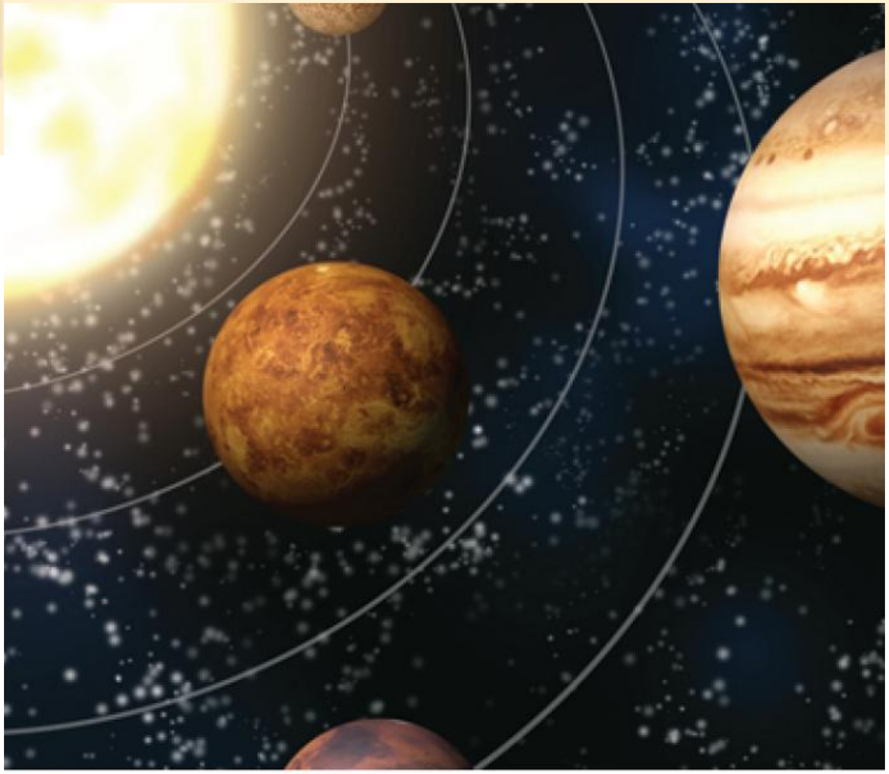
13 Vector Functions



13.1 Vector Functions and Space Curves

Vector Functions and Space Curves

In general, a function is a rule that assigns to each element in the domain an element in the range.

A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors.

We are most interested in vector functions $\mathbf{r}$ whose values are three-dimensional vectors.

This means that for every number t in the domain of $\mathbf{r}$ there is a unique vector in V_3 denoted by $\mathbf{r}(t)$.

Vector Functions and Space Curves

If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then f, g, and h are real-valued functions called the **component functions** of **r** and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

We use the letter t to denote the independent variable because it represents time in most applications of vector functions.

Example 1 – *Domain of a vector function*

If $\mathbf{r}(t) = \langle t^3, \ln(3 - t), \sqrt{t}\,\rangle$

then the component functions are

$$f(t) = t^3 \qquad g(t) = \ln(3 - t) \qquad h(t) = \sqrt{t}$$

By our usual convention, the domain of **r** consists of all values of t for which the expression for $\mathbf{r}(t)$ is defined.

The expressions t^3, $\ln(3 - t)$, and $\sqrt{t}$ are all defined when $3 - t > 0$ and $t \geq 0$.

Therefore the domain of **r** is the interval $[0, 3)$.

Vector Functions and Space Curves

The **limit** of a vector function **r** is defined by taking the limits of its component functions as follows.

$\boxed{1}$ If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

Limits of vector functions obey the same rules as limits of real-valued functions.

Vector Functions and Space Curves

A vector function **r** is **continuous at *a*** if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$

In view of Definition 1, we see that **r** is continuous at *a* if and only if its component functions *f*, *g*, and *h* are continuous at *a*.

There is a close connection between continuous vector functions and space curves.

Vector Functions and Space Curves

Suppose that f, g, and h are continuous real-valued functions on an interval I.

Then the set C of all points (x, y, z) in space, where

$$\boxed{2} \qquad x = f(t) \qquad y = g(t) \qquad z = h(t)$$

and t varies throughout the interval I, is called a **space curve**.

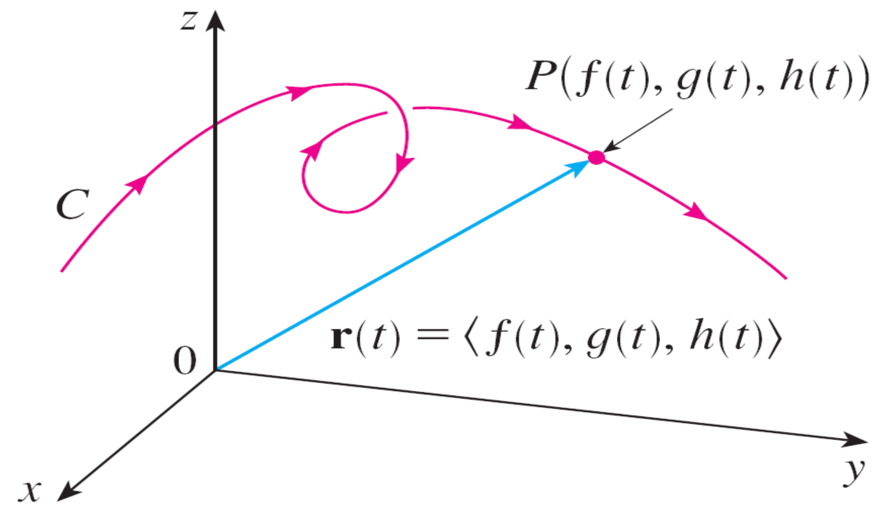
The equations in $\boxed{2}$ are called **parametric equations of *C*** and t is called a **parameter**.

We can think of C as being traced out by a moving particle whose position at time t is $(f(t), g(t), h(t))$.

Vector Functions and Space Curves

If we now consider the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then $\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C.

Thus any continuous vector function $\mathbf{r}$ defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$, as shown in Figure 1.



C is traced out by the tip of a moving position vector $\mathbf{r}(t)$.

Figure 1

Example 4 – *Sketching a helix*

Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$$

Solution:

The parametric equations for this curve are

$$x = \cos t \qquad y = \sin t \qquad z = t$$

Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the curve must lie on the circular cylinder $x^2 + y^2 = 1$.

The point (x, y, z) lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2 + y^2 = 1$ in the xy-plane.

Example 4 – *Solution*

cont'd

(The projection of the curve onto the *xy*-plane has vector equation $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$.) Since $z = t$, the curve spirals upward around the cylinder as t increases. The curve, shown in Figure 2, is called a **helix**.

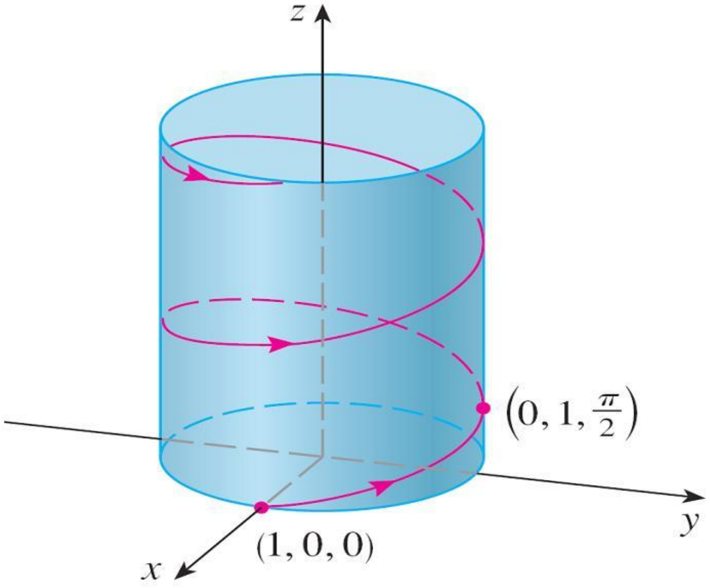


Figure 2

Vector Functions and Space Curves

The corkscrew shape of the helix in Example 4 is familiar from its occurrence in coiled springs.

It also occurs in the model of DNA (deoxyribonucleic acid, the genetic material of living cells).

In 1953 James Watson and Francis Crick showed that the structure of the DNA molecule is that of two linked, parallel helixes that are intertwined as in Figure 3.

A double helix

Figure 3

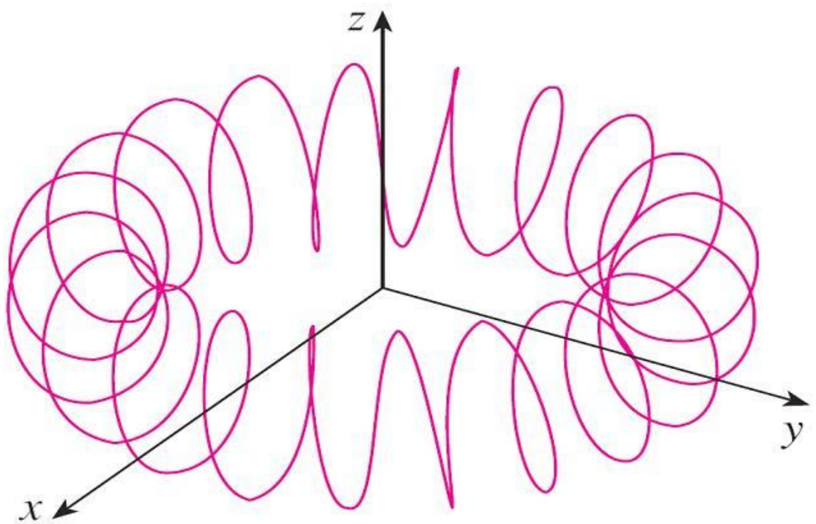
Using Computers to Draw Space Curves

Using Computers to Draw Space Curves

Space curves are inherently more difficult to draw by hand than plane curves; for an accurate representation we need to use technology.

For instance, Figure 7 shows a computer-generated graph of the curve with parametric equations

$x = (4 + \sin 20t) \cos t$
$y = (4 + \sin 20t) \sin t$
$z = \cos 20t$



A toroidal spiral

Figure 7

It's called a **toroidal spiral** because it lies on a torus.

Using Computers to Draw Space Curves

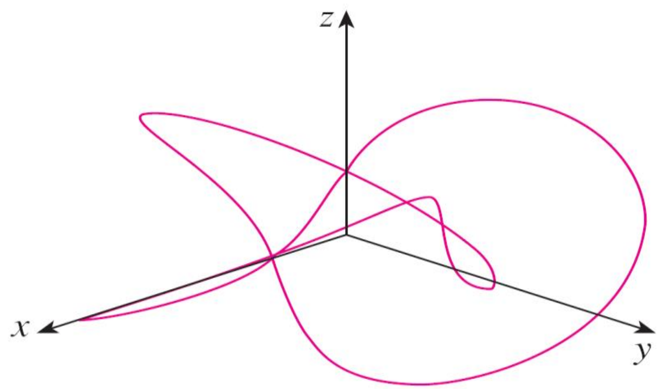
Another interesting curve, the **trefoil knot**, with equations

$$x = (2 + \cos 1.5t) \cos t$$

$$y = (2 + \cos 1.5t) \sin t$$

$$z = \sin 1.5t$$

is graphed in Figure 8. It wouldn't be easy to plot either of these curves by hand.



A trefoil knot

Figure 8

Using Computers to Draw Space Curves

Even when a computer is used to draw a space curve, optical illusions make it difficult to get a good impression of what the curve really looks like. (This is especially true in Figure 8.)

The next example shows how to cope with this problem.

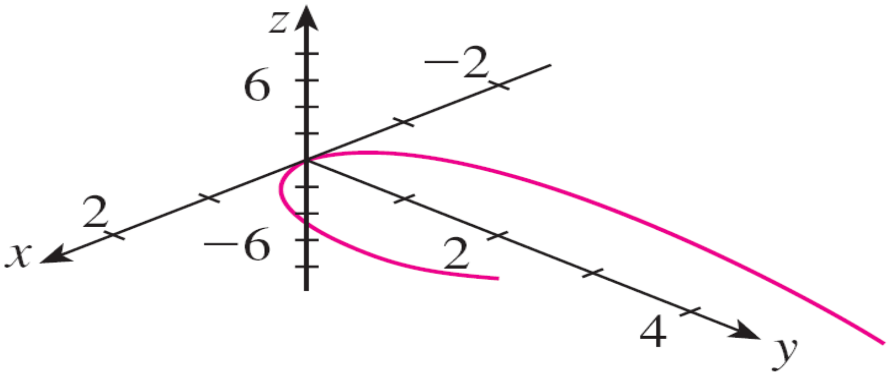
Example 7

Use a computer to draw the curve with vector equation $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$. This curve is called a **twisted cubic**.

Solution:

We start by using the computer to plot the curve with parametric equations $x = t$, $y = t^2$, $z = t^3$ for $-2 \leq t \leq 2$.

The result is shown in Figure 9(a), but it's hard to see the true nature of the curve from that graph alone.



View of the twisted cubic

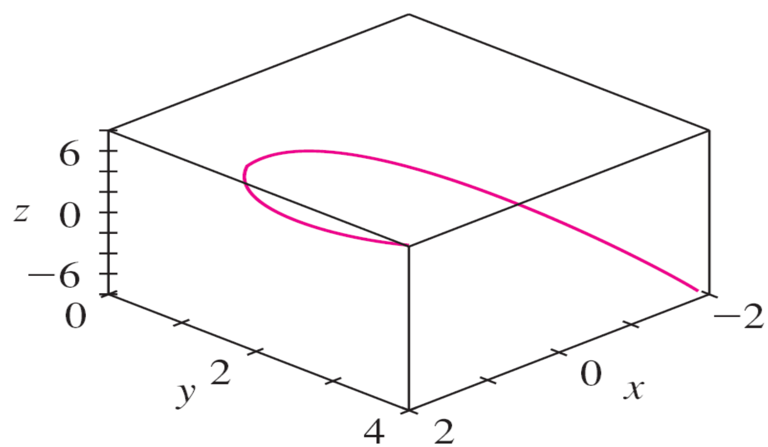
Figure 9(a)

Example 7 – *Solution*

cont'd

Most three-dimensional computer graphing programs allow the user to enclose a curve or surface in a box instead of displaying the coordinate axes.

When we look at the same curve in a box in Figure 9(b), we have a much clearer picture of the curve.



View of the twisted cubic

Figure 9(b)

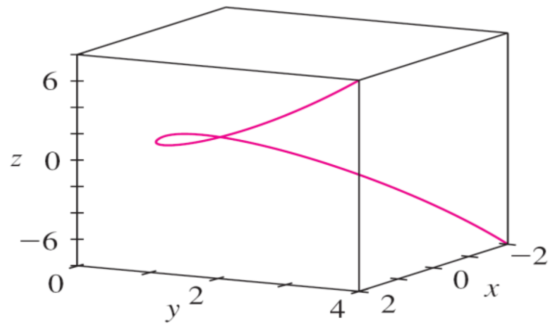
Example 7 – *Solution*

cont'd

We can see that it climbs from a lower corner of the box to the upper corner nearest us, and it twists as it climbs.

We get an even better idea of the curve when we view it from different vantage points.

Figure 9(c) shows the result of rotating the box to give another viewpoint.



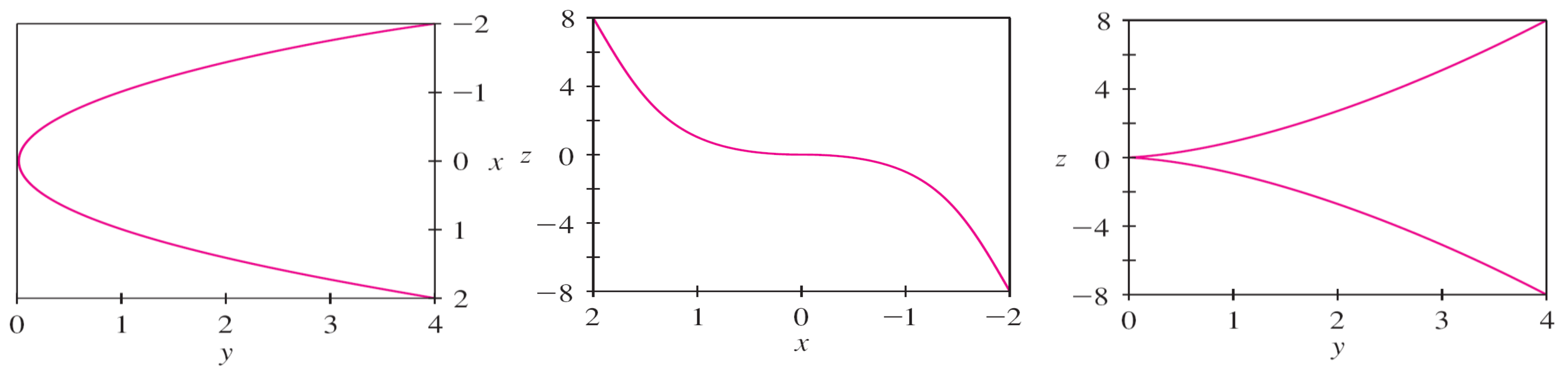
View of the twisted cubic

Figure 9(c)

Example 7 – *Solution*

cont'd

Figures 9(d), 9(e), and 9(f) show the views we get when we look directly at a face of the box.



Views of the twisted cubic

Figure 9(d) **Figure 9(e)** **Figure 9(f)**

In particular, Figure 9(d) shows the view from directly above the box.

Example 7 – *Solution*

cont'd

It is the projection of the curve on the *xy*-plane, namely, the parabola $y = x^2$.

Figure 9(e) shows the projection on the *xz*-plane, the cubic curve $z = x^3$.

It's now obvious why the given curve is called a twisted cubic.

Using Computers to Draw Space Curves

Another method of visualizing a space curve is to draw it on a surface.

For instance, the twisted cubic in Example 7 lies on the parabolic cylinder $y = x^2$. (Eliminate the parameter from the first two parametric equations, $x = t$ and $y = t^2$.)

Figure 10 shows both the cylinder and the twisted cubic, and we see that the curve moves upward from the origin along the surface of the cylinder.

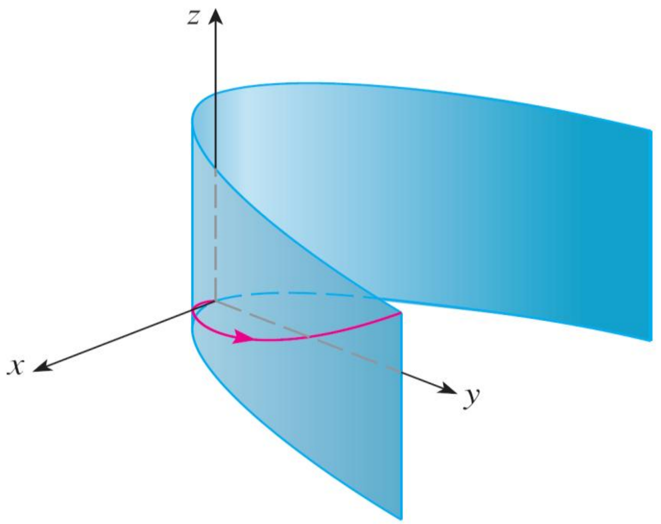


Figure 10

Using Computers to Draw Space Curves

We also used this method in Example 4 to visualize the helix lying on the circular cylinder.

A third method for visualizing the twisted cubic is to realize that it also lies on the cylinder $z = x^3$.

So it can be viewed as the curve of intersection of the cylinders $y = x^2$ and $z = x^3$. (See Figure 11.)

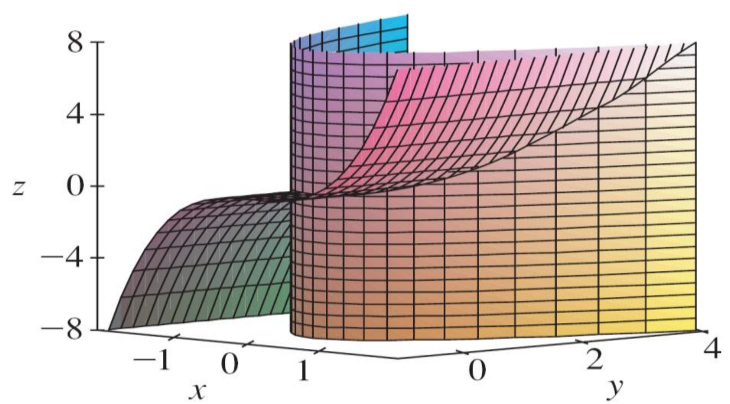


Figure 11

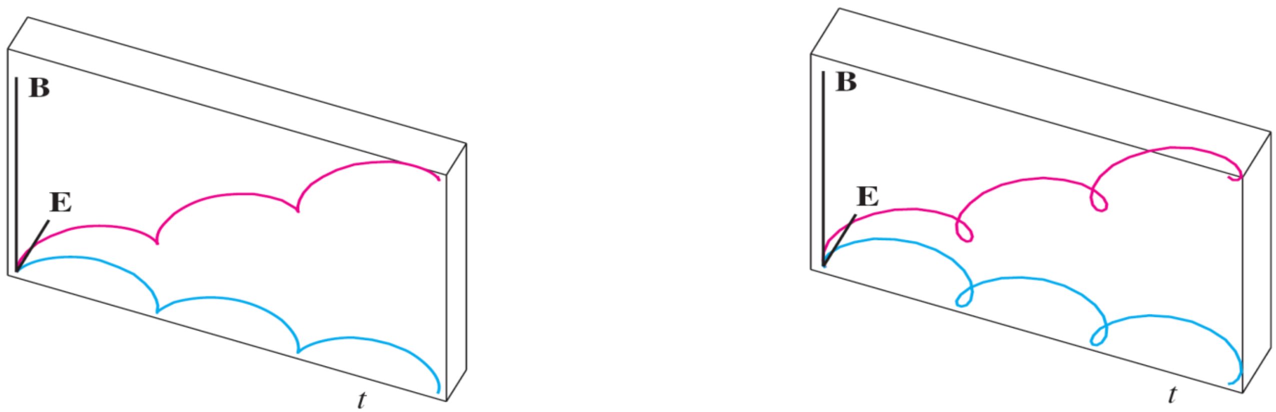
Using Computers to Draw Space Curves

We have seen that an interesting space curve, the helix, occurs in the model of DNA.

Another notable example of a space curve in science is the trajectory of a positively charged particle in orthogonally oriented electric and magnetic fields **E** and **B**.

Using Computers to Draw Space Curves

Depending on the initial velocity given the particle at the origin, the path of the particle is either a space curve whose projection on the horizontal plane is the cycloid [Figure 12(a)] or a curve whose projection is the trochoid [Figure 12(b)].



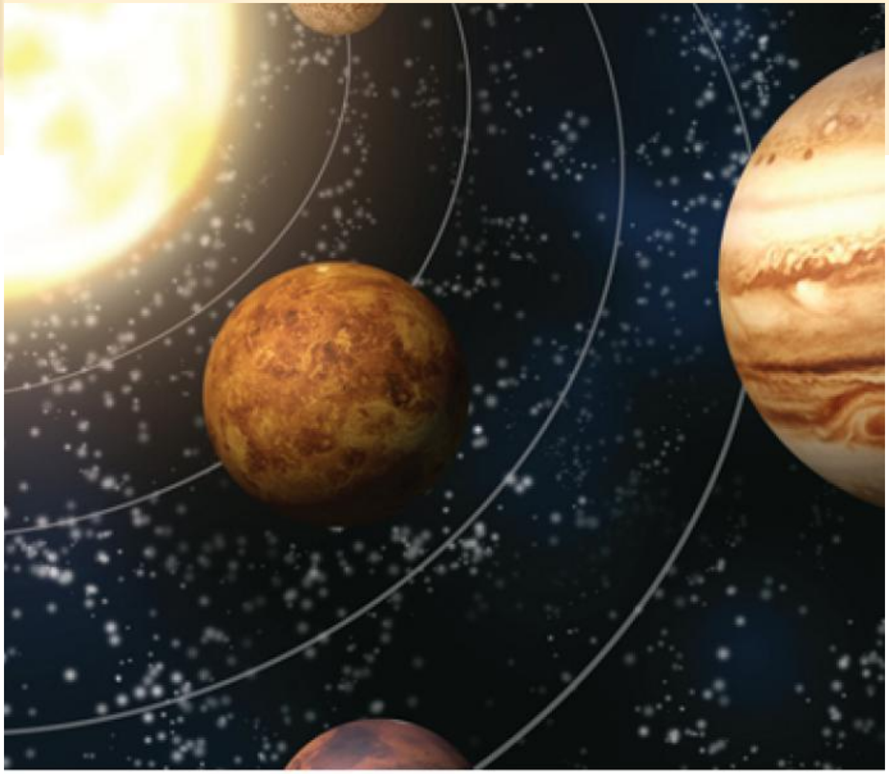
(a) $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t, t \rangle$

(b) $\mathbf{r}(t) = \left\langle t - \frac{3}{2}\sin t, 1 - \frac{3}{2}\cos t, t \right\rangle$

Motion of a charged particle in orthogonally oriented electric and magnetic fields

Figure 12

13 Vector Functions



13.2

Derivatives and Integrals of Vector Functions

Derivatives

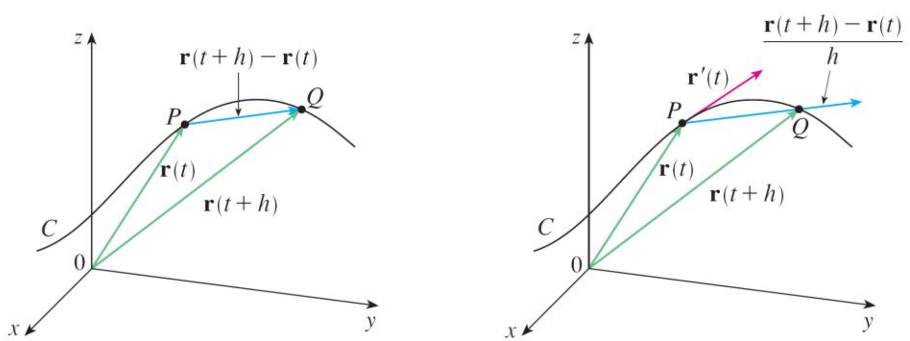
Derivatives

The **derivative r′** of a vector function **r** is defined in much the same way as for real valued functions:

1

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists. The geometric significance of this definition is shown in Figure 1.



(a) The secant vec $\overrightarrow{PQ}$ (b) The tangent vector r′(t)

Figure 1

Derivatives

If the points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t + h)$, then $\overrightarrow{PQ}$ represents the vector $\mathbf{r}(t + h) - \mathbf{r}(t)$, which can therefore be regarded as a secant vector.

If $h > 0$, the scalar multiple $(1/h)(\mathbf{r}(t + h) - \mathbf{r}(t))$ has the same direction as $\mathbf{r}(t + h) - \mathbf{r}(t)$. As $h \to 0$, it appears that this vector approaches a vector that lies on the tangent line.

For this reason, the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$.

Derivatives

The **tangent line** to *C* at *P* is defined to be the line through *P* parallel to the tangent vector $\mathbf{r}'(t)$.

We will also have occasion to consider the **unit tangent vector**, which is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Derivatives

The following theorem gives us a convenient method for computing the derivative of a vector function **r**: just differentiate each component of **r**.

$\boxed{2}$ **Theorem** If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

Example 1

(a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\,\mathbf{k}$.

(b) Find the unit tangent vector at the point where $t = 0$.

Solution:

(a) According to Theorem 2, we differentiate each component of **r**:

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + (1 - t)e^{-t}\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$$

Example 1 – *Solution*

cont'd

(b) Since $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$, the unit tangent vector at the point (1, 0, 0) is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|}$$

$$= \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1 + 4}}$$

$$= \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$$

Derivatives

Just as for real-valued functions, the **second derivative** of a vector function $\mathbf{r}$ is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$.

For instance, the second derivative of the function, $\mathbf{r}(t) = \langle 2 \cos t, \sin t, t \rangle$, is

$$\mathbf{r}''t = \langle -2 \cos t, -\sin t, 0 \rangle$$

Differentiation Rules

Differentiation Rules

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

3 Theorem Suppose **u** and **v** are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $$\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$$

2. $$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

3. $$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

4. $$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$$

5. $$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

6. $$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t)) \qquad \text{(Chain Rule)}$$

Example 4

Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

Solution:
Since

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

and c^2 is a constant, Formula 4 of Theorem 3 gives

$$0 = \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t)$$

Thus $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, which says that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Example 4 – *Solution*

cont'd

Geometrically, this result says that if a curve lies on a sphere with center the origin, then the tangent vector $\mathbf{r}'(t)$ is always perpendicular to the position vector $\mathbf{r}(t)$.

Integrals

Integrals

The **definite integral** of a continuous vector function $\mathbf{r}(t)$ can be defined in much the same way as for real-valued functions except that the integral is a vector.

But then we can express the integral of $\mathbf{r}$ in terms of the integrals of its component functions f, g, and h as follows.

$$\int_a^b \mathbf{r}(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\,\Delta t$$

$$= \lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} f(t_i^*)\,\Delta t \right) \mathbf{i} + \left(\sum_{i=1}^{n} g(t_i^*)\,\Delta t \right) \mathbf{j} + \left(\sum_{i=1}^{n} h(t_i^*)\,\Delta t \right) \mathbf{k} \right]$$

Integrals

and so

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

This means that we can evaluate an integral of a vector function by integrating each component function.

Integrals

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where **R** is an antiderivative of **r**, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$.

We use the notation $\int \mathbf{r}(t)\,dt$ for indefinite integrals (antiderivatives).

Example 5

If $\mathbf{r}(t) = 2 \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + 2t\, \mathbf{k}$, then

$$\int \mathbf{r}(t)\, dt = \left(\int 2 \cos t\, dt \right) \mathbf{i} + \left(\int \sin t\, dt \right) \mathbf{j} + \left(\int 2t\, dt \right) \mathbf{k}$$

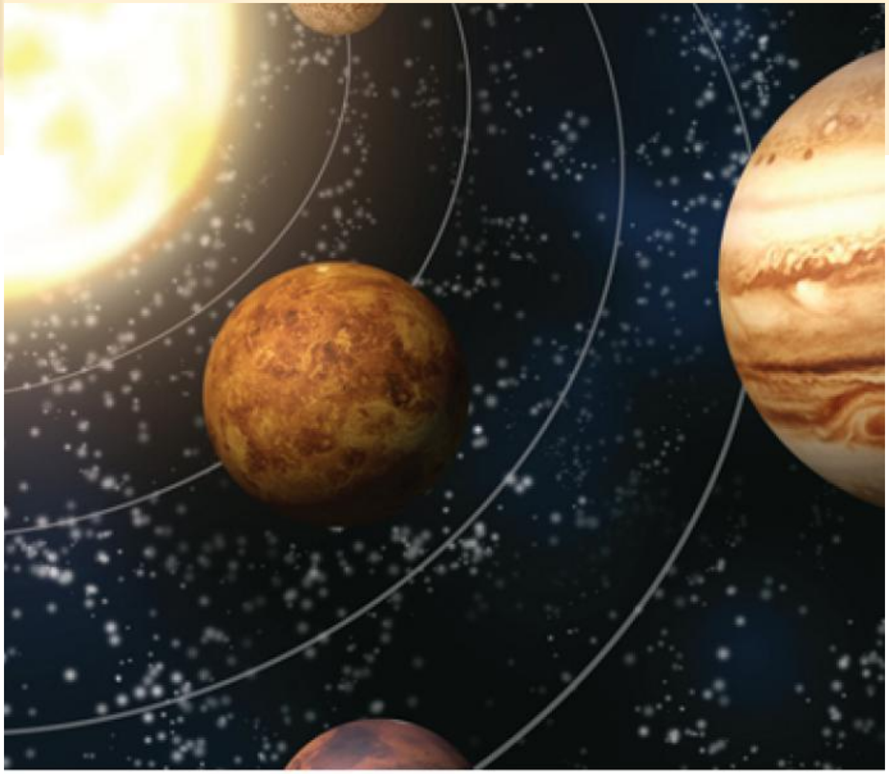
$$= 2 \sin t\, \mathbf{i} - \cos t\, \mathbf{j} + t^2\, \mathbf{k} + \mathbf{C}$$

where **C** is a vector constant of integration, and

$$\int_0^{\pi/2} \mathbf{r}(t)\, dt = \left[2 \sin t\, \mathbf{i} - \cos t\, \mathbf{j} + t^2\, \mathbf{k} \right]_0^{\pi/2}$$

$$= 2\,\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\mathbf{k}$$

13 Vector Functions



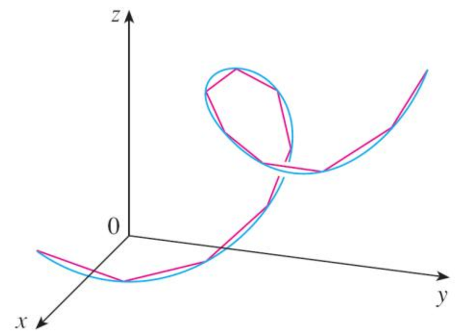
13.3 Arc Length and Curvature

Arc Length and Curvature

We have defined the length of a plane curve with parametric equations $x = f(t)$, $y = g(t)$, $a \le t \le b$, as the limit of lengths of inscribed polygons and, for the case where f' and g' are continuous, we arrived at the formula

1 $$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The length of a space curve is defined in exactly the same way (see Figure 1).



The length of a space curve is the limit of lengths of inscribed polygons.

Figure 1

Arc Length and Curvature

Suppose that the curve has the vector equation, $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f', g', and h' are continuous.

If the curve is traversed exactly once as t increases from a to b, then it can be shown that its length is

2

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$
$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Arc Length and Curvature

Notice that both of the arc length formulas $\boxed{1}$ and $\boxed{2}$ can be put into the more compact form

$$\boxed{3} \qquad L = \int_a^b |\mathbf{r}'(t)|\,dt$$

because, for plane curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$,

$$|\mathbf{r}'(t)| = |f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

and for space curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$,

$$|\mathbf{r}'(t)| = |f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Example 1

Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ from the point (1, 0, 0) to the point $(1, 0, 2\pi)$.

Solution:

Since $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, we have

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$$

The arc from (1, 0, 0) to $(1, 0, 2\pi)$ is described by the parameter interval $0 \le t \le 2\pi$ and so, from Formula 3, we have

$$L = \int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\,\pi$$

Arc Length and Curvature

A single curve C can be represented by more than one vector function. For instance, the twisted cubic

$$\boxed{4} \qquad \mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \qquad 1 \leq t \leq 2$$

could also be represented by the function

$$\boxed{5} \qquad \mathbf{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \qquad 0 \leq u \leq \ln 2$$

where the connection between the parameters t and u is given by $t = e^u$.

We say that Equations 4 and 5 are **parametrizations** of the curve C.

Arc Length and Curvature

If we were to use Equation 3 to compute the length of C using Equations 4 and 5, we would get the same answer.

In general, it can be shown that when Equation 3 is used to compute arc length, the answer is independent of the parametrization that is used.

Now we suppose that C is a curve given by a vector function

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} \qquad a \le t \le b$$

where $\mathbf{r}'$ is continuous and C is traversed exactly once as t increases from a to b.

Arc Length and Curvature

We define its **arc length function** *s* by

$$\boxed{6} \qquad s(t) = \int_a^t |\mathbf{r}'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

Thus $s(t)$ is the length of the part of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. (See Figure 3.

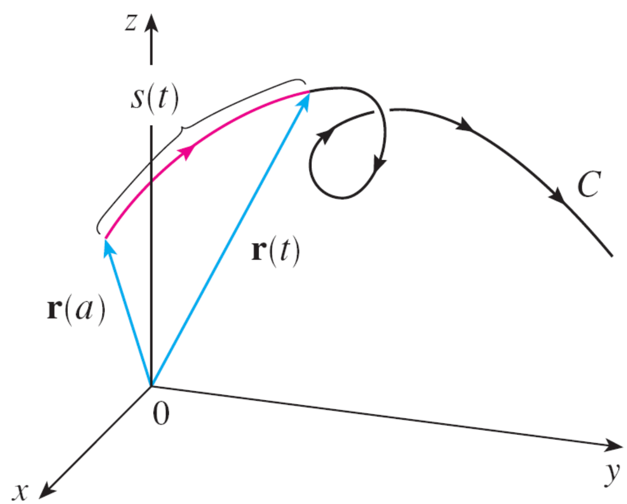


Figure 3

Arc Length and Curvature

If we differentiate both sides of Equation 6 using Part 1 of the Fundamental Theorem of Calculus, we obtain

$$\boxed{7} \qquad \frac{ds}{dt} = |\mathbf{r}'(t)|$$

It is often useful to **parametrize a curve with respect to arc length** because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system.

Arc Length and Curvature

If a curve $\mathbf{r}(t)$ is already given in terms of a parameter t and $s(t)$ is the arc length function given by Equation 6, then we may be able to solve for t as a function of s: $t = t(s)$.

Then the curve can be reparametrized in terms of s by substituting for t: $\mathbf{r} = \mathbf{r}(t(s))$.

Thus, if $s = 3$ for instance, $\mathbf{r}(t(3))$ is the position vector of the point 3 units of length along the curve from its starting point.

Curvature

Curvatures

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on I.

A curve is called **smooth** if it has a smooth parametrization. A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

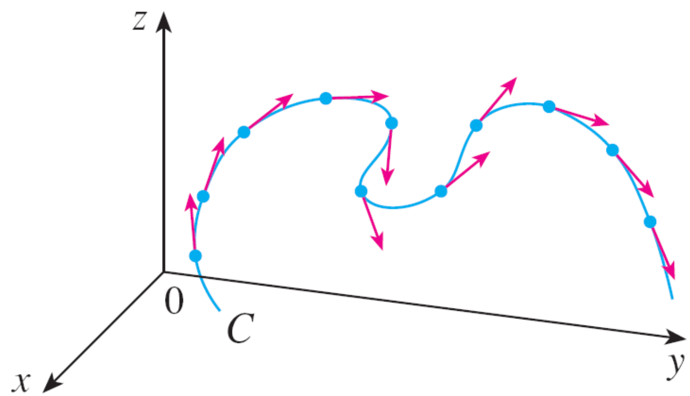
If C is a smooth curve defined by the vector function $\mathbf{r}$, recall that the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and indicates the direction of the curve.

Curvatures

From Figure 4 you can see that $\mathbf{T}(t)$ changes direction very slowly when C is fairly straight, but it changes direction more quickly when C bends or twists more sharply.



Unit tangent vectors at equally spaced points on C

Figure 4

Curvatures

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point.

Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. (We use arc length so that the curvature will be independent of the parametrization.)

8 Definition The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

Curvatures

The curvature is easier to compute if it is expressed in terms of the parameter *t* instead of *s*, so we use the Chain Rule to write

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \qquad \text{and} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right|$$

But $ds/dt = |\mathbf{r}'(t)|$ from Equation 7, so

9

$$\boxed{\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}}$$

Example 3

Show that the curvature of a circle of radius *a* is 1/*a*.

Solution:

We can take the circle to have center the origin, and then a parametrization is

$$\mathbf{r}(t) = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j}$$

Therefore $\mathbf{r}'(t) = -a \sin t\, \mathbf{i} + a \cos t\, \mathbf{j}$ and $|\mathbf{r}'(t)| = a$

so

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\sin t\, \mathbf{i} + \cos t\, \mathbf{j}$$

and

$$\mathbf{T}'(t) = -\cos t\, \mathbf{i} - \sin t\, \mathbf{j}$$

Example 3 – *Solution*

cont'd

This gives $|\mathbf{T}'(t)| = 1$, so using Equation 9, we have

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{a}$$

Curvature

The result of Example 3 shows that small circles have large curvature and large circles have small curvature, in accordance with our intuition.

We can see directly from the definition of curvature that the curvature of a straight line is always 0 because the tangent vector is constant.

Although Formula 9 can be used in all cases to compute the curvature, the formula given by the following theorem is often more convenient to apply.

10 **Theorem** The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Curvature

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $\mathbf{r}(x) = x\,\mathbf{i} + f(x)\,\mathbf{j}$.

Then $\mathbf{r}'(x) = \mathbf{i} + f'(x)\,\mathbf{j}$ and $\mathbf{r}''(x) = f''(x)\,\mathbf{j}$.

Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, it follows that

$$\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x)\,\mathbf{k}.$$

We also have $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$ and so, by Theorem 10,

11

$$\boxed{\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}}$$

The Normal and Binormal Vectors

The Normal and Binormal Vectors

At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$.

We single out one by observing that, because $|\mathbf{T}(t)| = 1$ for all t, we have $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$, so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$.

Note that $\mathbf{T}'(t)$ is itself not a unit vector.

But at any point where $\kappa \neq 0$ we can define the **principal unit normal vector** $\mathbf{N}(t)$ (or simply **unit normal**) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

The Normal and Binormal Vectors

The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the **binormal vector**.

It is perpendicular to both **T** and **N** and is also a unit vector. (See Figure 6.)

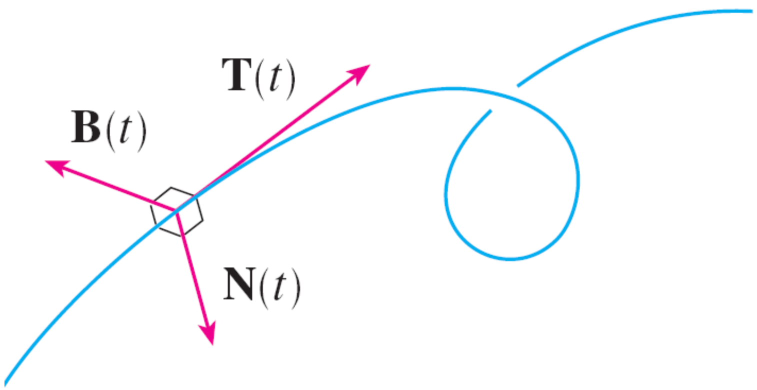


Figure 6

Example 6

Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

Solution:

We first compute the ingredients needed for the unit normal vector:

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k} \qquad |\mathbf{r}'(t)| = \sqrt{2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}}(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k})$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}(-\cos t\,\mathbf{i} - \sin t\,\mathbf{j}) \qquad |\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}$$

Example 6 – *Solution*

cont'd

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j} = \langle -\cos t, -\sin t, 0\rangle$$

This shows that the normal vector at a point on the helix is horizontal and points toward the *z*-axis.

The binormal vector is

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1\rangle$$

The Normal and Binormal Vectors

The plane determined by the normal and binormal vectors **N** and **B** at a point *P* on a curve *C* is called the **normal plane** of *C* at *P*.

It consists of all lines that are orthogonal to the tangent vector **T**.

The plane determined by the vectors **T** and **N** is called the **osculating plane** of *C* at *P*.

The name comes from the Latin *osculum*, meaning “kiss.” It is the plane that comes closest to containing the part of the curve near *P*. (For a plane curve, the osculating plane is simply the plane that contains the curve.)

The Normal and Binormal Vectors

The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which **N** points), and has radius $\rho = 1/\kappa$ (the reciprocal of the curvature) is called the **osculating circle** (or the **circle of curvature**) of C at P.

It is the circle that best describes how C behaves near P; it shares the same tangent, normal, and curvature at P.

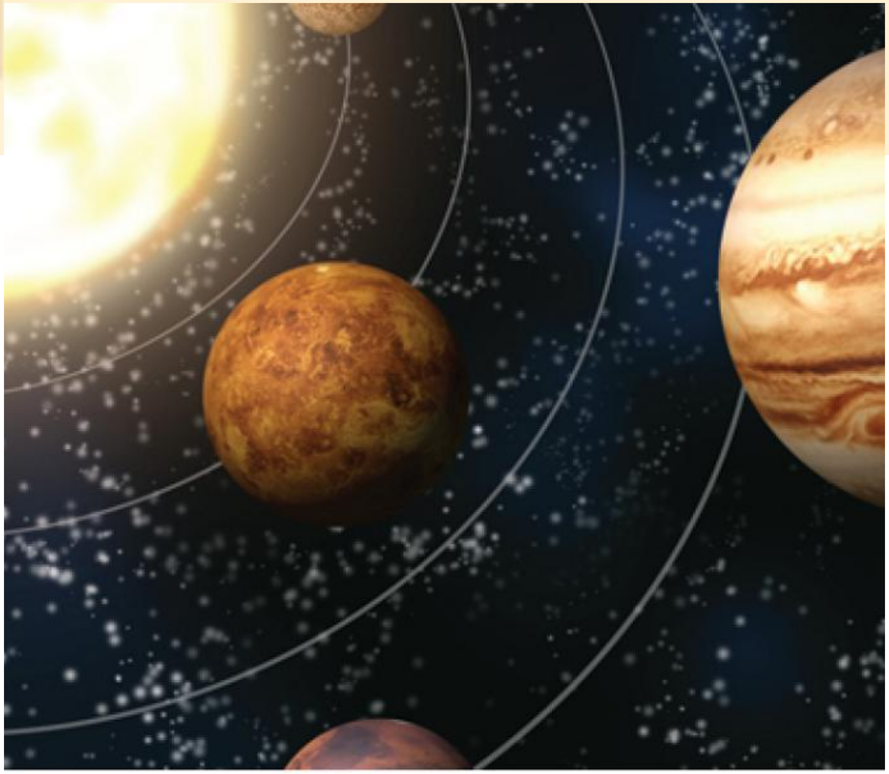
The Normal and Binormal Vectors

We summarize here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

13 Vector Functions



13.4 Motion in Space: Velocity and Acceleration

Motion in Space: Velocity and Acceleration

In this section we show how the ideas of tangent and normal vectors and curvature can be used in physics to study the motion of an object, including its velocity and acceleration, along a space curve.

In particular, we follow in the footsteps of Newton by using these methods to derive Kepler's First Law of planetary motion.

Motion in Space: Velocity and Acceleration

Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. Notice from Figure 1 that, for small values of h, the vector

$$\boxed{1} \qquad \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$

approximates the direction of the particle moving along the curve $\mathbf{r}(t)$.

Its magnitude measures the size of the displacement vector per unit time.

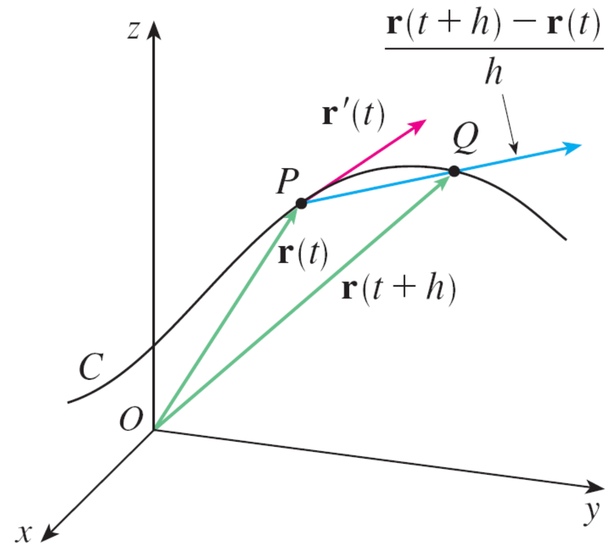


Figure 1

Motion in Space: Velocity and Acceleration

The vector $\boxed{1}$ gives the average velocity over a time interval of length *h* and its limit is the **velocity vector v**(*t*) at time *t*:

$\boxed{2}$

$$\mathbf{v}(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t)$$

Thus the velocity vector is also the tangent vector and points in the direction of the tangent line.

The **speed** of the particle at time *t* is the magnitude of the velocity vector, that is, $|\mathbf{v}(t)|$.

Motion in Space: Velocity and Acceleration

This is appropriate because, from $\boxed{2}$, we have

$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \dfrac{ds}{dt}$ = rate of change of distance with respect to time

As in the case of one-dimensional motion, the **acceleration** of the particle is defined as the derivative of the velocity:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

Example 1

The position vector of an object moving in a plane is given by $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j}$. Find its velocity, speed, and acceleration when $t = 1$ and illustrate geometrically.

Solution:

The velocity and acceleration at time t are

$$\mathbf{v}(t) = \mathbf{r}'(t) = 3t^2\,\mathbf{i} + 2t\,\mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = 6t\,\mathbf{i} + 2\,\mathbf{j}$$

and the speed is

$$|\,\mathbf{v}(t)\,| = \sqrt{(3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 4t^2}$$

Example 1 – *Solution*

cont'd

When $t = 1$, we have

$$\mathbf{v}(1) = 3\,\mathbf{i} + 2\,\mathbf{j} \qquad \mathbf{a}(1) = 6\,\mathbf{i} + 2\,\mathbf{j} \qquad |\mathbf{v}(1)| = \sqrt{13}$$

These velocity and acceleration vectors are shown in Figure 2.

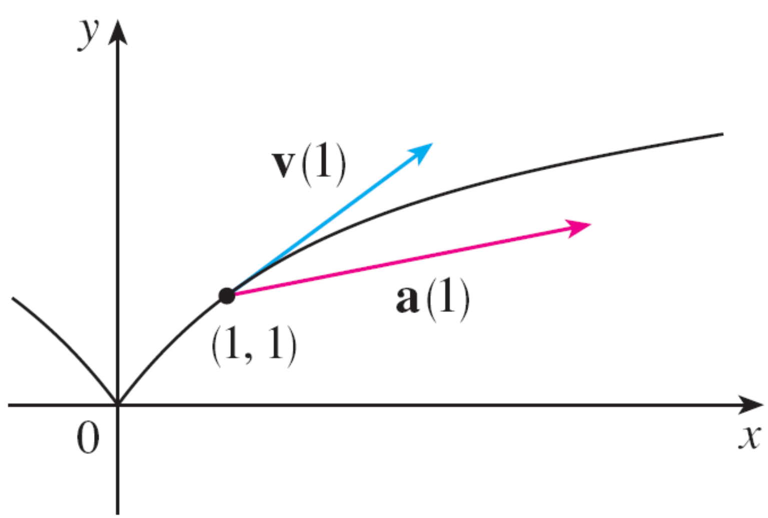


Figure 2

Motion in Space: Velocity and Acceleration

In general, vector integrals allow us to recover velocity when acceleration is known and position when velocity is known:

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(u)\, du \qquad \mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(u)\, du$$

If the force that acts on a particle is known, then the acceleration can be found from **Newton's Second Law of Motion**.

The vector version of this law states that if, at any time t, a force $\mathbf{F}(t)$ acts on an object of mass m producing an acceleration $\mathbf{a}(t)$, then

$$\mathbf{F}(t) = m\mathbf{a}(t)$$

Tangential and Normal Components of Acceleration

Tangential and Normal Components of Acceleration

When we study the motion of a particle, it is often useful to resolve the acceleration into two components, one in the direction of the tangent and the other in the direction of the normal.

If we write $v = |\mathbf{v}|$ for the speed of the particle, then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{v}}{v}$$

and so $\mathbf{v} = v\mathbf{T}$

If we differentiate both sides of this equation with respect to t, we get

$$\boxed{5} \qquad \mathbf{a} = \mathbf{v}' = v'\mathbf{T} + v\mathbf{T}'$$

Tangential and Normal Components of Acceleration

If we use the expression for the curvature, then we have

$$\boxed{6} \qquad \kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v} \qquad \text{so} \qquad |\mathbf{T}'| = \kappa v$$

The unit normal vector was defined in the preceding section as **N** = **T**′/|**T**′|, so $\boxed{6}$ gives

$$\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa v \mathbf{N}$$

and Equation 5 becomes

$$\boxed{7} \qquad \boxed{\mathbf{a} = v'\mathbf{T} + \kappa v^2 \mathbf{N}}$$

Tangential and Normal Components of Acceleration

Writing a_T and a_N for the tangential and normal components of acceleration, we have

$$\mathbf{a} = a_T\,\mathbf{T} + a_N\,\mathbf{N}$$

where

$$\boxed{8} \qquad a_T = v' \qquad \text{and} \qquad a_N = \kappa v^2$$

This resolution is illustrated in Figure 7.

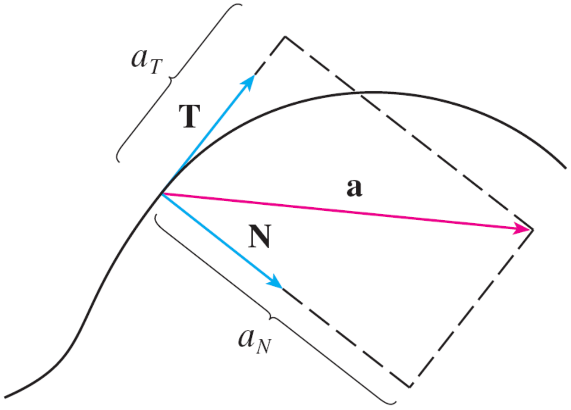


Figure 7

Tangential and Normal Components of Acceleration

Let's look at what Formula 7 says. The first thing to notice is that the binormal vector **B** is absent.

No matter how an object moves through space, its acceleration always lies in the plane of **T** and **N** (the osculating plane). (Recall that **T** gives the direction of motion and **N** points in the direction the curve is turning.)

Next we notice that the tangential component of acceleration is v', the rate of change of speed, and the normal component of acceleration is κv^2, the curvature times the square of the speed.

Tangential and Normal Components of Acceleration

This makes sense if we think of a passenger in a car—a sharp turn in a road means a large value of the curvature κ, so the component of the acceleration perpendicular to the motion is large and the passenger is thrown against a car door.

High speed around the turn has the same effect; in fact, if you double your speed, a_N is increased by a factor of 4.

Although we have expressions for the tangential and normal components of acceleration in Equations 8, it's desirable to have expressions that depend only on $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}''$.

Tangential and Normal Components of Acceleration

To this end we take the dot product of $\mathbf{v} = v\mathbf{T}$ with $\mathbf{a}$ as given by Equation 7:

$$\begin{aligned}\mathbf{v} \cdot \mathbf{a} &= v\mathbf{T} \cdot (v'\mathbf{T} + \kappa v^2\mathbf{N}) \\ &= vv'\mathbf{T} \cdot \mathbf{T} + \kappa v^3\mathbf{T} \cdot \mathbf{N} \\ &= vv' \qquad \text{(since } \mathbf{T} \cdot \mathbf{T} = 1 \text{ and } \mathbf{T} \cdot \mathbf{N} = 0\text{)}\end{aligned}$$

Therefore

$$\boxed{9} \qquad a_T = v' = \frac{\mathbf{v} \cdot \mathbf{a}}{v} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

Using the formula for curvature, we have

$$\boxed{10} \qquad a_N = \kappa v^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}|\mathbf{r}'(t)|^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

Example 7

A particle moves with position function $\mathbf{r}(t) = \langle t^2, t^2, t^3 \rangle$. Find the tangential and normal components of acceleration.

Solution:

$$\mathbf{r}(t) = t^2\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$$

$$\mathbf{r}'(t) = 2t\,\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$$

$$\mathbf{r}''(t) = 2\,\mathbf{i} + 2\,\mathbf{j} + 6t\,\mathbf{k}$$

$$|\mathbf{r}'(t)| = \sqrt{8t^2 + 9t^4}$$

Example 7 – *Solution*

cont'd

Therefore Equation 9 gives the tangential component as

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

$$= \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}$$

Since $$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix}$$

$$= 6t^2\,\mathbf{i} - 6t^2\,\mathbf{j}$$

Example 7 – *Solution*

cont'd

Equation 10 gives the normal component as

$$a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

$$= \frac{6\sqrt{2}\,t^2}{\sqrt{8t^2 + 9t^4}}$$

Kepler’s Laws of Planetary Motion

Kepler's Laws of Planetary Motion

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Since the gravitational force of the sun on a planet is so much larger than the forces exerted by other celestial bodies, we can safely ignore all bodies in the universe except the sun and one planet revolving about it.

We use a coordinate system with the sun at the origin and we let $\mathbf{r} = \mathbf{r}(t)$ be the position vector of the planet. (Equally well, $\mathbf{r}$ could be the position vector of the moon or a satellite moving around the earth or a comet moving around a star.)

Kepler's Laws of Planetary Motion

The velocity vector is $\mathbf{v} = \mathbf{r}'$ and the acceleration vector is $\mathbf{a} = \mathbf{r}''$.

We use the following laws of Newton:

Second Law of Motion: $\mathbf{F} = m\mathbf{a}$

Law of Gravitation: $\mathbf{F} = -\dfrac{GMm}{r^3}\mathbf{r} = -\dfrac{GMm}{r^2}\mathbf{u}$

where **F** is the gravitational force on the planet, m and M are the masses of the planet and the sun, G is the gravitational constant, $r = |\mathbf{r}|$, and $\mathbf{u} = (1/r)\mathbf{r}$ is the unit vector in the direction of **r**.

Kepler's Laws of Planetary Motion

We first show that the planet moves in one plane.

By equating the expressions for **F** in Newton's two laws, we find that

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

and so **a** is parallel to **r**.

It follows that $\mathbf{r} \times \mathbf{a} = \mathbf{0}$.

Kepler's Laws of Planetary Motion

We use the formula

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

to write

$$\begin{aligned} \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) &= \mathbf{r}' \times \mathbf{v} + \mathbf{r} \times \mathbf{v}' \\ &= \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} \\ &= \mathbf{0} + \mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

Kepler's Laws of Planetary Motion

Therefore

$$\mathbf{r} \times \mathbf{v} = \mathbf{h}$$

where **h** is a constant vector. (We may assume that $\mathbf{h} \neq 0$; that is, **r** and **v** are not parallel.)

This means that the vector $\mathbf{r} = \mathbf{r}(t)$ is perpendicular to **h** for all values of t, so the planet always lies in the plane through the origin perpendicular to **h**.

Thus the orbit of the planet is a plane curve.

Kepler's Laws of Planetary Motion

To prove Kepler's First Law we rewrite the vector **h** as follows:

$$\begin{aligned}\mathbf{h} &= \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \mathbf{r}' = r\mathbf{u} \times (r\mathbf{u})' \\ &= r\mathbf{u} \times (r\mathbf{u}' + r'\mathbf{u}) = r^2(\mathbf{u} \times \mathbf{u}') + rr'(\mathbf{u} \times \mathbf{u}) \\ &= r^2(\mathbf{u} \times \mathbf{u}')\end{aligned}$$

Then

$$\begin{aligned}\mathbf{a} \times \mathbf{h} &= \frac{-GM}{r^2}\mathbf{u} \times (r^2\mathbf{u} \times \mathbf{u}') = -GM\,\mathbf{u} \times (\mathbf{u} \times \mathbf{u}') \\ &= -GM\,[(\mathbf{u} \cdot \mathbf{u}')\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{u}']\end{aligned}$$

by Formula
$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Kepler's Laws of Planetary Motion

But $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 = 1$ and, since $|\mathbf{u}(t)| = 1$, it follows that $\mathbf{u} \cdot \mathbf{u}' = 0$.

Therefore

$$\mathbf{a} \times \mathbf{h} = GM\,\mathbf{u}'$$

and so

$$\begin{aligned}(\mathbf{v} \times \mathbf{h})' &= \mathbf{v}' \times \mathbf{h} \\ &= \mathbf{a} \times \mathbf{h} \\ &= GM\,\mathbf{u}'\end{aligned}$$

Kepler's Laws of Planetary Motion

Integrating both sides of this equation, we get

$$\boxed{11} \qquad \mathbf{v} \times \mathbf{h} = GM\,\mathbf{u} + \mathbf{c}$$

where **c** is a constant vector.

At this point it is convenient to choose the coordinate axes so that the standard basis vector **k** points in the direction of the vector **h**.

Then the planet moves in the *xy*-plane. Since both $\mathbf{v} \times \mathbf{h}$ and **u** are perpendicular to **h**, Equation 11 shows that **c** lies in the *xy*-plane.

Kepler's Laws of Planetary Motion

This means that we can choose the *x*- and *y*-axes so that the vector **i** lies in the direction of **c**, as shown in Figure 8.

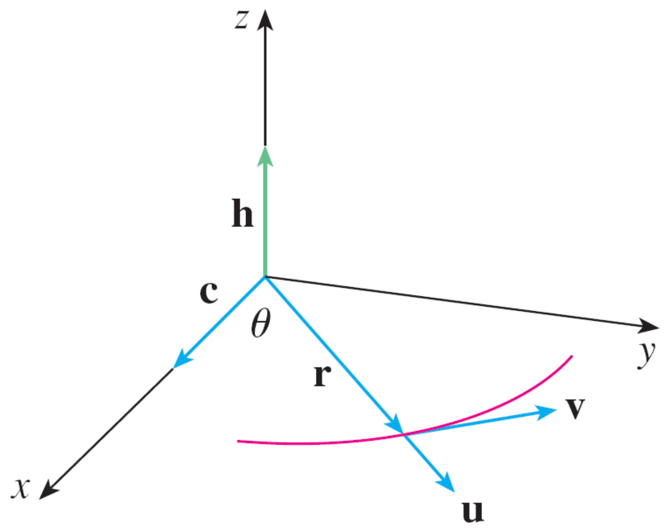


Figure 8

Kepler's Laws of Planetary Motion

If θ is the angle between **c** and **r**, then (r, θ) are polar coordinates of the planet.

From Equation 11 we have

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = \mathbf{r} \cdot (GM\,\mathbf{u} + \mathbf{c}) = GM\,\mathbf{r} \cdot \mathbf{u} + \mathbf{r} \cdot \mathbf{c}$$

$$= GMr\,\mathbf{u} \cdot \mathbf{u} + |\mathbf{r}|\,|\mathbf{c}| \cos\theta$$

$$= GMr + rc \cos\theta$$

where $c = |\mathbf{c}|$.

Kepler's Laws of Planetary Motion

Then

$$r = \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{GM + c \cos\theta} = \frac{1}{GM} \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{1 + e \cos\theta}$$

where $e = c/(GM)$.

But

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = |\mathbf{h}|^2 = h^2$$

where $h = |\mathbf{h}|$.

Kepler's Laws of Planetary Motion

So

$$r = \frac{h^2/(GM)}{1 + e\cos\theta} = \frac{eh^2/c}{1 + e\cos\theta}$$

Writing $d = h^2/c$, we obtain the equation

$$\boxed{12} \qquad r = \frac{ed}{1 + e\cos\theta}$$

we see that Equation 12 is the polar equation of a conic section with focus at the origin and eccentricity *e*. We know that the orbit of a planet is a closed curve and so the conic must be an ellipse.