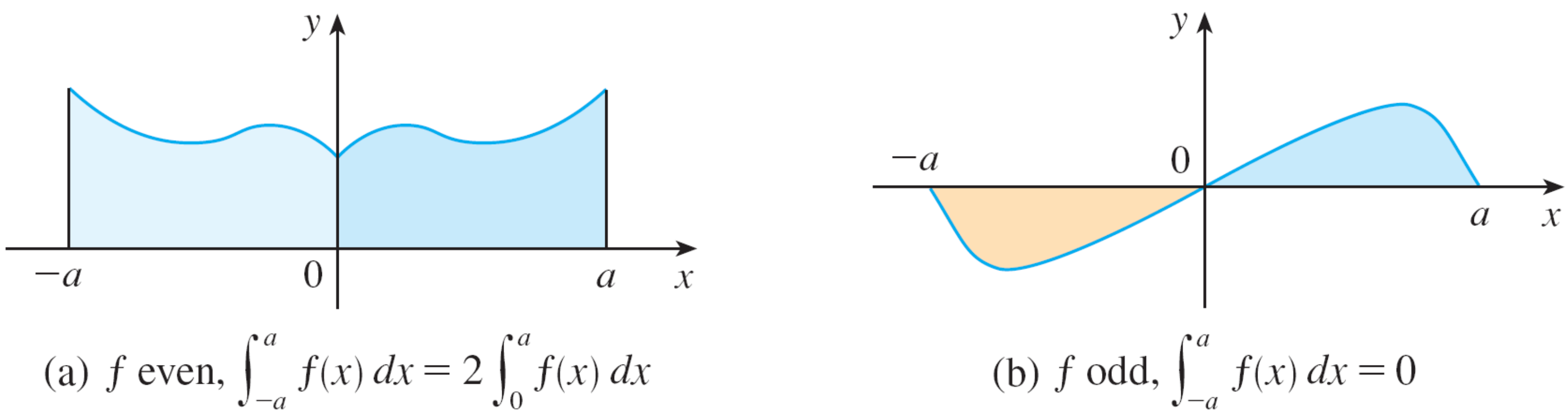


Figure 3

For the case where *f* is positive and even, part (a) says that the area under $y = f(x)$ from $-a$ to a is twice the area from 0 to a because of symmetry.

Symmetry

We know that an integral $\int_a^b f(x)\,dx$ can be expressed as the area above the x-axis and below $y = f(x)$ minus the area below the axis and above the curve.

Thus part (b) says the integral is 0 because the areas cancel.

Example 10

Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so

$$\int_{-2}^{2} (x^6 + 1)\,dx = 2\int_{0}^{2} (x^6 + 1)\,dx$$

$$= 2\left[\tfrac{1}{7}x^7 + x\right]_0^2$$

$$= 2\left(\tfrac{128}{7} + 2\right)$$

$$= \frac{284}{7}$$

Example 11

Since $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\, dx = 0$$