12

Vectors and the Geometry of Space

12.1 Three-Dimensional Coordinate Systems

Three-Dimensional Coordinate Systems

To locate a point in a plane, two numbers are necessary.

We know that any point in the plane can be represented as an ordered pair (a, b) of real numbers, where a is the x-coordinate and b is the y-coordinate.

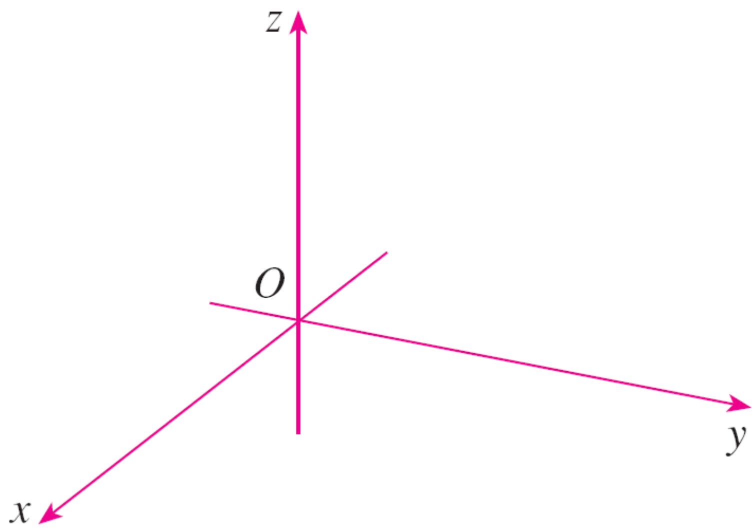
For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required.

We represent any point in space by an ordered triple (a, b, c) of real numbers.

Three-Dimensional Coordinate Systems

In order to represent points in space, we first choose a fixed point *O* (the origin) and three directed lines through *O* that are perpendicular to each other, called the **coordinate axes** and labeled the *x*-axis, *y*-axis, and *z*-axis.

Usually we think of the *x*- and *y*-axes as being horizontal and the *z*-axis as being vertical, and we draw the orientation of the axes as in Figure 1.

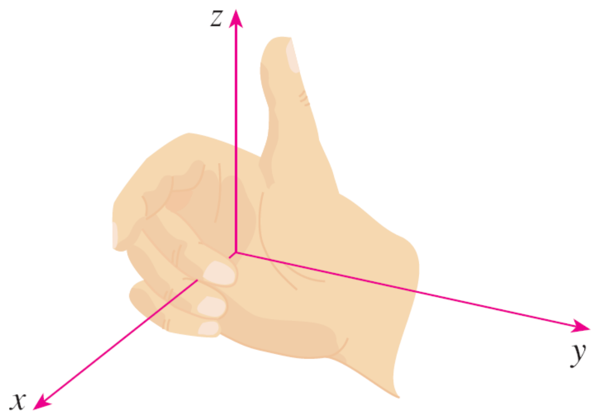


Coordinate axes

Figure 1

Three-Dimensional Coordinate Systems

The direction of the z-axis is determined by the **right-hand rule** as illustrated in Figure 2:



Right-hand rule

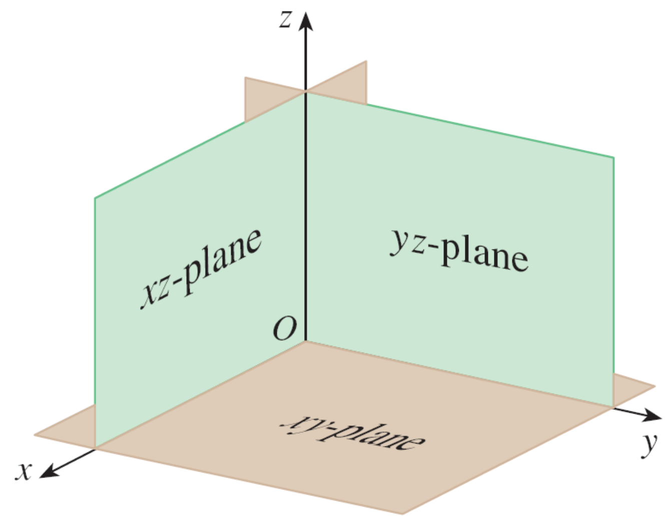
Figure 2

If you curl the fingers of your right hand around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis.

Three-Dimensional Coordinate Systems

The three coordinate axes determine the three **coordinate planes** illustrated in Figure 3(a).

The *xy*-plane is the plane that contains the *x*- and *y*-axes; the *yz*-plane contains the *y*- and *z*-axes; the *xz*-plane contains the *x*- and *z*-axes.



Coordinate planes

Figure 3(a)

These three coordinate planes divide space into eight parts, called **octants**. The **first octant**, in the foreground, is determined by the positive axes.

Three-Dimensional Coordinate Systems

Because many people have some difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following [see Figure 3(b)].

Look at any bottom corner of a room and call the corner the origin.

The wall on your left is in the *xz*-plane, the wall on your right is in the *yz*-plane, and the floor is in the *xy*-plane.

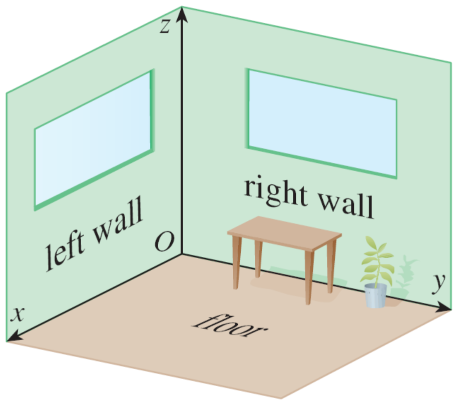


Figure 3(b)

Three-Dimensional Coordinate Systems

The x-axis runs along the intersection of the floor and the left wall.

The y-axis runs along the intersection of the floor and the right wall.

The z-axis runs up from the floor toward the ceiling along the intersection of the two walls.

You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants (three on the same floor and four on the floor below), all connected by the common corner point O.

Three-Dimensional Coordinate Systems

Now if P is any point in space, let a be the (directed) distance from the yz-plane to P, let b be the distance from the xz-plane to P, and let c be the distance from the xy-plane to P.

We represent the point P by the ordered triple (a, b, c) of real numbers and we call a, b, and c the **coordinates** of P; a is the x-coordinate, b is the y-coordinate, and c is the z-coordinate.

Three-Dimensional Coordinate Systems

Thus, to locate the point (a, b, c), we can start at the origin O and move a units along the x-axis, then b units parallel to the y-axis, and then c units parallel to the z-axis as in Figure 4.

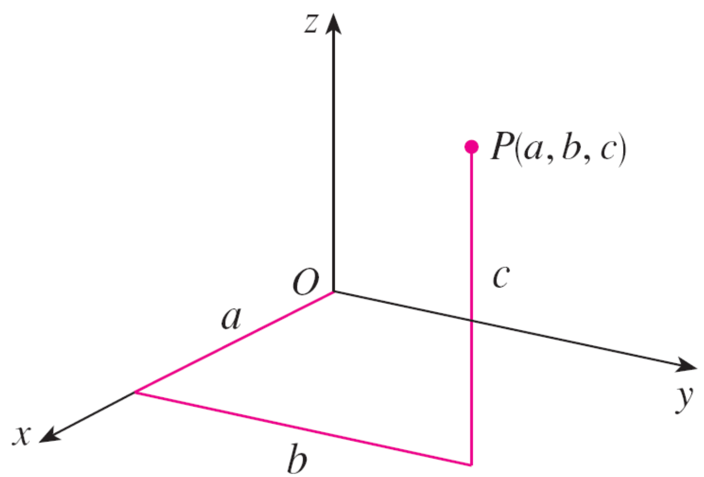


Figure 4

Three-Dimensional Coordinate Systems

The point $P(a, b, c)$ determines a rectangular box as in Figure 5.

If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a, b, 0)$ called the **projection** of P onto the xy-plane.

Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P onto the yz-plane and xz-plane, respectively.

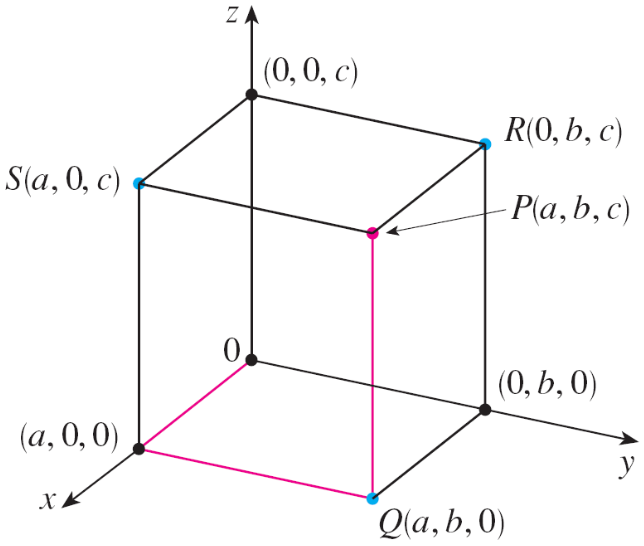


Figure 5

Three-Dimensional Coordinate Systems

As numerical illustrations, the points (–4, 3, –5) and (3, –2, –6) are plotted in Figure 6.

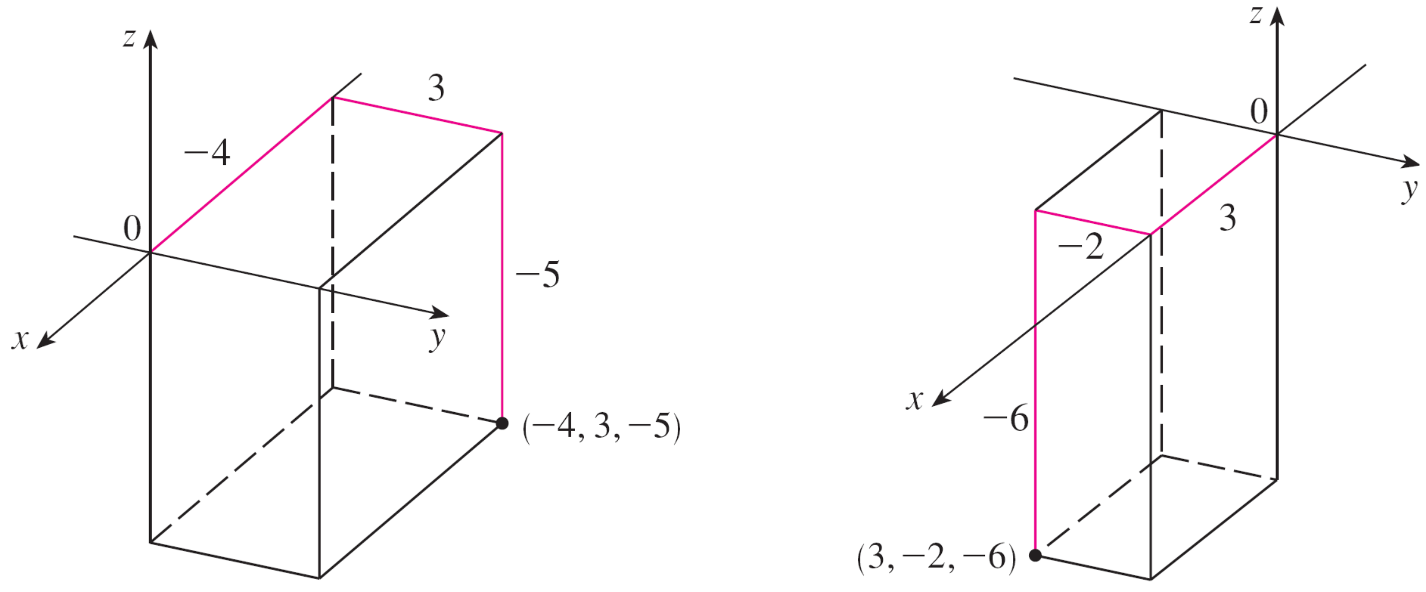


Figure 6

Three-Dimensional Coordinate Systems

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$.

We have given a one-to-one correspondence between points P in space and ordered triples (a, b, c) in $\mathbb{R}^3$. It is called a **three-dimensional rectangular coordinate system**.

Notice that, in terms of coordinates, the first octant can be described as the set of points whose coordinates are all positive.

Three-Dimensional Coordinate Systems

In two-dimensional analytic geometry, the graph of an equation involving *x* and *y* is a curve in $\mathbb{R}^2$.

In three-dimensional analytic geometry, an equation in *x*, *y*, and *z* represents a *surface* in $\mathbb{R}^3$.

Example 1

What surfaces in $\mathbb{R}^3$ are represented by the following equations?

(a) $z = 3$ **(b)** $y = 5$

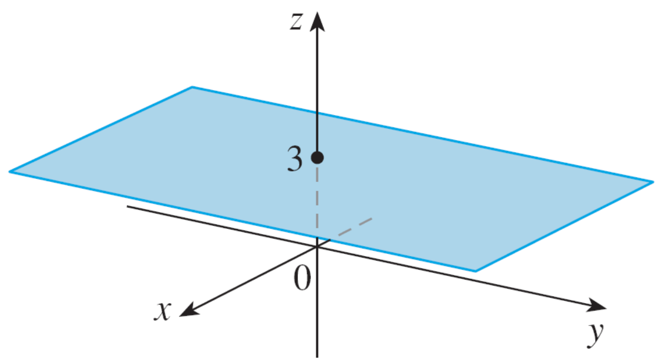
Solution:

(a) The equation $z = 3$ represents the set $\{(x, y, z) \mid z = 3\}$, which is the set of all points in $\mathbb{R}^3$ whose z-coordinate is 3.

Example 1 – *Solution*

cont'd

This is the horizontal plane that is parallel to the *xy*–plane and three units above it as in Figure 7(a).



$z = 3$, a plane in $\mathbb{R}^3$

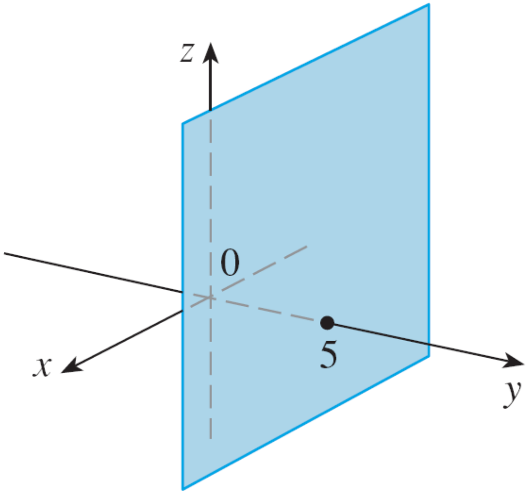
Figure 7(a)

Example 1 – *Solution*

cont'd

(b) The equation $y = 5$ represents the set of all points in $\mathbb{R}^3$ whose y-coordinate is 5.

This is the vertical plane that is parallel to the xz-plane and five units to the right of it as in Figure 7(b).



$y = 5$, a plane in $\mathbb{R}^3$

Figure 7(b)

Three-Dimensional Coordinate Systems

In general, if k is a constant, then $x = k$ represents a plane parallel to the yz-plane, $y = k$ is a plane parallel to the xz-plane, and $z = k$ is a plane parallel to the xy-plane.

In Figure 5, the faces of the rectangular box are formed by the three coordinate planes $x = 0$ (the yz-plane), $y = 0$ (the xz-plane), and $z = 0$ (the xy-plane), and the planes $x = a$, $y = b$, and $z = c$.

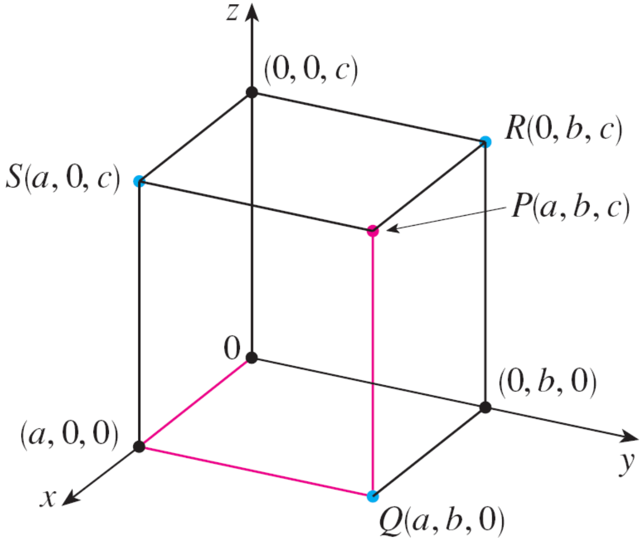


Figure 5

Three-Dimensional Coordinate Systems

The familiar formula for the distance between two points in a plane is easily extended to the following three-dimensional formula.

Distance Formula in Three Dimensions The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 5

Find an equation of a sphere with radius *r* and center *C*(*h*, *k*, *l*).

Solution:

By definition, a sphere is the set of all points *P*(*x*, *y*, *z*) whose distance from *C* is *r*. (See Figure 12.)

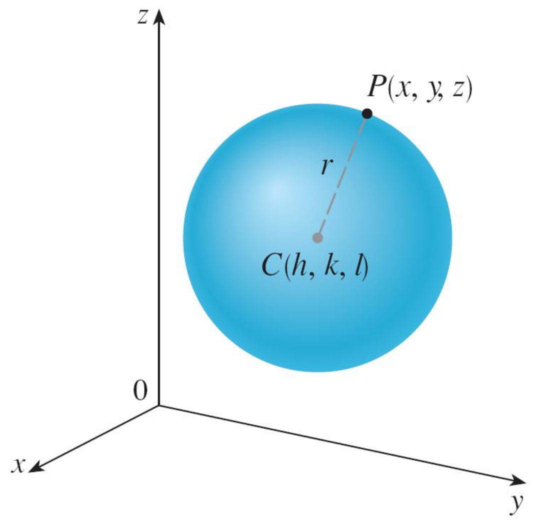


Figure 12

Example 5 – *Solution*

cont'd

Thus P is on the sphere if and only if $|PC| = r$.

Squaring both sides, we have

$$|PC|^2 = r^2$$

or

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

Three-Dimensional Coordinate Systems

The result of Example 5 is worth remembering.

Equation of a Sphere An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

12

Vectors and the Geometry of Space

12.2 Vectors

Vectors

The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction.

A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector.

We denote a vector by printing a letter in boldface (**v**) or by putting an arrow above the letter $(\vec{v})$.

Vectors

For instance, suppose a particle moves along a line segment from point *A* to point *B*.

The corresponding **displacement vector v**, shown in Figure 1, has **initial point** *A* (the tail) and **terminal point** *B* (the tip) and we indicate this by writing $\mathbf{v} = \overrightarrow{AB}$

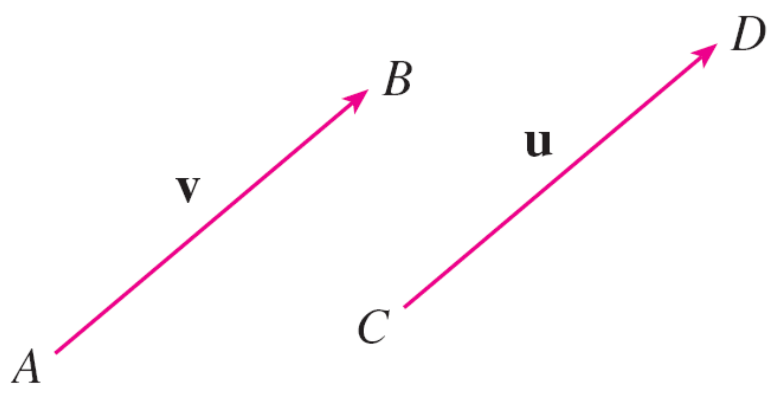


Figure 1

Equivalent vectors

Vectors

Notice that the vector $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as **v** even though it is in a different position.

We say that **u** and **v** are **equivalent** (or **equal**) and we write **u** = **v**.

The **zero vector**, denoted by **0**, has length 0. It is the only vector with no specific direction.

Combining Vectors

Combining Vectors

Suppose a particle moves from *A* to *B*, so its displacement vector is $\overrightarrow{AB}$. Then the particle changes direction and moves from *B* to *C*, with displacement vector $\overrightarrow{BC}$ as in Figure 2.

The combined effect of these displacements is that the particle has moved from *A* to *C*.

The resulting displacement vector $\overrightarrow{AC}$ is called the *sum* of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

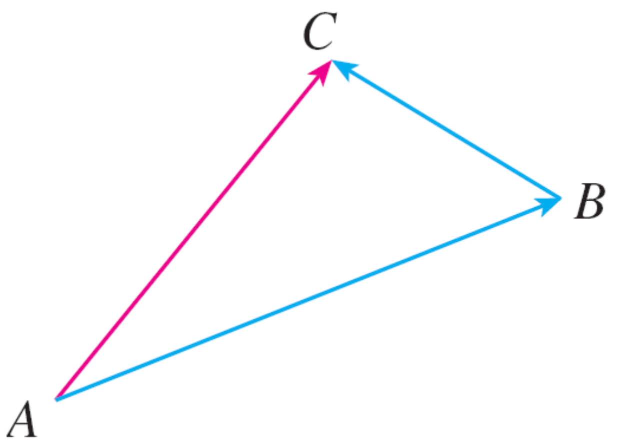


Figure 2

Combining Vectors

In general, if we start with vectors **u** and **v**, we first move **v** so that its tail coincides with the tip of **u** and define the sum of **u** and **v** as follows.

Definition of Vector Addition If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum u + v** is the vector from the initial point of **u** to the terminal point of **v**.

The definition of vector addition is illustrated in Figure 3. You can see why this definition is sometimes called the **Triangle Law**.

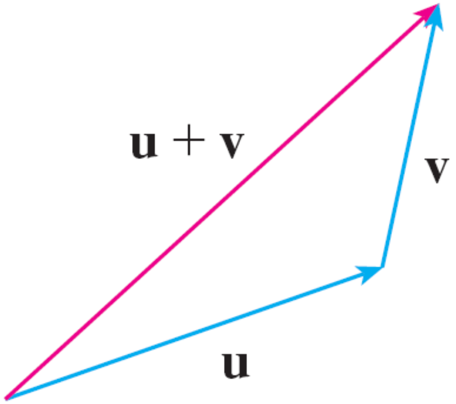


Figure 3

The Triangle Law

Combining Vectors

In Figure 4 we start with the same vectors **u** and **v** as in Figure 3 and draw another copy of **v** with the same initial point as **u**.

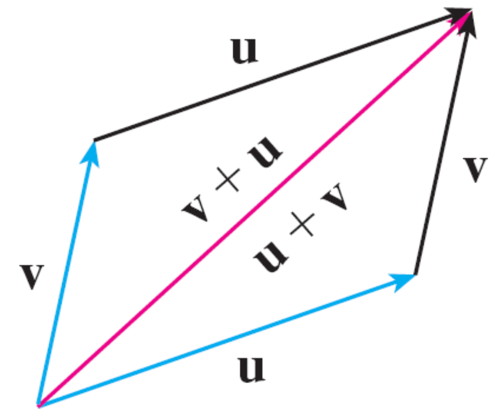


Figure 4

The Parallelogram Law

Completing the parallelogram, we see that **u** + **v** = **v** + **u**.

This also gives another way to construct the sum: If we place **u** and **v** so they start at the same point, then **u** + **v** lies along the diagonal of the parallelogram with **u** and **v** as sides. (This is called the **Parallelogram Law**.)

Example 1

Draw the sum of the vectors **a** and **b** shown in Figure 5.

Figure 5

Solution:

First we translate **b** and place its tail at the tip of **a**, being careful to draw a copy of **b** that has the same length and direction.

Example 1 – *Solution*

cont'd

Then we draw the vector **a** + **b** [see Figure 6(a)] starting at the initial point of **a** and ending at the terminal point of the copy of **b**.

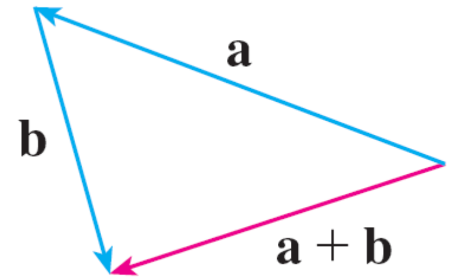


Figure 6(a)

Alternatively, we could place **b** so it starts where **a** starts and construct **a** + **b** by the Parallelogram Law as in Figure 6(b).

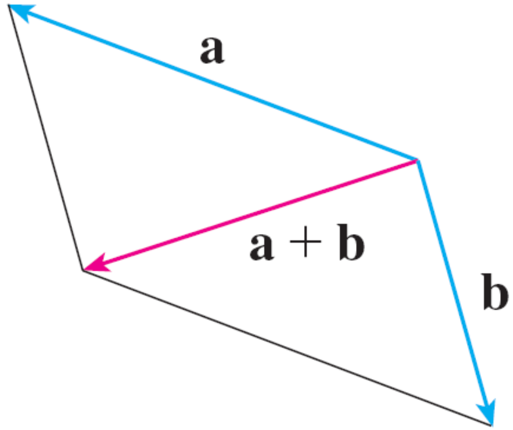


Figure 6(b)

Combining Vectors

It is possible to multiply a vector by a real number c. (In this context we call the real number c a **scalar** to distinguish it from a vector.)

For instance, we want 2**v** to be the same vector as **v** + **v**, which has the same direction as **v** but is twice as long. In general, we multiply a vector by a scalar as follows.

Definition of Scalar Multiplication If c is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

Combining Vectors

This definition is illustrated in Figure 7.

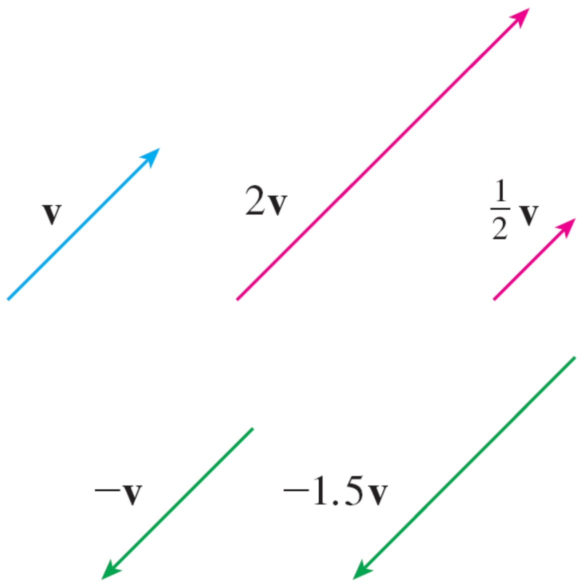


Figure 7

Scalar multiples of **v**

We see that real numbers work like scaling factors here; that's why we call them scalars.

Combining Vectors

Notice that two nonzero vectors are **parallel** if they are scalar multiples of one another.

In particular, the vector $-\mathbf{v} = (-1)\mathbf{v}$ has the same length as $\mathbf{v}$ but points in the opposite direction. We call it the **negative** of $\mathbf{v}$.

By the **difference** $\mathbf{u} - \mathbf{v}$ of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

Combining Vectors

So we can construct **u** – **v** by first drawing the negative of **v**, –**v**, and then adding it to **u** by the Parallelogram Law as in Figure 8(a).

Alternatively, since **v** + (**u** – **v**) = **u**, the vector **u** – **v**, when added to **v**, gives **u**. So we could construct **u** – **v** as in Figure 8(b) by means of the Triangle Law.

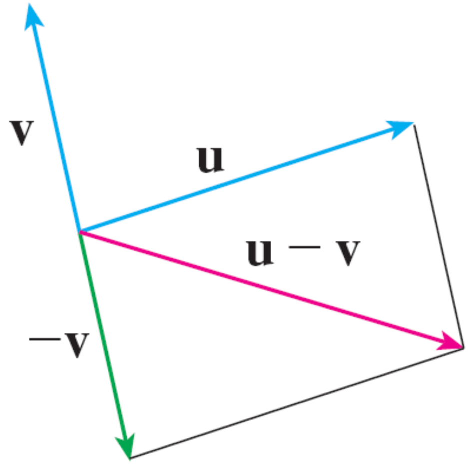


Figure 8(a)

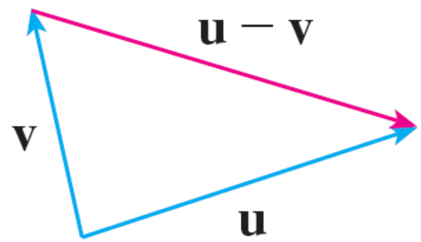


Figure 8(b)

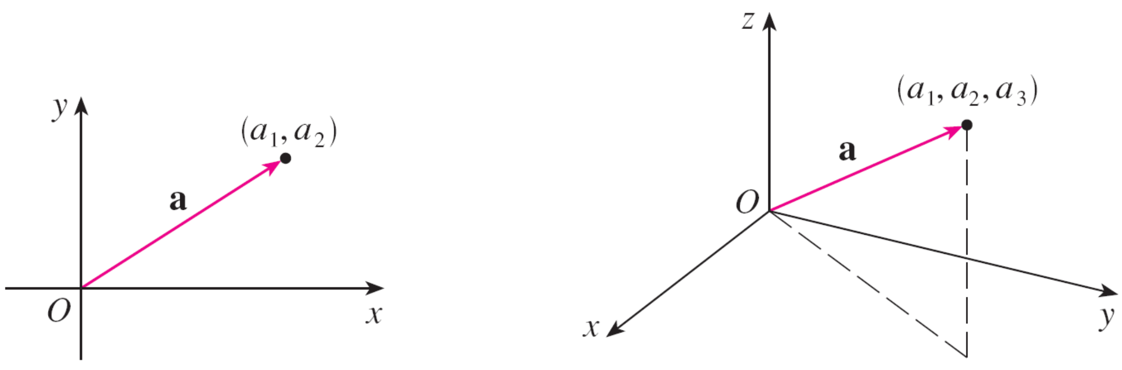
Drawing **u – v**

Components

Components

For some purposes it's best to introduce a coordinate system and treat vectors algebraically.

If we place the initial point of a vector **a** at the origin of a rectangular coordinate system, then the terminal point of **a** has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3), depending on whether our coordinate system is two- or three-dimensional (see Figure 11).



$\mathbf{a} = \langle a_1, a_2 \rangle$ $\quad$ $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Figure 11

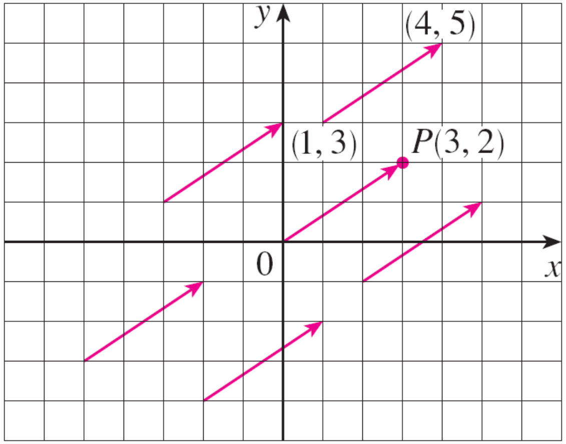
Components

These coordinates are called the **components** of **a** and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \qquad \text{or} \qquad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

We use the notation $\langle a_1, a_2 \rangle$ for the ordered pair that refers to a vector so as not to confuse it with the ordered pair (a_1, a_2) that refers to a point in the plane.

For instance, the vectors shown in Figure 12 are all equivalent to the vector $\overrightarrow{OP} = \langle 3, 2 \rangle$ whose terminal point is $P(3, 2)$.



Representations of the vector **a** = ⟨3, 2⟩

Figure 12

Components

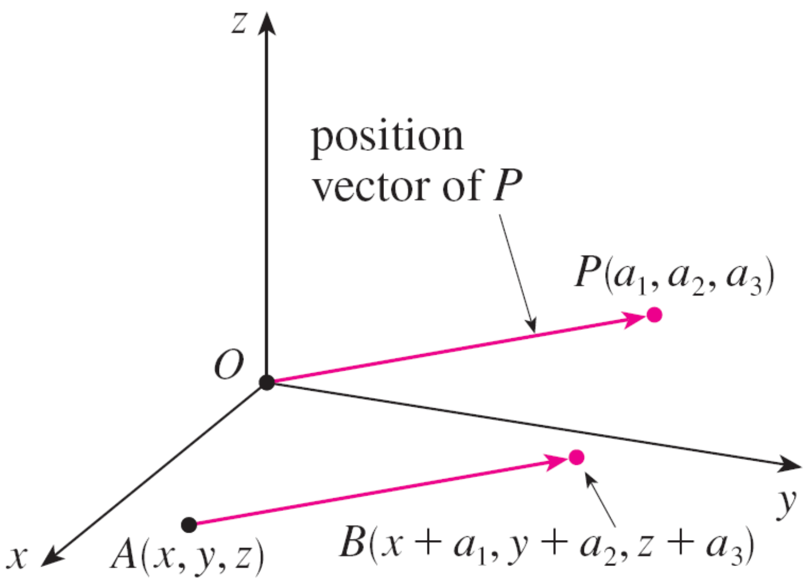
What they have in common is that the terminal point is reached from the initial point by a displacement of three units to the right and two upward.

We can think of all these geometric vectors as **representations** of the algebraic vector **a** = $\langle 3, 2 \rangle$.

The particular representation $\overrightarrow{OP}$ from the origin to the point $P(3, 2)$ is called the **position vector** of the point P.

Components

In three dimensions, the vector **a** = $\overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is the **position vector** of the point $P(a_1, a_2, a_3)$. (See Figure 13.)



Representations of **a** = $\langle a_1, a_2, a_3 \rangle$

Figure 13

Components

Let's consider any other representation $\overrightarrow{AB}$ of **a**, where the initial point is $A(x_1, y_1, z_1)$ and the terminal point is $B(x_2, y_2, z_2)$.

Then we must have $x_1 + a_1 = x_2$, $y_1 + a_2 = y_2$, and $z_1 + a_3 = z_2$ and so $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, and $a_3 = z_2 - z_1$.

Thus we have the following result.

1 Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Example 3

Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

Solution:

By $\boxed{1}$, the vector corresponding to $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle -2 - 2, 1 - (-3), 1 - 4 \rangle$$

$$= \langle -4, 4, -3 \rangle$$

Components

The **magnitude** or **length** of the vector **v** is the length of any of its representations and is denoted by the symbol |**v**| or ||**v**||. By using the distance formula to compute the length of a segment *OP*, we obtain the following formulas.

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Components

How do we add vectors algebraically? Figure 14 shows that if $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then the sum is $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, at least for the case where the components are positive.

In other words, *to add algebraic vectors we add their components*. Similarly, *to subtract vectors we subtract components*.

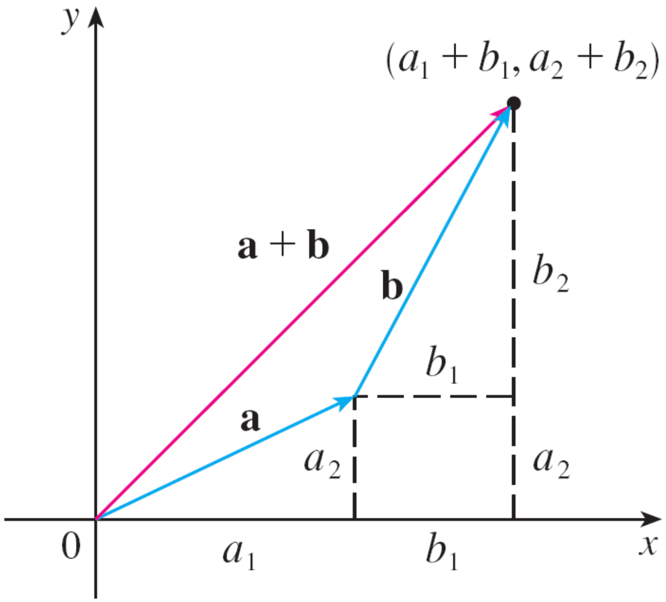


Figure 14

Components

From the similar triangles in Figure 15 we see that the components of $c\mathbf{a}$ are ca_1 and ca_2.

So *to multiply a vector by a scalar we multiply each component by that scalar*.

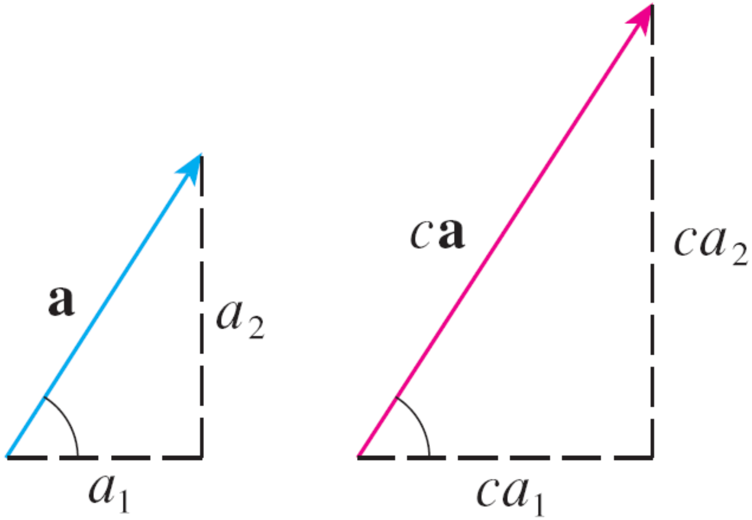


Figure 15

Components

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

Components

We denote by V_2 the set of all two-dimensional vectors and by V_3 the set of all three-dimensional vectors.

More generally, we will consider the set V_n of all n-dimensional vectors.

An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$$

where $a_1, a_2, \ldots, a_n$ are real numbers that are called the components of **a**.

Components

Addition and scalar multiplication are defined in terms of components just as for the cases *n* = 2 and *n* = 3.

Properties of Vectors If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Components

Three vectors in V_3 play a special role. Let

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

These vectors **i**, **j**, and **k** are called the **standard basis vectors**. They have length 1 and point in the directions of the positive *x*-, *y*-, and *z*-axes. Similarly, in two dimensions we define $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. (See Figure 17.)

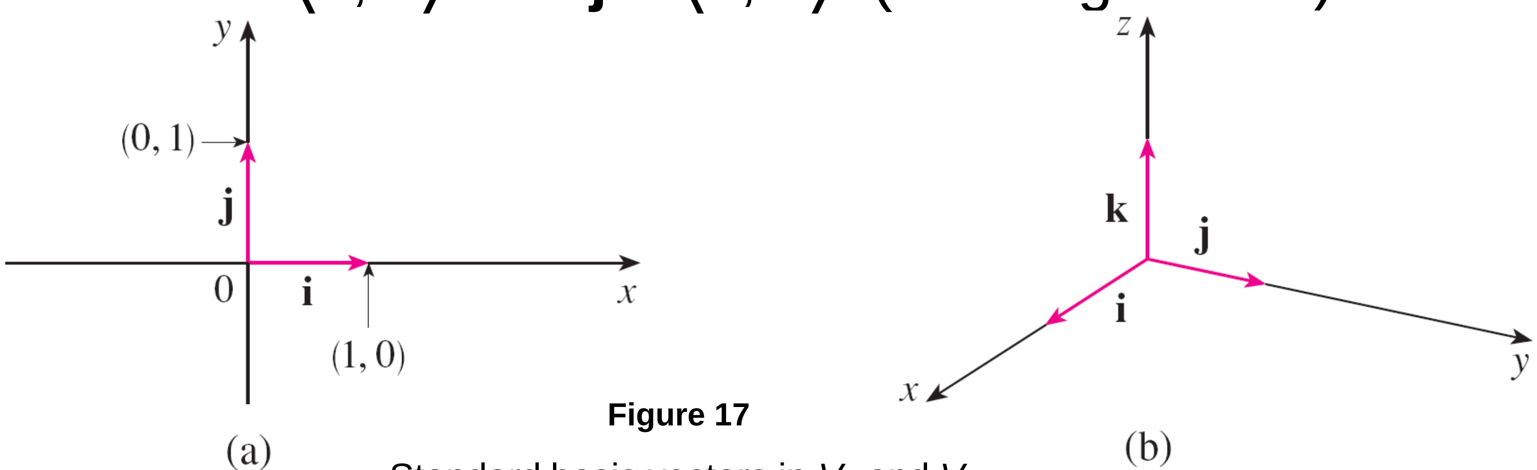


Figure 17

Standard basis vectors in V_2 and V_3

Components

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$

$$= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle$$

2 $$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Thus any vector in V_3 can be expressed in terms of **i**, **j**, and **k**. For instance,

$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Similarly, in two dimensions, we can write

3 $$\mathbf{a} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$$

Components

See Figure 18 for the geometric interpretation of Equations 3 and 2 and compare with Figure 17.

Standard basis vectors in V_2 and V_3

Figure 17

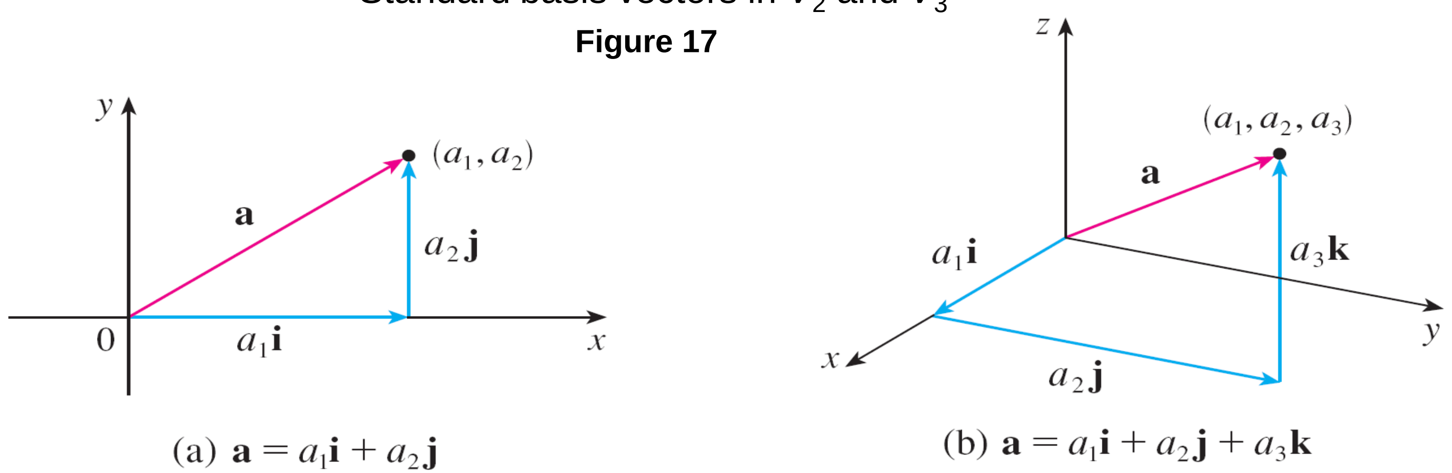


Figure 18

Components

A **unit vector** is a vector whose length is 1. For instance, **i**, **j**, and **k** are all unit vectors. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as **a** is

$$\boxed{4} \qquad \mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

In order to verify this, we let $c = 1/|\mathbf{a}|$. Then $\mathbf{u} = c\mathbf{a}$ and c is a positive scalar, so **u** has the same direction as **a**. Also

$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1$$

Applications

Applications

Vectors are useful in many aspects of physics and engineering. Here we look at forces.

A force is represented by a vector because it has both a magnitude (measured in pounds or newtons) and a direction.

If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

Example 7

A 100-lb weight hangs from two wires as shown in Figure 19. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and the magnitudes of the tensions.

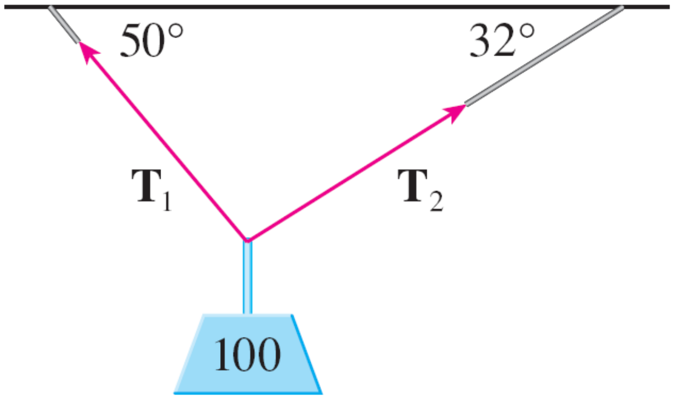


Figure 19

Example 7 – *Solution*

We first express $\mathbf{T}_1$ and $\mathbf{T}_2$ in terms of their horizontal and vertical components. From Figure 20 we see that

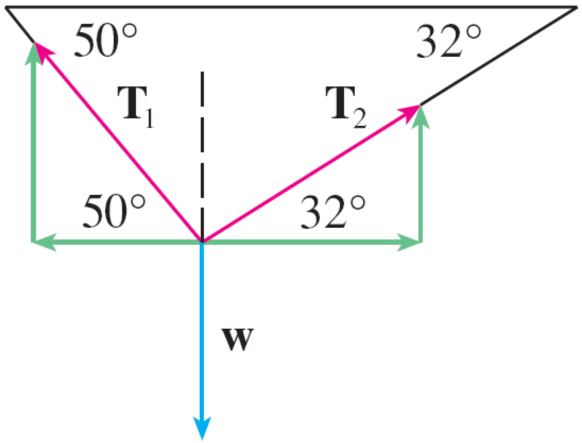


Figure 20

5 $$\mathbf{T}_1 = -|\mathbf{T}_1| \cos 50° \, \mathbf{i} + |\mathbf{T}_1| \sin 50° \, \mathbf{j}$$

6 $$\mathbf{T}_2 = |\mathbf{T}_2| \cos 32° \, \mathbf{i} + |\mathbf{T}_2| \sin 32° \, \mathbf{j}$$

The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w}$ and so we must have

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100 \, \mathbf{j}$$

Example 7 – *Solution*

cont'd

Thus

$(-|\mathbf{T}_1|\cos 50^\circ + |\mathbf{T}_2|\cos 32^\circ)\,\mathbf{I} + (|\mathbf{T}_1|\sin 50^\circ + |\mathbf{T}_2|\sin 32^\circ)\,\mathbf{j}$

$= 100\,\mathbf{j}$

Equating components, we get

$$-|\mathbf{T}_1|\cos 50^\circ + |\mathbf{T}_2|\cos 32^\circ = 0$$

$$|\mathbf{T}_1|\sin 50^\circ + |\mathbf{T}_2|\sin 32^\circ = 100$$

Solving the first of these equations for $|\mathbf{T}_2|$ and substituting into the second, we get

$$|\mathbf{T}_1|\sin 50^\circ + \frac{|\mathbf{T}_1|\cos 50^\circ}{\cos 32^\circ}\sin 32^\circ = 100$$

Example 7 – *Solution*

cont'd

So the magnitudes of the tensions are

$$|\mathbf{T}_1| = \frac{100}{\sin 50^\circ + \tan 32^\circ \cos 50^\circ}$$

$$\approx 85.64 \text{ lb}$$

and $$|\mathbf{T}_2| = \frac{|\mathbf{T}_1| \cos 50^\circ}{\cos 32^\circ}$$

$$\approx 64.91 \text{ lb}$$

Substituting these values in $\boxed{5}$ and $\boxed{6}$ we obtain the tension vectors

$$\mathbf{T}_1 \approx -55.05\mathbf{i} + 65.60\mathbf{j} \qquad \mathbf{T}_2 \approx 55.05\mathbf{i} + 34.40\mathbf{j}$$

12

Vectors and the Geometry of Space

12.3 The Dot Product

The Dot Product

So far we have added two vectors and multiplied a vector by a scalar. The question arises: Is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, whose definition follows.

1 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Thus, to find the dot product of **a** and **b**, we multiply corresponding components and add.

The Dot Product

The result is not a vector. It is a real number, that is, a scalar. For this reason, the dot product is sometimes called the **scalar product** (or **inner product**).

Although Definition 1 is given for three-dimensional vectors, the dot product of two-dimensional vectors is defined in a similar fashion:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

Example 1

$$\langle 2, 4\rangle \cdot \langle 3, -1\rangle = 2(3) + 4(-1)$$

$$= 2$$

$$\langle -1, 7, 4\rangle \cdot \langle 6, 2, -\tfrac{1}{2}\rangle = (-1)(6) + 7(2) + 4\left(-\tfrac{1}{2}\right)$$

$$= 6$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1)$$

$$= 7$$

The Dot Product

The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated in the following theorem.

$\boxed{2}$ **Properties of the Dot Product** If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

The Dot Product

These properties are easily proved using Definition 1. For instance, here are the proofs of Properties 1 and 3:

1. $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$

3. $$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\ &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\ &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \end{aligned}$$

The Dot Product

The dot product **a** · **b** can be given a geometric interpretation in terms of the **angle** θ **between a and b**, which is defined to be the angle between the representations of **a** and **b** that start at the origin, where $0 \leq \theta \leq \pi$.

In other words, θ is the angle between the line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in Figure 1. Note that if **a** and **b** are parallel vectors, then $\theta = 0$ or $\theta = \pi$.

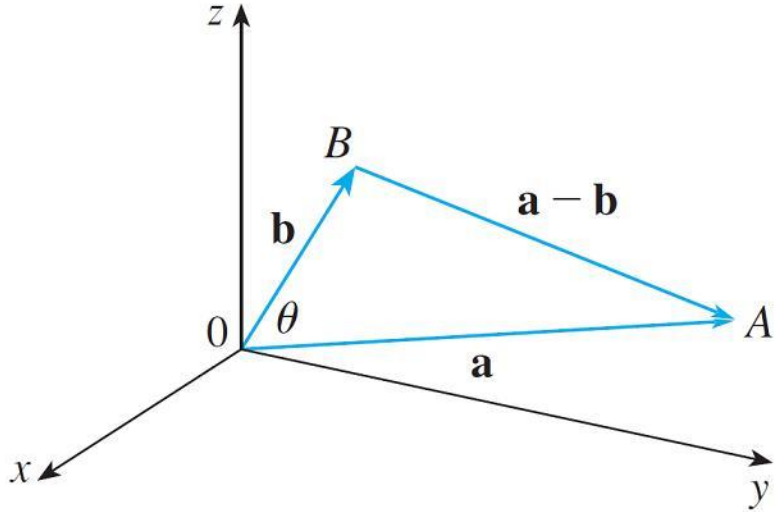


Figure 1

The Dot Product

The formula in the following theorem is used by physicists as the *definition* of the dot product.

3 Theorem If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

Example 2

If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

Solution:

Using Theorem 3, we have

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\pi/3)$$

$$= 4 \cdot 6 \cdot \frac{1}{2}$$

$$= 12$$

The Dot Product

The formula in Theorem 3 also enables us to find the angle between two vectors.

6 Corollary If θ is the angle between the nonzero vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Example 3

Find the angle between the vectors **a** = ⟨2, 2, –1⟩ and **b** = ⟨5, –3, 2⟩.

Solution:

Since

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \quad \text{and} \quad |\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since

$$\mathbf{a} \cdot \mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$$

Example 3 – *Solution*

cont'd

We have, from Corollary 6,

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

So the angle between **a** and **b** is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right)$$

$$\approx 1.46 \quad (\text{or } 84^\circ)$$

The Dot Product

Two nonzero vectors **a** and **b** are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$. Then Theorem 3 gives

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\pi/2) = 0$$

and conversely if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos\theta = 0$, so $\theta = \pi/2$. The zero vector **0** is considered to be perpendicular to all vectors.

Therefore we have the following method for determining whether two vectors are orthogonal.

7 Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Example 4

Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

Solution:

Since

$$(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 2(5) + 2(-4) + (-1)(2) = 0$$

these vectors are perpendicular by $\boxed{7}$.

The Dot Product

Because $\cos\theta > 0$ if $0 \le \theta < \pi/2$ and $\cos\theta < 0$ if $\pi/2 < \theta \le \pi$, we see that $\mathbf{a} \cdot \mathbf{b}$ is positive for $\theta < \pi/2$ and negative for $\theta > \pi/2$. We can think of $\mathbf{a} \cdot \mathbf{b}$ as measuring the extent to which **a** and **b** point in the same direction.

The dot product $\mathbf{a} \cdot \mathbf{b}$ is positive if **a** and **b** point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions (see Figure 2).

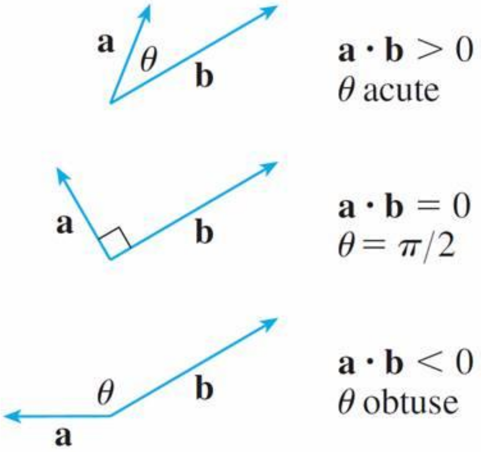


Figure 2

The Dot Product

In the extreme case where **a** and **b** point in exactly the same direction, we have $\theta = 0$, so $\cos\theta = 1$ and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$$

If **a** and **b** point in exactly opposite directions, then $\theta = \pi$ and so $\cos\theta = -1$ and $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}|\,|\mathbf{b}|$.

Direction Angles and Direction Cosines

Direction Angles and Direction Cosines

The **direction angles** of a nonzero vector **a** are the angles α, β, and γ (in the interval $[0, \pi]$) that **a** makes with the positive *x*-, *y*-, and *z*-axes. (See Figure 3.)

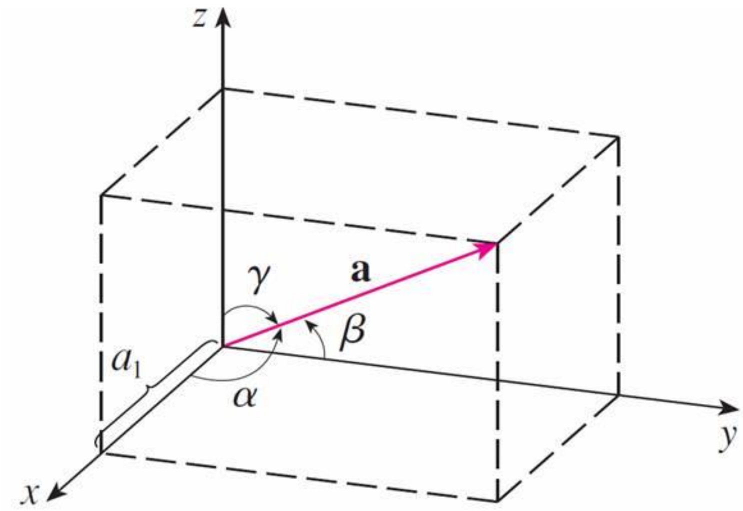


Figure 3

Direction Angles and Direction Cosines

The cosines of these direction angles, cos α, cos β, and cos γ, are called the **direction cosines** of the vector **a**. Using Corollary 6 with **b** replaced by **i**, we obtain

$$\boxed{8} \qquad \cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

(This can also be seen directly from Figure 3.) Similarly, we also have

$$\boxed{9} \qquad \cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

Direction Angles and Direction Cosines

By squaring the expressions in Equations 8 and 9 and adding, we see that

$$\boxed{10} \qquad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

We can also use Equations 8 and 9 to write

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle |\mathbf{a}| \cos \alpha, |\mathbf{a}| \cos \beta, |\mathbf{a}| \cos \gamma \rangle$$

$$= |\mathbf{a}| \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$

Direction Angles and Direction Cosines

Therefore

$$\boxed{11} \qquad \frac{1}{|\mathbf{a}|}\mathbf{a} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

which says that the direction cosines of **a** are the components of the unit vector in the direction of **a**.

Example 5

Find the direction angles of the vector **a** = ⟨1, 2, 3⟩.

Solution:

Since $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, Equations 8 and 9 give

$$\cos\alpha = \frac{1}{\sqrt{14}} \qquad \cos\beta = \frac{2}{\sqrt{14}} \qquad \cos\gamma = \frac{3}{\sqrt{14}}$$

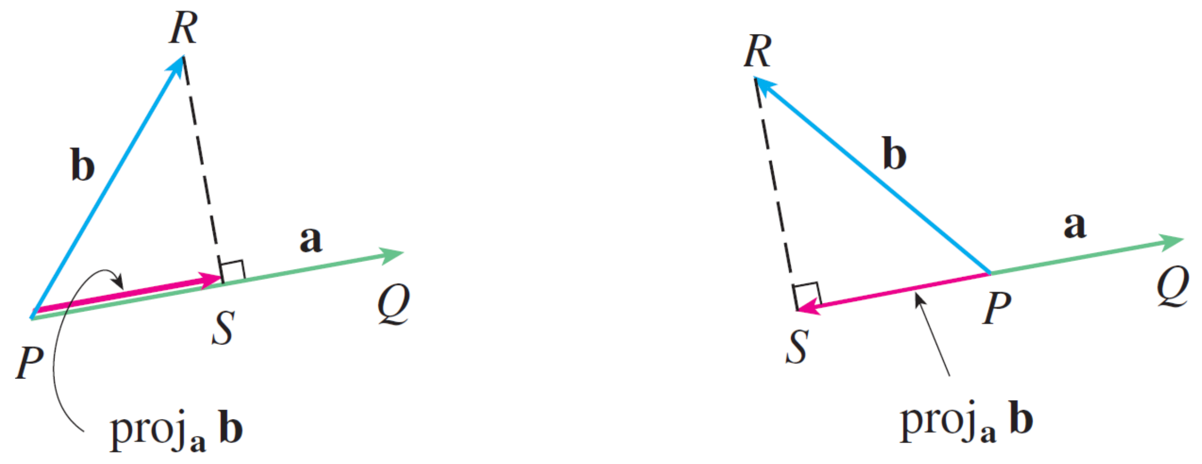
and so

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74^\circ \qquad \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58^\circ \qquad \gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37^\circ$$

Projections

Projections

Figure 4 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with the same initial point *P*. If *S* is the foot of the perpendicular from *R* to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of **b** onto **a** and is denoted by $\text{proj}_{\mathbf{a}}\, \mathbf{b}$.
(You can think of it as a shadow of **b**).

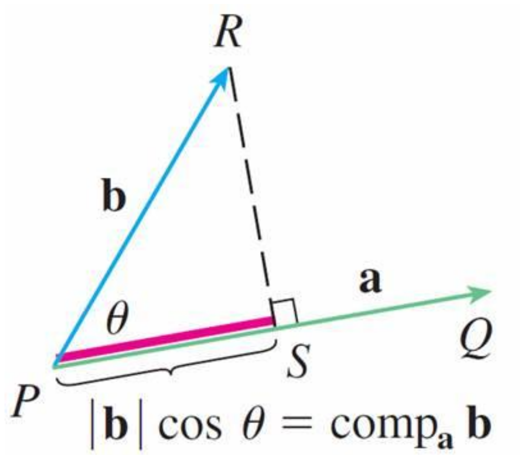


Vector projections

Figure 4

Projections

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}| \cos \theta$, where θ is the angle between **a** and **b**.
(See Figure 5.)



Scalar projection

Figure 5

Projections

This is denoted by $\text{comp}_{\mathbf{a}}\,\mathbf{b}$. Observe that it is negative if $\pi/2 < \theta \leq \pi$. The equation

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \theta = |\mathbf{a}|(|\mathbf{b}| \cos \theta)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**. Since

$$|\mathbf{b}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**.

Projections

We summarize these ideas as follows.

Scalar projection of **b** onto **a**: $\qquad \operatorname{comp}_{\mathbf{a}} \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\qquad \operatorname{proj}_{\mathbf{a}} \mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\, \mathbf{a}$

Notice that the vector projection is the scalar projection times the unit vector in the direction of **a**.

Example 6

Find the scalar projection and vector projection of **b** = ⟨1, 1, 2⟩ onto **a** = ⟨–2, 3, 1⟩.

Solution:

Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of **b** onto **a** is

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

$$= \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}}$$

$$= \frac{3}{\sqrt{14}}$$

Example 6 – *Solution*

cont'd

The vector projection is this scalar projection times the unit vector in the direction of **a**:

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{3}{\sqrt{14}}\frac{\mathbf{a}}{|\mathbf{a}|}$$

$$= \frac{3}{14}\mathbf{a}$$

$$= \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

Projections

The work done by a constant force *F* in moving an object through a distance *d* as *W* = *Fd*, but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction, as in Figure 6.

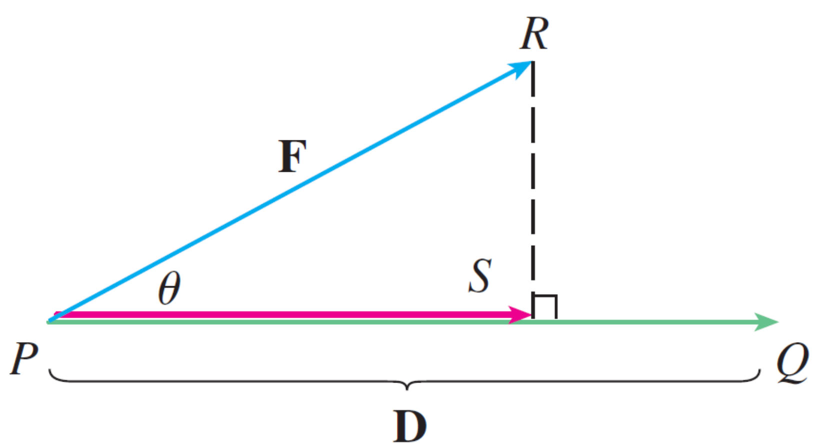


Figure 6

Projections

If the force moves the object from P to Q, then the **displacement vector** is $\mathbf{D} = \overrightarrow{PQ}$. The **work** done by this force is defined to be the product of the component of the force along **D** and the distance moved:

$$W = (|\mathbf{F}| \cos \theta)\, |\mathbf{D}|$$

But then, from Theorem 3, we have

$$\boxed{12} \qquad W = |\mathbf{F}|\, |\mathbf{D}| \cos \theta = \mathbf{F} \cdot \mathbf{D}$$

Thus the work done by a constant force **F** is the dot product **F** · **D**, where **D** is the displacement vector.

Example 7

A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.

Solution:

If **F** and **D** are the force and displacement vectors, as pictured in Figure 7, then the work done is

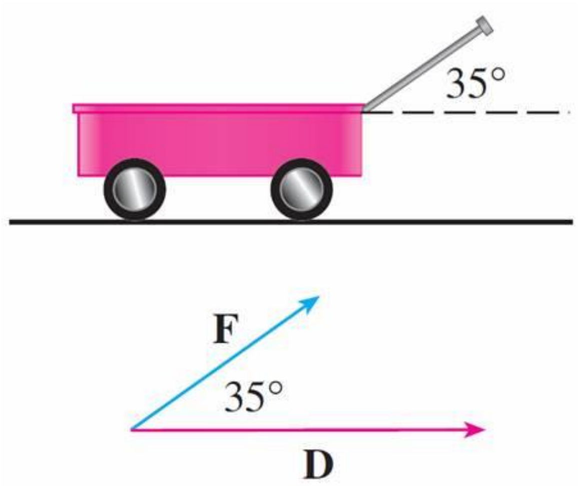


Figure 7

$$W = \mathbf{F} \cdot \mathbf{D}$$

$$= |\mathbf{F}|\,|\mathbf{D}| \cos 35°$$

Example 7 – *Solution*

cont'd

$= (70)(100) \cos 35°$

≈ 5734 N•m

$= 5734$ J

12

Vectors and the Geometry of Space

12.4 The Cross Product

The Cross Product

Given two nonzero vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$, it is very useful to be able to find a nonzero vector **c** that is perpendicular to both **a** and **b**.

If $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is such a vector, then $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$ and so

$$\boxed{1} \qquad a_1c_1 + a_2c_2 + a_3c_3 = 0$$

$$\boxed{2} \qquad b_1c_1 + b_2c_2 + b_3c_3 = 0$$

The Cross Product

To eliminate c_3 we multiply $\boxed{1}$ by b_3 and $\boxed{2}$ by a_3 and subtract:

$$\boxed{3} \qquad (a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0$$

Equation 3 has the form $pc_1 + qc_2 = 0$, for which an obvious solution is $c_1 = q$ and $c_2 = -p$. So a solution of $\boxed{3}$ is

$$c_1 = a_2b_3 - a_3b_2 \qquad\qquad c_2 = a_3b_1 - a_1b_3$$

The Cross Product

Substituting these values into $\boxed{1}$ and $\boxed{2}$, we then get

$$c_3 = a_1b_2 - a_2b_1$$

This means that a vector perpendicular to both **a** and **b** is

$$\langle c_1, c_2, c_3\rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

The resulting vector is called the *cross product* of **a** and **b** and is denoted by **a** × **b**.

The Cross Product

4 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Notice that the **cross product a × b** of two vectors **a** and **b**, unlike the dot product, is a vector. For this reason it is also called the **vector product**.

Note that **a** × **b** is defined only when **a** and **b** are *three-dimensional* vectors.

The Cross Product

In order to make Definition 4 easier to remember, we use the notation of determinants.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

The Cross Product

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$\boxed{5} \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Observe that each term on the right side of Equation 5 involves a number a_i in the first row of the determinant, and a_i is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which a_i appears.

The Cross Product

Notice also the minus sign in the second term. For example,

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0 – 4) – 2(6 + 5) + (–1)(12 – 0)$$

$$= –38$$

The Cross Product

If we now rewrite Definition 4 using second-order determinants and the standard basis vectors **i**, **j**, and **k**, we see that the cross product of the vectors

$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$ is

$$\boxed{6} \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

The Cross Product

In view of the similarity between Equations 5 and 6, we often write

$$\boxed{7} \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Although the first row of the symbolic determinant in Equation 7 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 5, we obtain Equation 6.

The symbolic formula in Equation 7 is probably the easiest way of remembering and computing cross products.

Example 1

If **a** = ⟨1, 3, 4⟩ and **b** = ⟨2, 7, –5⟩, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

$$= (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k}$$

$$= -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$

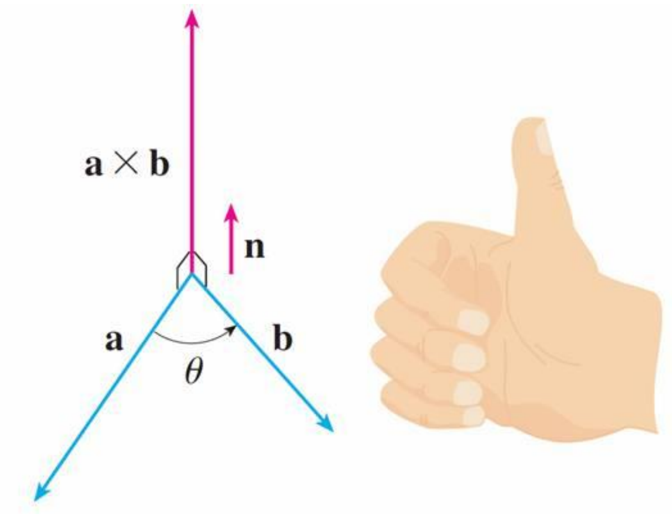
The Cross Product

We constructed the cross product $\mathbf{a} \times \mathbf{b}$ so that it would be perpendicular to both **a** and **b**. This is one of the most important properties of a cross product.

$\boxed{8}$ **Theorem** The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**.

The Cross Product

If **a** and **b** are represented by directed line segments with the same initial point (as in Figure 1), then Theorem 8 says that the cross product **a** × **b** points in a direction perpendicular to the plane through **a** and **b**.



The right-hand rule gives the direction of **a** × **b**.

Figure 1

The Cross Product

It turns out that the direction of $\mathbf{a} \times \mathbf{b}$ is given by the *right-hand rule*: If the fingers of your right hand curl in the direction of a rotation (through an angle less than 180°) from to $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

Now that we know the direction of the vector $\mathbf{a} \times \mathbf{b}$, the remaining thing we need to complete its geometric description is its length $|\mathbf{a} \times \mathbf{b}|$. This is given by the following theorem.

9 Theorem If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

The Cross Product

Since a vector is completely determined by its magnitude and direction, we can now say that $\mathbf{a} \times \mathbf{b}$ is the vector that is perpendicular to both **a** and **b**, whose orientation is determined by the right-hand rule, and whose length is $|\mathbf{a}|\,|\mathbf{b}|\sin\theta$. In fact, that is exactly how physicists *define* $\mathbf{a} \times \mathbf{b}$.

10 Corollary Two nonzero vectors **a** and **b** are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

The Cross Product

The geometric interpretation of Theorem 9 can be seen by looking at Figure 2.

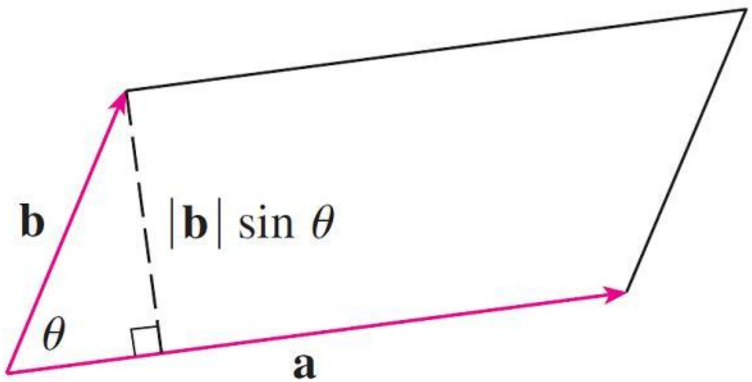


Figure 2

The Cross Product

If **a** and **b** are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $|\mathbf{b}|\sin\theta$, and area

$$A = |\mathbf{a}|(|\mathbf{b}|\sin\theta) = |\mathbf{a} \times \mathbf{b}|$$

Thus we have the following way of interpreting the magnitude of a cross product.

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by **a** and **b**.

Example 4

Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Solution:

In Example 3 we computed that $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle$. The area of the parallelogram with adjacent sides PQ and PR is the length of this cross product:

$$\left| \overrightarrow{PQ} \times \overrightarrow{PR} \right| = \sqrt{(-40)^2 + (-15)^2 + 15^2}$$

$$= 5\sqrt{82}$$

The area A of the triangle PQR is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$.

The Cross Product

If we apply Theorems 8 and 9 to the standard basis vectors **i**, **j**, and **k** using $\theta = \pi/2$, we obtain

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Observe that

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$

The Cross Product

Thus the cross product is not commutative. Also

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

whereas

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = 0 \times \mathbf{j} = 0$$

So the associative law for multiplication does not usually hold; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

However, some of the usual laws of algebra *do* hold for cross products.

The Cross Product

The following theorem summarizes the properties of vector products.

11 Theorem If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

The Cross Product

These properties can be proved by writing the vectors in terms of their components and using the definition of a cross product.

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then

$$\boxed{12} \qquad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

Triple Products

Triple Products

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that occurs in Property 5 is called the **scalar triple product** of the vectors **a**, **b**, and **c**. Notice from Equation 12 that we can write the scalar triple product as a determinant:

$$\boxed{13} \qquad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Triple Products

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors **a**, **b**, and **c**. (See Figure 3.)

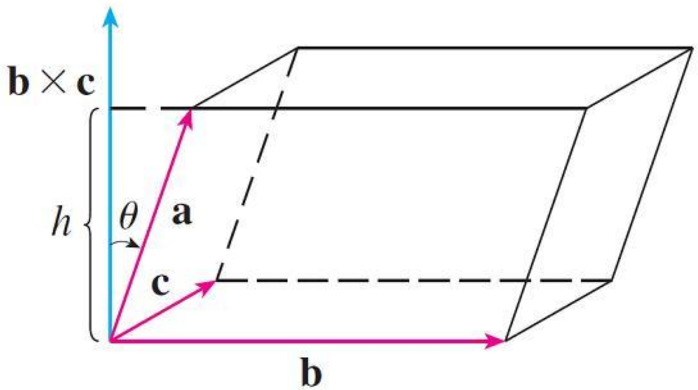


Figure 3

The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$.

Triple Products

If θ is the angle between **a** and **b** × **c**, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. (We must use $|\cos\theta|$ instead of $\cos\theta$ in case $\theta > \pi/2$.) Therefore the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Thus we have proved the following formula.

$\boxed{14}$ The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Triple Products

If we use the formula in $\boxed{14}$ and discover that the volume of the parallelepiped determined by **a**, **b**, and **c** is 0, then the vectors must lie in the same plane; that is, they are **coplanar**.

Example 5

Use the scalar triple product to show that the vectors **a** = ⟨1, 4, –7⟩, **b** = ⟨2, –1, 4⟩, and **c** = ⟨0, –9, 18⟩ are coplanar.

Solution:

We use Equation 13 to compute their scalar triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

Example 5 – *Solution*

cont'd

$$= 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7\begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$= 1(18) - 4(36) - 7(-18)$

$= 0$

Therefore, by $\boxed{14}$, the volume of the parallelepiped determined by **a**, **b**, and **c** is 0. This means that **a**, **b**, and **c** are coplanar.

Triple Products

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ that occurs in Property 6 is called the **vector triple product** of **a**, **b**, and **c**.

Torque

Torque

The idea of a cross product occurs often in physics.
In particular, we consider a force **F** acting on a rigid body at a point given by a position vector **r**. (For instance, if we tighten a bolt by applying a force to a wrench as in Figure 4, we produce a turning effect.)

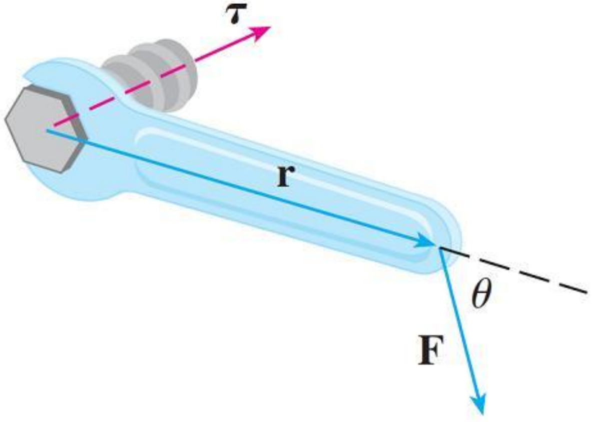


Figure 4

Torque

The **torque** $\boldsymbol{\tau}$ (relative to the origin) is defined to be the cross product of the position and force vectors

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation.

$\boxed{9}$ **Theorem** If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

Torque

According to Theorem 9, the magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin \theta$$

where θ is the angle between the position and force vectors. Observe that the only component of **F** that can cause a rotation is the one perpendicular to **r**, that is, $|\mathbf{F}| \sin \theta$.

The magnitude of the torque is equal to the area of the parallelogram determined by **r** and **F**.

Example 6

A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 5.

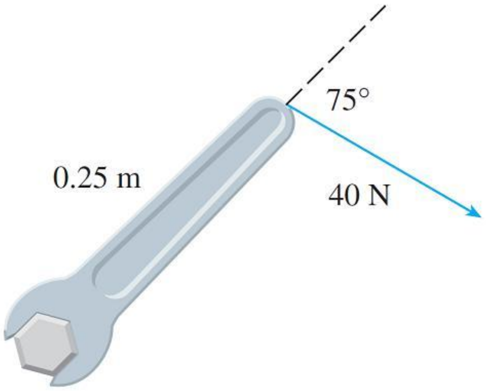


Figure 5

Find the magnitude of the torque about the center of the bolt.

Example 6 – *Solution*

The magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin 75°$$

$$= (0.25)(40) \sin 75°$$

$$= 10 \sin 75° \approx 9.66 \text{ N}\cdot\text{m}$$

If the bolt is right-threaded, then the torque vector itself is

$$\boldsymbol{\tau} = |\boldsymbol{\tau}|\, \mathbf{n} \approx 9.66\, \mathbf{n}$$

where **n** is a unit vector directed down into the page.

12

Vectors and the Geometry of Space

12.5 Equations of Lines and Planes

Equations of Lines and Planes

A line in the *xy*-plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given.

The equation of the line can then be written using the point-slope form.

Likewise, a line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L. In three dimensions the direction of a line is conveniently described by a vector, so we let **v** be a vector parallel to L.

Equations of Lines and Planes

Let $P(x, y, z)$ be an arbitrary point on L and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P (that is, they have representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$).

If $\mathbf{a}$ is the vector with representation $\overrightarrow{P_0P}$, as in Figure 1, then the Triangle Law for vector addition gives $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$.

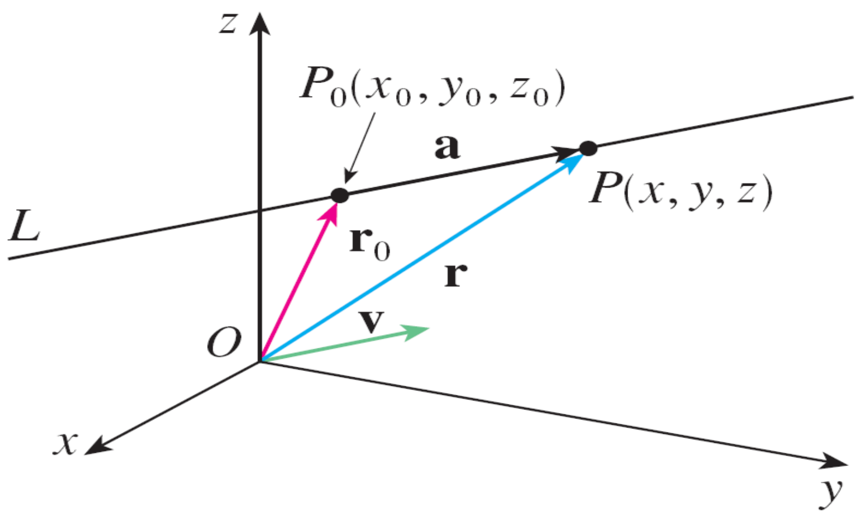


Figure 1

Equations of Lines and Planes

But, since **a** and **v** are parallel vectors, there is a scalar *t* such that **a** = *t***v**. Thus

1

$$\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v}$$

which is a **vector equation** of *L*.

Each value of the **parameter** *t* gives the position vector **r** of a point on *L*. In other words, as *t* varies, the line is traced out by the tip of the vector **r**.

Equations of Lines and Planes

As Figure 2 indicates, positive values of t correspond to points on L that lie on one side of P_0, whereas negative values of t correspond to points that lie on the other side of P_0.

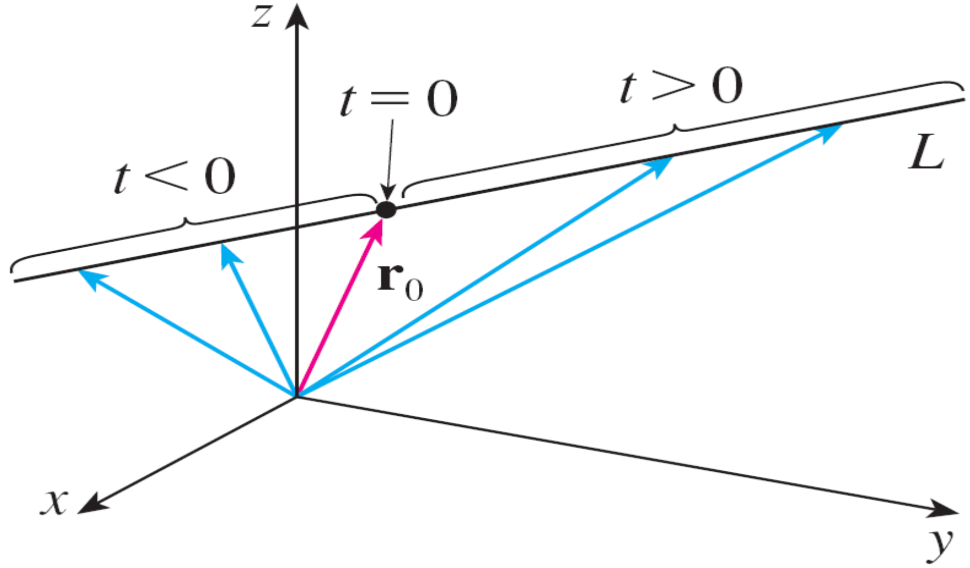


Figure 2

Equations of Lines and Planes

If the vector **v** that gives the direction of the line L is written in component form as $\mathbf{v} = \langle a, b, c \rangle$, then we have $t\mathbf{v} = \langle ta, tb, tc \rangle$.

We can also write $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, so the vector equation $\boxed{1}$ becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Two vectors are equal if and only if corresponding components are equal.

Equations of Lines and Planes

Therefore we have the three scalar equations:

2

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

where $t \in \mathbb{R}$.

These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$.

Each value of the parameter t gives a point (x, y, z) on L.

Example 1

(a) Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

(b) Find two other points on the line.

Solution:

(a) Here $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so the vector equation $\boxed{1}$ becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or

$$\mathbf{r} = (5 + t)\,\mathbf{i} + (1 + 4t)\,\mathbf{j} + (3 - 2t)\,\mathbf{k}$$

Example 1 – *Solution*

cont'd

Parametric equations are

$$x = 5 + t \qquad y = 1 + 4t \qquad z = 3 - 2t$$

(b) Choosing the parameter value $t = 1$ gives $x = 6$, $y = 5$, and $z = 1$, so (6, 5, 1) is a point on the line.

Similarly, $t = -1$ gives the point (4, –3, 5).

Equations of Lines and Planes

The vector equation and parametric equations of a line are not unique. If we change the point or the parameter or choose a different parallel vector, then the equations change.

For instance, if, instead of (5, 1, 3), we choose the point (6, 5, 1) in Example 1, then the parametric equations of the line become

$$x = 6 + t \qquad y = 5 + 4t \qquad z = 1 - 2t$$

Equations of Lines and Planes

Or, if we stay with the point (5, 1, 3) but choose the parallel vector 2**i** + 8**j** – 4**k**, we arrive at the equations

$$x = 5 + 2t \qquad y = 1 + 8t \qquad z = 3 - 4t$$

In general, if a vector **v** = $\langle a, b, c \rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers** of L.

Since any vector parallel to **v** could also be used, we see that any three numbers proportional to a, b, and c could also be used as a set of direction numbers for L.

Equations of Lines and Planes

Another way of describing a line *L* is to eliminate the parameter *t* from Equations 2.

If none of *a*, *b*, or *c* is 0, we can solve each of these equations for *t*, equate the results, and obtain

3

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These equations are called **symmetric equations** of *L*.

Equations of Lines and Planes

Notice that the numbers *a*, *b*, and *c* that appear in the denominators of Equations 3 are direction numbers of *L*, that is, components of a vector parallel to *L*.

If one of *a*, *b*, or *c* is 0, we can still eliminate *t*. For instance, if *a* = 0, we could write the equations of *L* as

$$x = x_0 \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This means that *L* lies in the vertical plane $x = x_0$.

Equations of Lines and Planes

In general, we know from Equation 1 that the vector equation of a line through the (tip of the) vector $\mathbf{r}_0$ in the direction of a vector $\mathbf{v}$ is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$.

If the line also passes through (the tip of) $\mathbf{r}_1$, then we can take $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$ and so its vector equation is

$$\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$$

The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the parameter interval $0 \leq t \leq 1$.

4 The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \leqslant t \leqslant 1$$

Planes

Planes

Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe.

A single vector parallel to a plane is not enough to convey the “direction” of the plane, but a vector perpendicular to the plane does completely specify its direction.

Thus a plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector **n** that is orthogonal to the plane. This orthogonal vector **n** is called a **normal vector**.

Planes

Let $P(x, y, z)$ be an arbitrary point in the plane, and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P.

Then the vector $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$. Figure 6.)

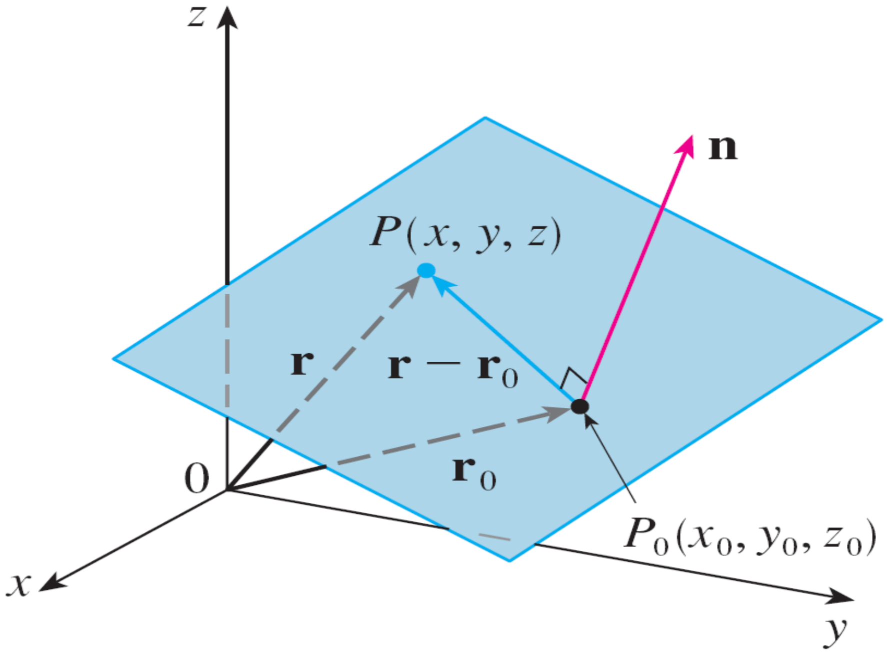


Figure 6

Planes

The normal vector **n** is orthogonal to every vector in the given plane. In particular, **n** is orthogonal to $\mathbf{r} - \mathbf{r}_0$ and so we have

5 $$\boxed{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0}$$

which can be rewritten as

6 $$\boxed{\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0}$$

Either Equation 5 or Equation 6 is called a **vector equation of the plane**.

Planes

To obtain a scalar equation for the plane, we write

$\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$.

Then the vector equation $\boxed{5}$ becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or

7 $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Equation 7 is the **scalar equation of the plane through** $P_0(x_0, y_0, z_0)$ **with normal vector** $\mathbf{n} = \langle a, b, c \rangle$.

Example 4

Find an equation of the plane through the point (2, 4, –1) with normal vector **n** = $\langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.

Solution:

Putting $a = 2$, $b = 3$, $c = 4$, $x_0 = 2$, $y_0 = 4$, and $z_0 = -1$ in Equation 7, we see that an equation of the plane is

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

or

$$2x + 3y + 4z = 12$$

To find the x-intercept we set $y = z = 0$ in this equation and obtain $x = 6$.

Example 4 – *Solution*

cont'd

Similarly, the y-intercept is 4 and the z-intercept is 3. This enables us to sketch the portion of the plane that lies in the first octant (see Figure 7).

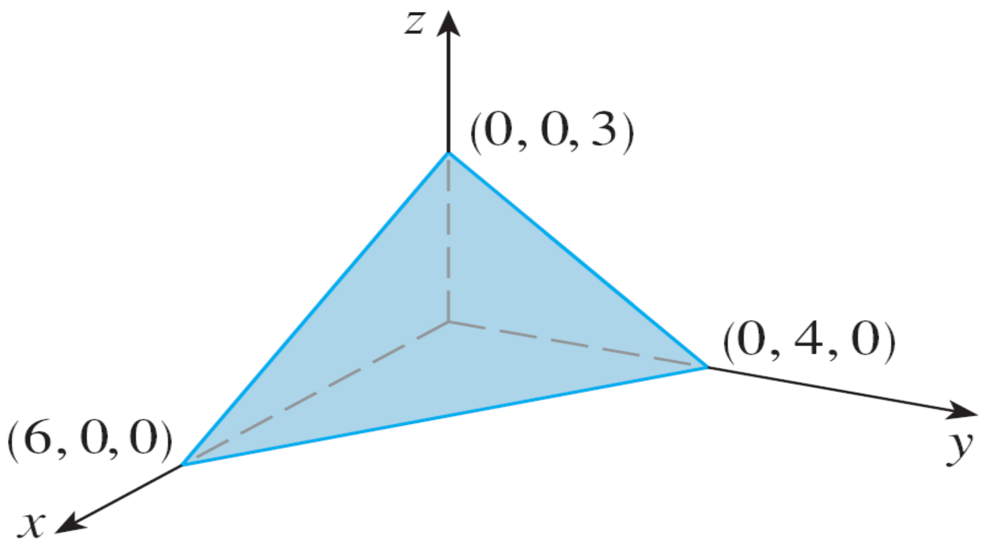


Figure 7

Planes

By collecting terms in Equation 7 as we did in Example 4, we can rewrite the equation of a plane as

8
$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$.

Equation 8 is called a **linear equation** in x, y, and z. Conversely, it can be shown that if a, b, and c are not all 0, then the linear equation $\boxed{8}$ represents a plane with normal vector $\langle a, b, c\rangle$.

Planes

Two planes are **parallel** if their normal vectors are parallel.

For instance, the planes $x + 2y - 3z = 4$ and $2x + 4y - 6z = 3$ are parallel because their normal vectors are $\mathbf{n}_1 = \langle 1, 2, -3 \rangle$ and $\mathbf{n}_2 = \langle 2, 4, -6 \rangle$ and $\mathbf{n}_2 = 2\mathbf{n}_1$.

If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors (see angle θ in Figure 9).

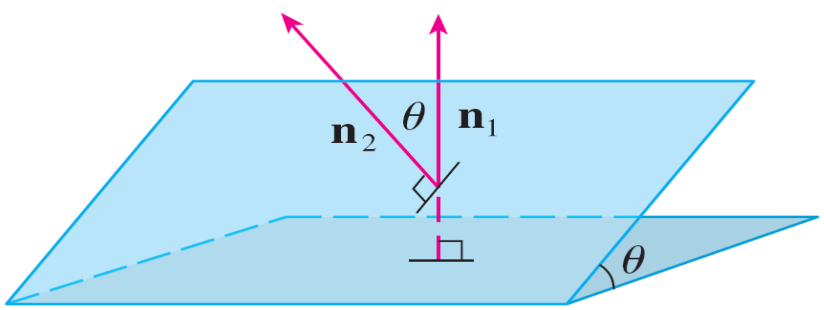


Figure 9

12

Vectors and the Geometry of Space

12.6 Cylinders and Quadric Surfaces

Cylinders and Quadric Surfaces

We have already looked at two special types of surfaces: planes and spheres.

Here we investigate two other types of surfaces: cylinders and quadric surfaces: cylinders and quadric surfaces.

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes.

These curves are called **traces** (or cross-sections) of the surface.

Cylinders

Cylinders

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

Example 1

Sketch the graph of the surface $z = x^2$.

Solution:

Notice that the equation of the graph, $z = x^2$, doesn't involve y.

This means that any vertical plane with equation $y = k$(parallel to the xz-plane) intersects the graph in a curve with equation $z = x^2$.

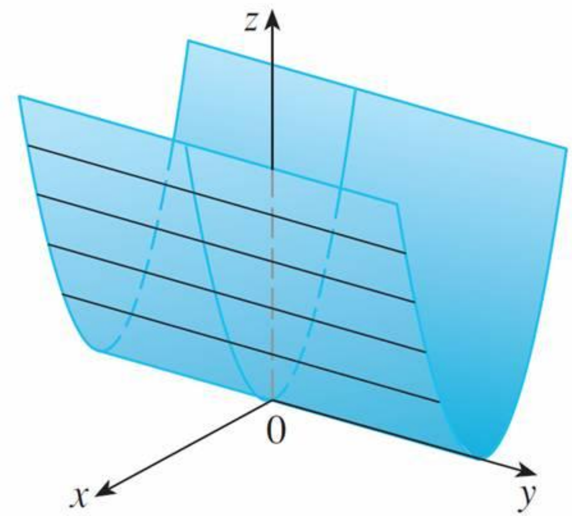
So these vertical traces are parabolas.

Example 1 – *Solution*

cont'd

Figure 1 shows how the graph is formed by taking the parabola $z = x^2$ in the xz-plane and moving it in the direction of the y-axis.

The graph is a surface, called a **parabolic cylinder**, made up of infinitely many shifted copies of the same parabola. Here the rulings of the cylinder are parallel to the y-axis.



The surface $z = x^2$ is a parabolic cylinder.

Figure 1

Cylinders

We noticed that the variable y is missing from the equation of the cylinder in Example 1. This is typical of a surface whose rulings are parallel to one of the coordinate axes.

If one of the variables x, y or z is missing from the equation of a surface, then the surface is a cylinder.

Note:
When you are dealing with surfaces, it is important to recognize that an equation like $x^2 + y^2 = 1$ represents a cylinder and not a circle. The trace of the cylinder $x^2 + y^2 = 1$ in the xy-plane is the circle with equations $x^2 + y^2 = 1$, $z = 0$.

Quadric Surfaces

Quadric Surfaces

A **quadric surface** is the graph of a second-degree equation in three variables *x*, *y*, and *z*. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

Where $A, B, C, \ldots, J$ are constants, but by translation and rotation it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

Quadric surfaces are the counterparts in three dimensions of the conic sections in the plane.

Example 3

Use traces to sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

Solution:

By substituting $z = 0$, we find that the trace in the xy-plane is $x^2 + y^2/9 = 1$, which we recognize as an equation of an ellipse. In general, the horizontal trace in the plane $z = k$ is

$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4} \qquad z = k$$

which is an ellipse, provided that $k^2 < 4$, that is, $-2 < k < 2$.

Example 3 – *Solution*

cont'd

Similarly, the vertical traces are also ellipses:

$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2 \qquad x = k$$ (if $-1 < k < 1$)

$$x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9} \qquad y = k$$ (if $-3 < k < 3$)

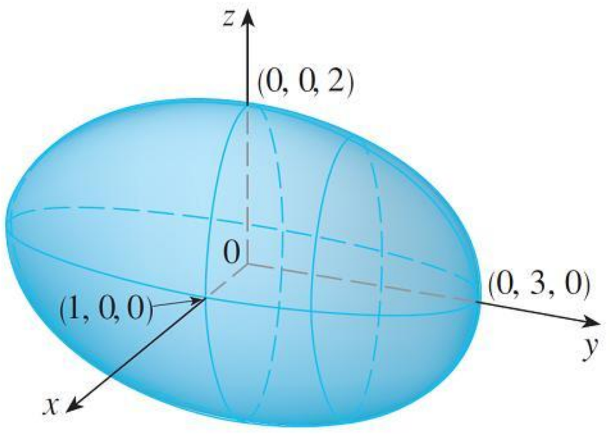
Example 3 – *Solution*

cont'd

Figure 4 shows how drawing some traces indicates the shape of the surface.

It's called an **ellipsoid** because all of its traces are ellipses.

Notice that it is symmetric with respect to each coordinate plane; this is a reflection of the fact that its equation involves only even powers of *x*, *y*, and *z*.



The ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

Figure 4

Example 4

Use traces to sketch the surface $z = 4x^2 + y^2$.

Solution:

If we put $x = 0$, we get $z = y^2$, so the yz-plane intersects the surface in a parabola. If we put $x = k$ (a constant), we get $z = y^2 + 4k^2$.

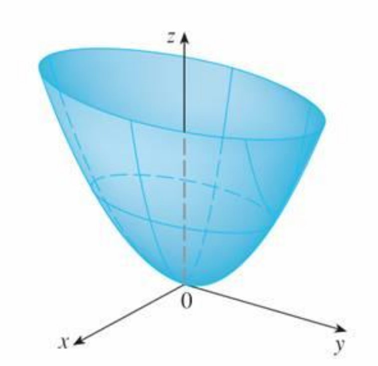
This means that if we slice the graph with any plane parallel to the yz-plane, we obtain a parabola that opens upward.

Similarly, if $y = k$, the trace is $z = 4x^2 + k^2$, which is again a parabola that opens upward.

Example 4 – *Solution*

cont'd

If we put $z = k$, we get the horizontal traces $4x^2 + y^2 = k$, which we recognize as a family of ellipses. Knowing the shapes of the traces, we can sketch the graph in Figure 5.



The surface $z = 4x^2 + y^2$ is an elliptic paraboloid. Horizontal traces are ellipses; vertical traces are parabolas.

Figure 5

Because of the elliptical and parabolic traces, the quadric surface $z = 4x^2 + y^2$ is called an **elliptic paraboloid**.

Example 5

Sketch the surface $z = y^2 - x^2$.

Solution:

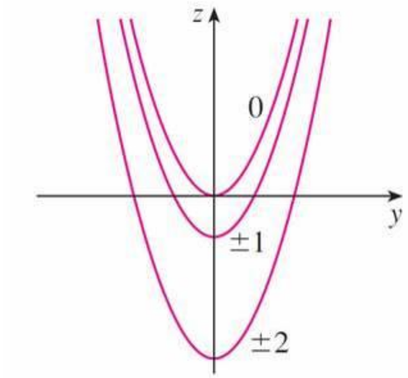
The traces in the vertical planes $x = k$ are the parabolas $z = y^2 - k^2$, which open upward. The traces in $y = k$ are the parabolas $z = -x^2 + k^2$, which open downward.

The horizontal traces are $y^2 - x^2 = k$, a family of hyperbolas.

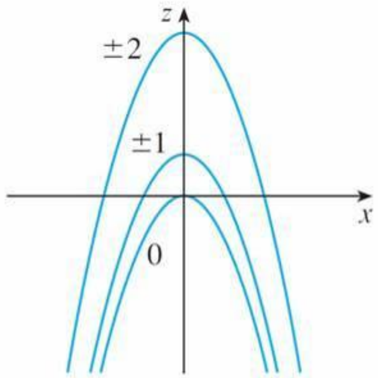
Example 5 – *Solution*

cont'd

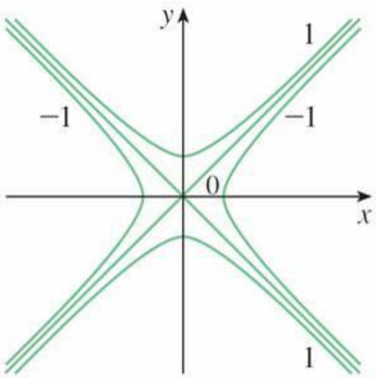
We draw the families of traces in Figure 6, and we show how the traces appear when placed in their correct planes in Figure 7.



Traces in $x = k$ are $z = y^2 - k^2$



Traces in $y = k$ are $z = -x^2 + k^2$



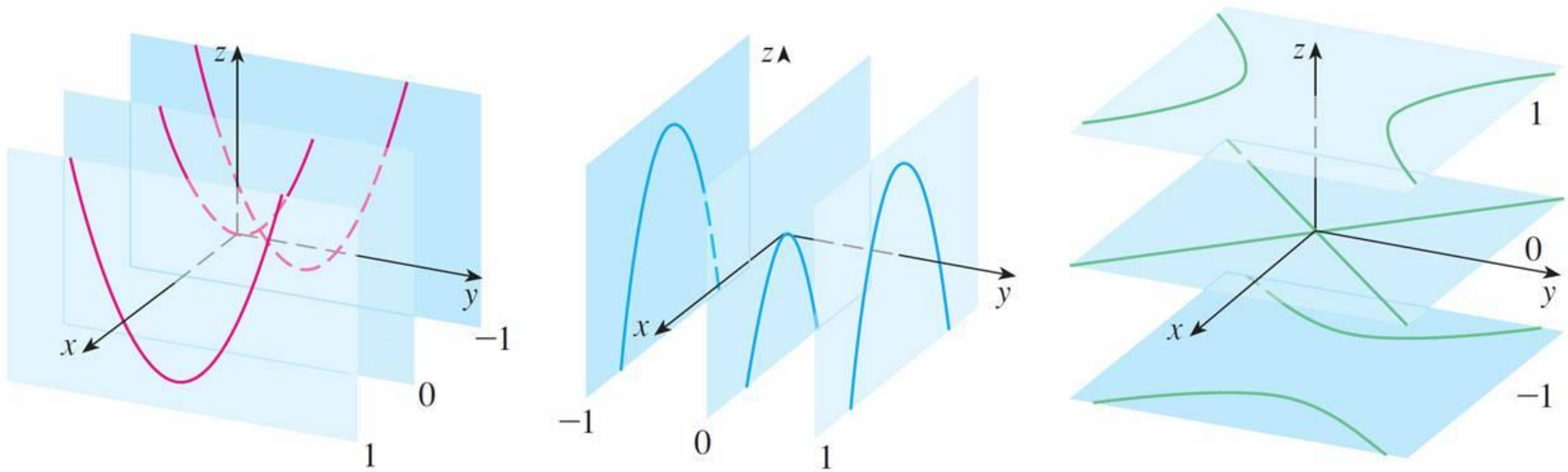
Traces in $z = k$ are $y^2 - x^2 = k$

Vertical traces are parabolas; horizontal traces are hyperbolas. All traces are labeled with the value of *k*.

Figure 6

Example 5 – *Solution*

cont'd



Traces in $x = k$ Traces in $y = k$ Traces in $z = k$

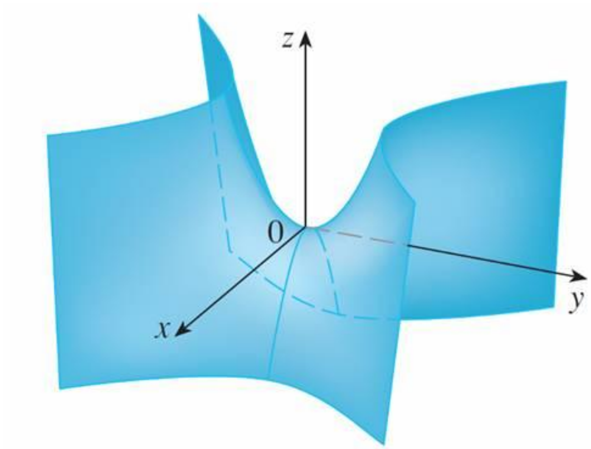
Traces moved to their correct planes

Figure 7

Example 5 – *Solution*

cont'd

In Figure 8 we fit together the traces from Figure 7 to form the surface $z = y^2 - x^2$, a **hyperbolic paraboloid**.



The surface $z = y^2 - x^2$ is a hyperbolic paraboloid.

Figure 8

Notice that the shape of the surface near the origin resembles that of a saddle.

Example 6

Sketch the surface $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1.$

Solution:

The trace in any horizontal plane $z = k$ is the ellipse

$$\frac{x^2}{4} + y^2 = 1 + \frac{k^2}{4} \qquad z = k$$

but the traces in the *xz*- and *yz*-planes are the hyperbolas

$$\frac{x^2}{4} - \frac{z^2}{4} = 1 \quad y = 0 \qquad \text{and} \qquad y^2 - \frac{z^2}{4} = 1 \quad x = 0$$

Example 6 – *Solution*

This surface is called a **hyperboloid of one sheet** and is sketched in Figure 9.

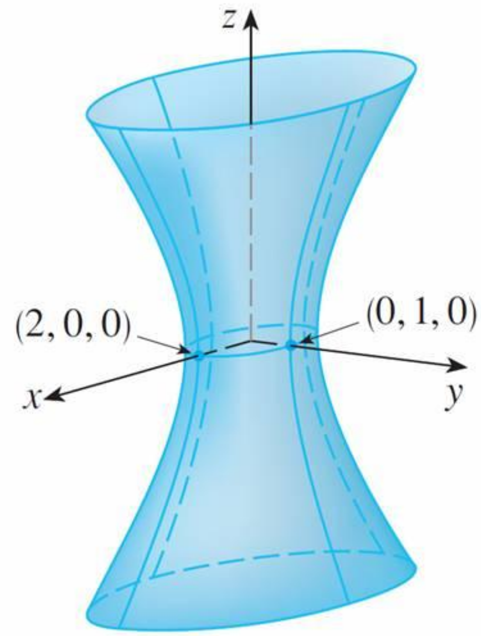


Figure 9

Quadric Surfaces

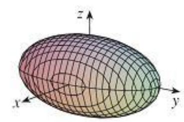
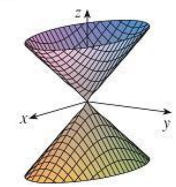
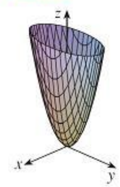
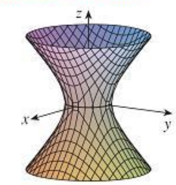
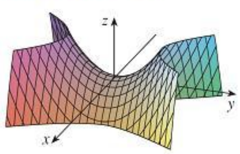
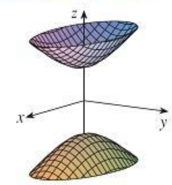
The idea of using traces to draw a surface is employed in three-dimensional graphing software for computers.

In most such software, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k, and parts of the graph are eliminated using hidden line removal.

Quadric Surfaces

Table 1 shows computer-drawn graphs of the six basic types of quadric surfaces in standard form.

All surfaces are symmetric with respect to the *z*-axis.
If a quadric surface is symmetric about a different axis, its equation changes accordingly.

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Graphs of quadric surfaces

Table 1

Applications of Quadric Surfaces

Applications of Quadric Surfaces

Examples of quadric surfaces can be found in the world around us. In fact, the world itself is a good example.

Although the earth is commonly modeled as a sphere, a more accurate model is an ellipsoid because the earth's rotation has caused a flattening at the poles.

Circular paraboloids, obtained by rotating a parabola about its axis, are used to collect and reflect light, sound, and radio and television signals.

Applications of Quadric Surfaces

In a radio telescope, for instance, signals from distant stars that strike the bowl are all reflected to the receiver at the focus and are therefore amplified.

The same principle applies to microphones and satellite dishes in the shape of paraboloids.

Cooling towers for nuclear reactors are usually designed in the shape of hyperboloids of one sheet for reasons of structural stability.

Pairs of hyperboloids are used to transmit rotational motion between skew axes.