

14

Partial Derivatives

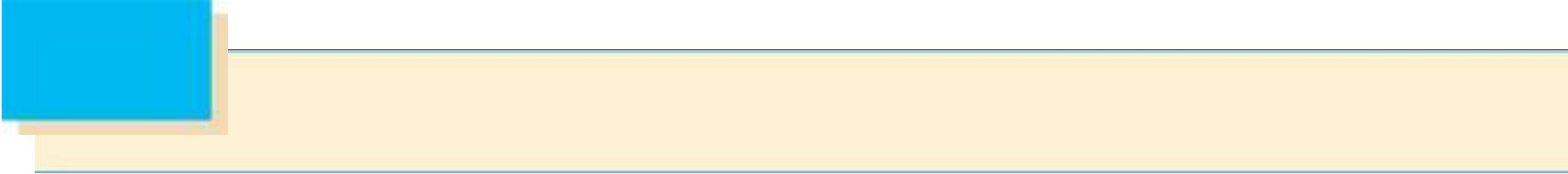


14.1 Functions of Several Variables

Functions of Several Variables

In this section we study functions of two or more variables from four points of view:

- verbally (by a description in words)
- numerically (by a table of values)
- algebraically (by an explicit formula)
- visually (by a graph or level curves)



Functions of Two Variables

Functions of Several Variables

The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point.

We can think of T as being a function of the two variables x and y , or as a function of the pair (x, y) . We indicate this functional dependence by writing $T = f(x, y)$.

The volume V of a circular cylinder depends on its radius r and its height h . In fact, we know that $V = \pi r^2 h$. We say that V is a function of r and h , and we write $V(r, h) = \pi r^2 h$.

Functions of Several Variables

Definition A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y) .

The variables x and y are **independent variables** and z is the **dependent variable**.

[Compare this with the notation $y = f(x)$ for functions of a single variable.]

Example 2

In regions with severe winter weather, the *wind-chill index* is often used to describe the apparent severity of the cold.

This index W is a subjective temperature that depends on the actual temperature T and the wind speed v .

So W is a function of T and v , and we can write $W = f(T, v)$.

Example 2

cont'd

Table 1 records values of W compiled by the National Weather Service of the US and the Meteorological Service of Canada.

		Wind speed (km/h)										
Actual temperature (°C)	$T \backslash v$	5	10	15	20	25	30	40	50	60	70	80
	5	4	3	2	1	1	0	-1	-1	-2	-2	-3
	0	-2	-3	-4	-5	-6	-6	-7	-8	-9	-9	-10
	-5	-7	-9	-11	-12	-12	-13	-14	-15	-16	-16	-17
	-10	-13	-15	-17	-18	-19	-20	-21	-22	-23	-23	-24
	-15	-19	-21	-23	-24	-25	-26	-27	-29	-30	-30	-31
	-20	-24	-27	-29	-30	-32	-33	-34	-35	-36	-37	-38
	-25	-30	-33	-35	-37	-38	-39	-41	-42	-43	-44	-45
	-30	-36	-39	-41	-43	-44	-46	-48	-49	-50	-51	-52
	-35	-41	-45	-48	-49	-51	-52	-54	-56	-57	-58	-60
	-40	-47	-51	-54	-56	-57	-59	-61	-63	-64	-65	-67

Wind-chill index as a function of air temperature and wind speed

Table 1

Example 2

cont'd

For instance, the table shows that if the temperature is -5°C and the wind speed is 50 km/h, then subjectively it would feel as cold as a temperature of about -15°C with no wind.

So

$$f(-5, 50) = -15$$

Example 3

In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899–1922.

They considered a simplified view of the economy in which production output is determined by the amount of labor involved and the amount of capital invested.

While there are many other factors affecting economic performance, their model proved to be remarkably accurate.

Example 3

cont'd

The function they used to model production was of the form

1

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is the amount of capital invested (the monetary worth of all machinery, equipment, and buildings).

Example 3

cont'd

Cobb and Douglas used economic data published by the government to obtain Table 2.

Year	P	L	K
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208

Year	P	L	K
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	266
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

Table 2

Example 3

cont'd

They took the year 1899 as a baseline and P , L , and K for 1899 were each assigned the value 100.

The values for other years were expressed as percentages of the 1899 figures.

Cobb and Douglas used the method of least squares to fit the data of Table 2 to the function

2

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

Example 3

cont'd

If we use the model given by the function in Equation 2 to compute the production in the years 1910 and 1920, we get the values

$$P(147, 208) = 1.01(147)^{0.75}(208)^{0.25} \approx 161.9$$

$$P(194, 407) = 1.01(194)^{0.75}(407)^{0.25} \approx 235.8$$

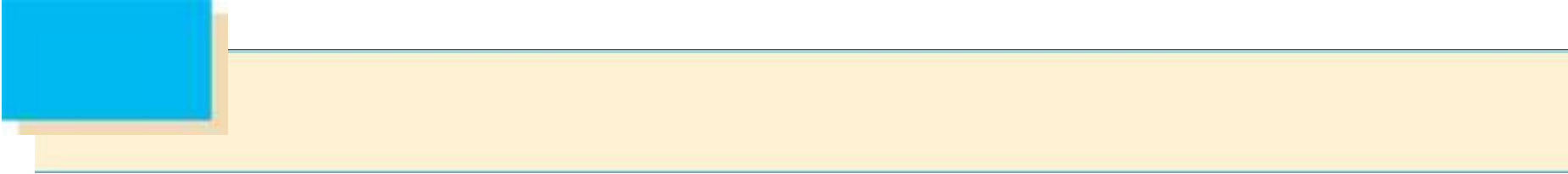
which are quite close to the actual values, 159 and 231.

The production function $\boxed{1}$ has subsequently been used in many settings, ranging from individual firms to global economics. It has become known as the **Cobb-Douglas production function**.

Example 3

cont'd

Its domain is $\{(L, K) \mid L \geq 0, K \geq 0\}$ because L and K represent labor and capital and are therefore never negative.



Graphs

Graphs

Another way of visualizing the behavior of a function of two variables is to consider its graph.

Definition If f is a function of two variables with domain D , then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D .

Just as the graph of a function f of one variable is a curve C with equation $y = f(x)$, so the graph of a function f of two variables is a surface S with equation $z = f(x, y)$.

Graphs

We can visualize the graph S of f as lying directly above or below its domain D in the xy -plane (see Figure 5).

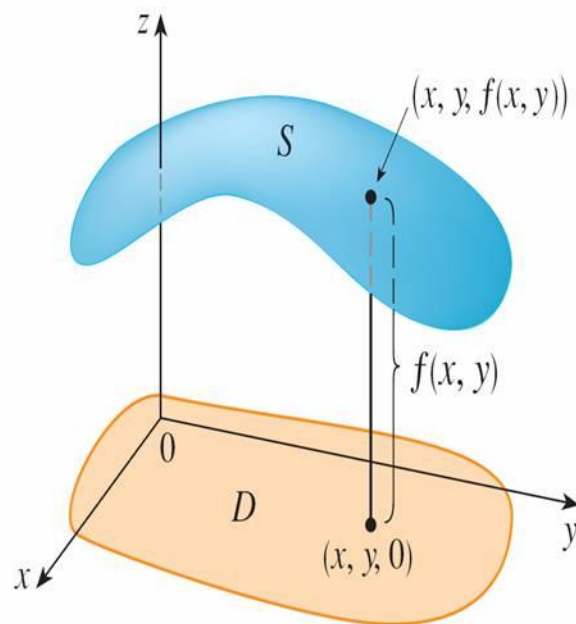


Figure 5

Graphs

The function $f(x, y) = ax + by + c$ is called as a **linear function**.

The graph of such a function has the equation

$$z = ax + by + c \quad \text{or} \quad ax + by - z + c = 0$$

so it is a plane. In much the same way that linear functions of one variable are important in single-variable calculus, we will see that linear functions of two variables play a central role in multivariable calculus.

Example 6

Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution:

The graph has equation $z = \sqrt{9 - x^2 - y^2}$. We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$, which we recognize as an equation of the sphere with center the origin and radius 3.

But, since $z \geq 0$, the graph of g is just the top half of this sphere (see Figure 7).

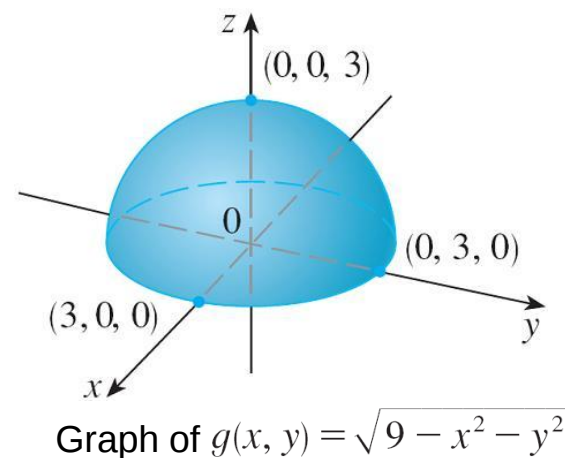
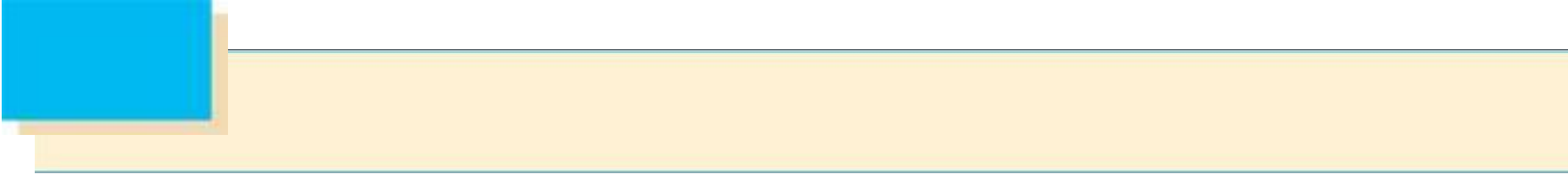


Figure 7



Level Curves

Level Curves

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour lines*, or *level curves*.

Definition The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k .

In other words, it shows where the graph of f has height k .

Level Curves

You can see from Figure 11 the relation between level curves and horizontal traces.

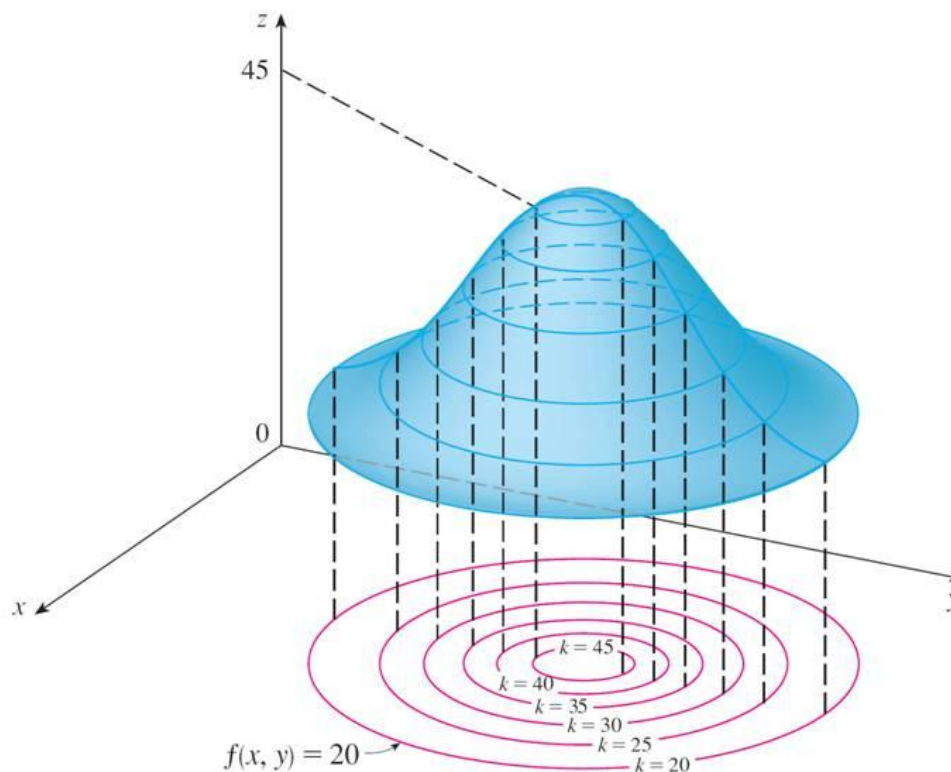


Figure 11

Level Curves

The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy -plane.

So if you draw the level curves of a function and visualize them being lifted up to the surface at the indicated height, then you can mentally piece together a picture of the graph.

The surface is steep where the level curves are close together. It is somewhat flatter where they are farther apart.

Level Curves

One common example of level curves occurs in topographic maps of mountainous regions, such as the map in Figure 12.

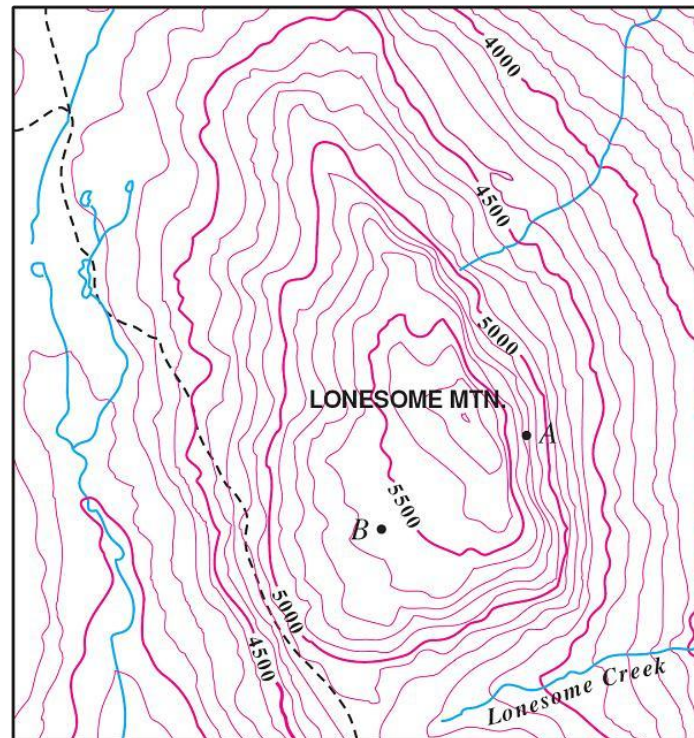


Figure 12

Level Curves

The level curves are curves of constant elevation above sea level.

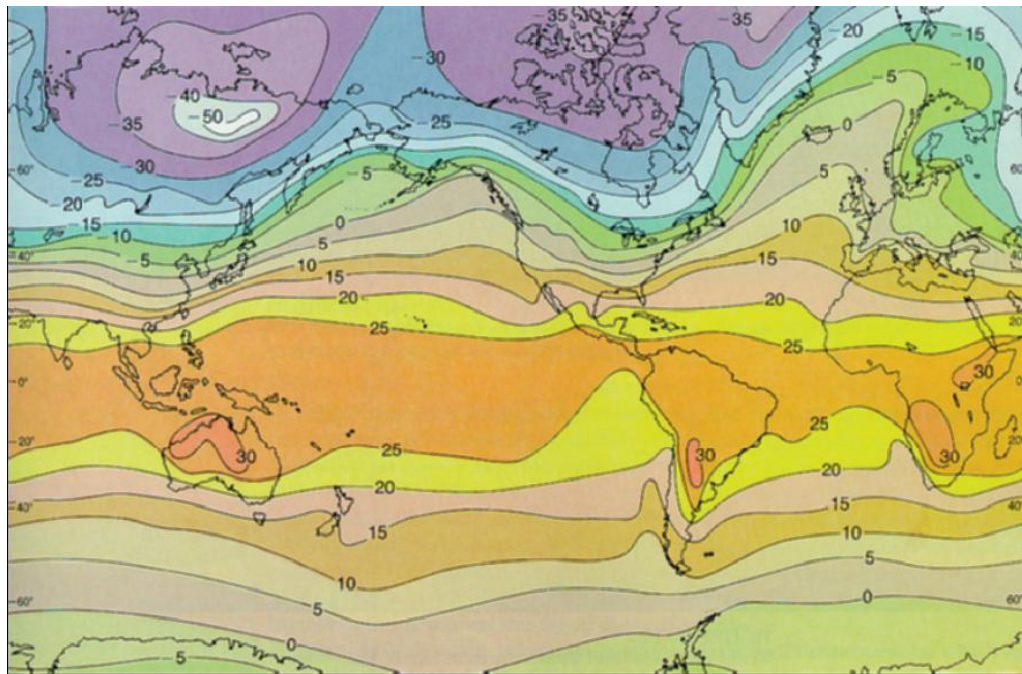
If you walk along one of these contour lines, you neither ascend nor descend.

Another common example is the temperature function introduced in the opening paragraph of this section.

Here the level curves are called **isothermals** and join locations with the same temperature.

Level Curves

Figure 13 shows a weather map of the world indicating the average January temperatures. The isothermals are the curves that separate the colored bands.

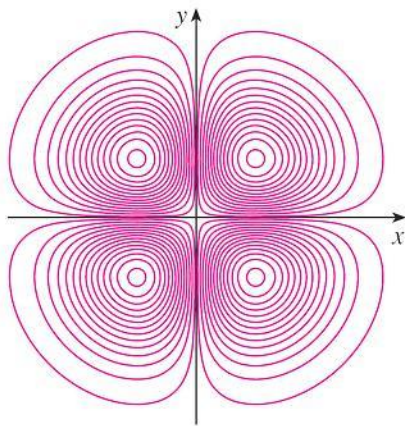


World mean sea-level temperatures in January in degrees Celsius

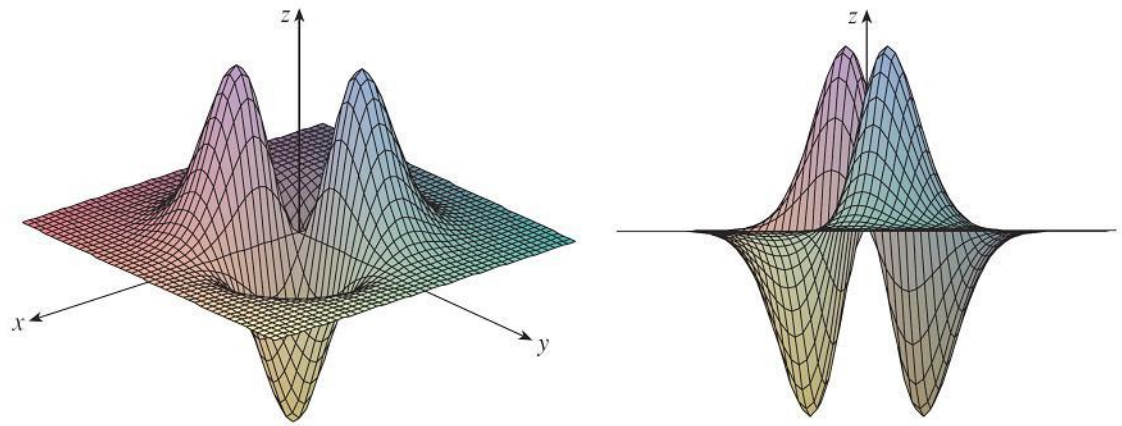
Figure 13

Level Curves

For some purposes, a contour map is more useful than a graph. It is true in estimating function values. Figure 19 shows some computer-generated level curves together with the corresponding computer-generated graphs.



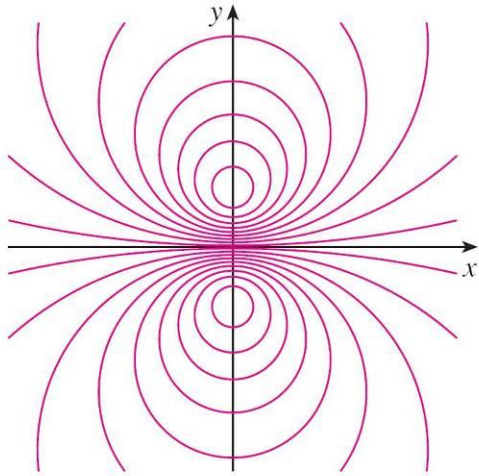
(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$



(b) Two views of $f(x, y) = -xye^{-x^2-y^2}$

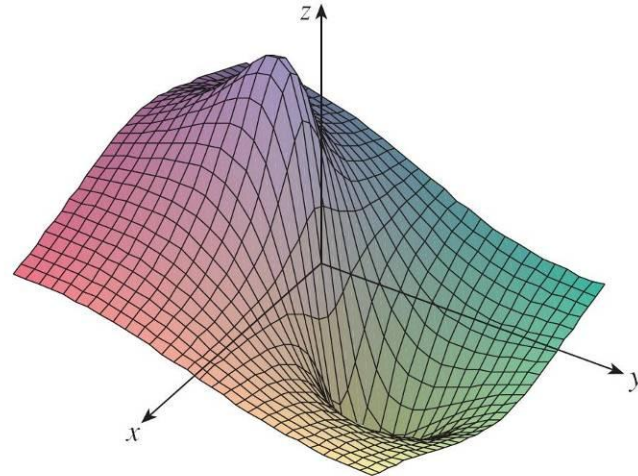
Figure 19

Level Curves



Level curves of $f(x, y) = \frac{-3y}{x^2 + y^2 + 1}$

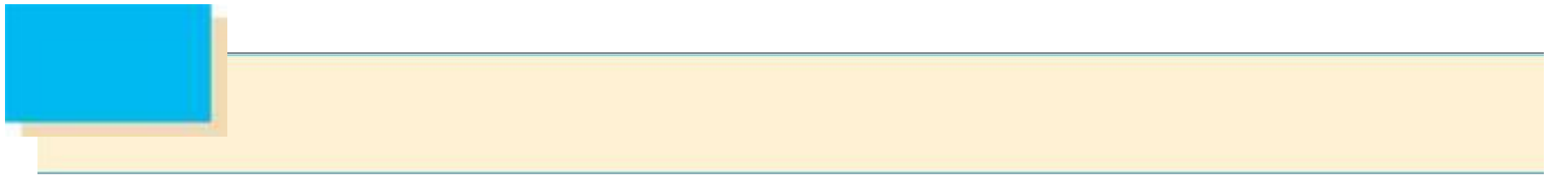
Figure 19(c)



$f(x, y) = \frac{-3y}{x^2 + y^2 + 1}$

Figure 19(d)

Notice that the level curves in Figure 19(c) crowd together near the origin. That corresponds to the fact that the graph in Figure 19(d) is very steep near the origin.



Functions of Three or More Variables

Functions of Three or More Variables

A **function of three variables**, f , is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t , so we could write $T = f(x, y, t)$.

Example 14

Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

Solution:

The expression for $f(x, y, z)$ is defined as long as $z - y > 0$, so the domain of f is

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid z > y\}$$

This is a **half-space** consisting of all points that lie above the plane $z = y$.

Functions of Three or More Variables

It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space.

However, we do gain some insight into f by examining its **level surfaces**, which are the surfaces with equations $f(x, y, z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

Functions of any number of variables can be considered. A **function of n variables** is a rule that assigns a number $z = f(x_1, x_2, \dots, x_n)$ to an n -tuple $(x_1, x_2, \dots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n -tuples.

Functions of Three or More Variables

For example, if a company uses n different ingredients in making a food product, c_i is the cost per unit of the i th ingredient, and x_i units of the i th ingredient are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, \dots, x_n$:

$$\boxed{3} \quad C = f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

The function f is a real-valued function whose domain is a subset of $\mathbb{R}^n$.

Functions of Three or More Variables

Sometimes we will use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \dots, x_n)$.

With this notation we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where $\mathbf{c} = \langle c_1, c_2, \dots, c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors $\mathbf{c}$ and $\mathbf{x}$ in V_n .

Functions of Three or More Variables

In view of the one-to-one correspondence between points $(x_1, x_2, \dots, x_n)$ in $\mathbb{R}^n$ and their position vectors

$\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$ in V_n , we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

1. As a function of n real variables $x_1, x_2, \dots, x_n$
2. As a function of a single point variable $(x_1, x_2, \dots, x_n)$
3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$

14

Partial Derivatives



14.2

Limits and Continuity

Limits and Continuity

Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

and

$$g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as x and y both approach 0 [and therefore the point (x, y) approaches the origin].

Limits and Continuity

Tables 1 and 2 show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin. (Notice that neither function is defined at the origin.)

$x \backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
-0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
-0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Values of $f(x, y)$

Table 1

Limits and Continuity

$x \backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
-0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
-0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0	-1.000	-1.000	-1.000		-1.000	-1.000	-1.000
0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

Values of $g(x, y)$

Table 2

Limits and Continuity

It appears that as (x, y) approaches $(0, 0)$, the values of $f(x, y)$ are approaching 1 whereas the values of $g(x, y)$ aren't approaching any number. It turns out that these guesses based on numerical evidence are correct, and we write

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1 \quad \text{and}$$

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 - y^2}{x^2 + y^2} \quad \text{does not exist}$$

Limits and Continuity

In general, we use the notation

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f .

Limits and Continuity

In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b) , but not equal to (a, b) . A more precise definition follows.

1 Definition Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b) . Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if $(x, y) \in D$ and $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$ then $|f(x, y) - L| < \varepsilon$

Limits and Continuity

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = L \quad \text{and}$$

$$f(x, y) \rightarrow L \text{ as } (x, y) \rightarrow (a, b)$$

For functions of a single variable, when we let x approach a , there are only two possible directions of approach, from the left or from the right.

We recall that if $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$, then $\lim_{x \rightarrow a} f(x)$ does not exist.

Limits and Continuity

For functions of two variables the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see Figure 3) as long as (x, y) stays within the domain of f .

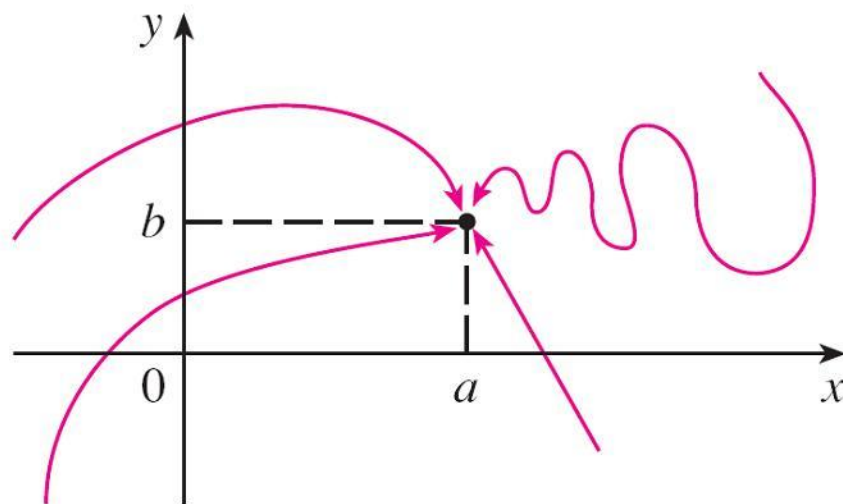


Figure 3

Limits and Continuity

Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0).

The definition refers only to the *distance* between (x, y) and (a, b) . It does not refer to the direction of approach.

Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b) .

Limits and Continuity

Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x, y) \rightarrow (a, b)} f(x, y)$ does not exist.

If $f(x, y) \rightarrow L_1$ as $(x, y) \rightarrow (a, b)$ along a path C_1 and $f(x, y) \rightarrow L_2$ as $(x, y) \rightarrow (a, b)$ along a path C_2 , where $L_1 \neq L_2$, then $\lim_{(x, y) \rightarrow (a, b)} f(x, y)$ does not exist.

Example 1

Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution:

Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$.

First let's approach $(0, 0)$ along the x -axis.

Then $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$, so

$f(x, y) \rightarrow 1$ as $(x, y) \rightarrow (0, 0)$ along the x -axis

Example 1 – Solution

cont'd

We now approach along the y -axis by putting $x = 0$.

Then $f(0, y) = \frac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$f(x, y) \rightarrow -1$ as $(x, y) \rightarrow (0, 0)$ along the y -axis

(See Figure 4.)

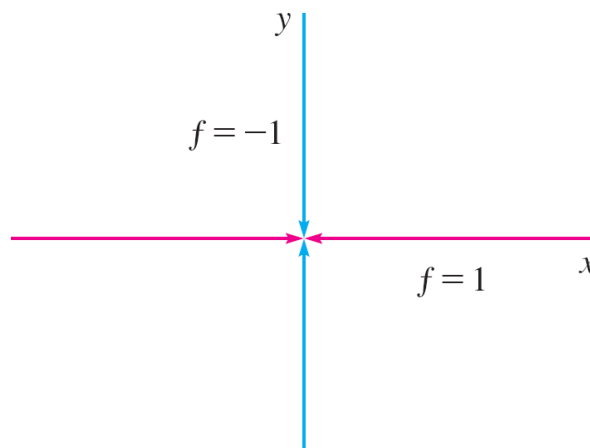


Figure 4

Example 1 – *Solution*

cont'd

Since f has two different limits along two different lines, the given limit does not exist. (This confirms the conjecture we made on the basis of numerical evidence at the beginning of this section.)

Limits and Continuity

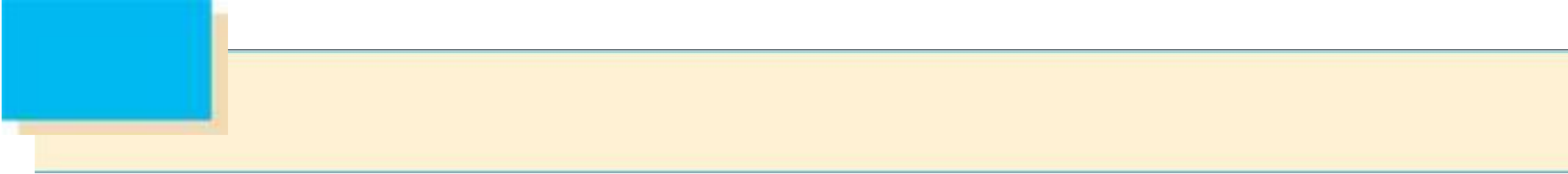
Now let's look at limits that *do* exist. Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits.

The Limit Laws can be extended to functions of two variables: The limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and so on.

In particular, the following equations are true.

$$\boxed{2} \quad \lim_{(x,y) \rightarrow (a,b)} x = a \qquad \lim_{(x,y) \rightarrow (a,b)} y = b \qquad \lim_{(x,y) \rightarrow (a,b)} c = c$$

The Squeeze Theorem also holds.



Continuity

Continuity

Recall that evaluating limits of *continuous* functions of a single variable is easy.

It can be accomplished by direct substitution because the defining property of a continuous function is

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Continuous functions of two variables are also defined by the direct substitution property.

4 Definition A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D .

Continuity

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount.

This means that a surface that is the graph of a continuous function has no hole or break.

Using the properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains.

Let's use this fact to give examples of continuous functions.

Continuity

A **polynomial function of two variables** (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers.

A **rational function** is a ratio of polynomials.

For instance,

$$f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

Continuity

The limits in [2] show that the functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = c$ are continuous.

Since any polynomial can be built up out of the simple functions f , g , and h by multiplication and addition, it follows that *all polynomials are continuous on $\mathbb{R}^2$* .

Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions.

Example 5

Evaluate $\lim_{(x,y) \rightarrow (1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

Solution:

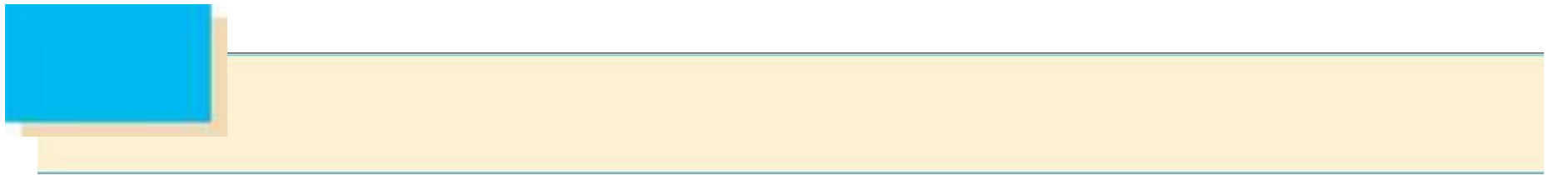
Since $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, it is continuous everywhere, so we can find the limit by direct substitution:

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) &= 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 \\ &\quad + 2 \cdot 2 \\ &= 11 \end{aligned}$$

Continuity

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third.

In fact, it can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f , then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.



Functions of Three or More Variables

Functions of Three or More Variables

Everything that we have done in this section can be extended to functions of three or more variables.

The notation

$$\lim_{(x, y, z) \rightarrow (a, b, c)} f(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) along any path in the domain of f .

Functions of Three or More Variables

Because the distance between two points (x, y, z) and (a, b, c) in $\mathbb{R}^3$ is given by $\sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$, we can write the precise definition as follows: For every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if (x, y, z) is in the domain of f and

$$0 < \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} < \delta$$

$$\text{then } |f(x, y, z) - L| < \varepsilon$$

Functions of Three or More Variables

The function f is **continuous** at (a, b, c) if

$$\lim_{(x, y, z) \rightarrow (a, b, c)} f(x, y, z) = f(a, b, c)$$

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in $\mathbb{R}^3$ except where $x^2 + y^2 + z^2 = 1$. In other words, it is discontinuous on the sphere with center the origin and radius 1.

Functions of Three or More Variables

We can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

5 If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } \mathbf{x} \in D \text{ and } 0 < |\mathbf{x} - \mathbf{a}| < \delta \text{ then } |f(\mathbf{x}) - L| < \varepsilon$$

14

Partial Derivatives



14.3

Partial Derivatives

Partial Derivatives

On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates.

The National Weather Service has devised the *heat index* (also called the temperature-humidity index, or humidex, in some countries) to describe the combined effects of temperature and humidity.

The heat index I is the perceived air temperature when the actual temperature is T and the relative humidity is H . So I is a function of T and H and we can write $I = f(T, H)$.

Partial Derivatives

The following table of values of I is an excerpt from a table compiled by the National Weather Service.

		Relative humidity (%)								
Actual temperature (°F)	$T \backslash H$	50	55	60	65	70	75	80	85	90
	90	96	98	100	103	106	109	112	115	119
	92	100	103	105	108	112	115	119	123	128
	94	104	107	111	114	118	122	127	132	137
	96	109	113	116	121	125	130	135	141	146
	98	114	118	123	127	133	138	144	150	157
	100	119	124	129	135	141	147	154	161	168

Heat index I as a function of
temperature and humidity

Table 1

Partial Derivatives

If we concentrate on the highlighted column of the table, which corresponds to a relative humidity of $H = 70\%$, we are considering the heat index as a function of the single variable T for a fixed value of H . Let's write $g(T) = f(T, 70)$.

Then $g(T)$ describes how the heat index I increases as the actual temperature T increases when the relative humidity is 70%.

The derivative of g when $T = 96^\circ\text{F}$ is the rate of change of I with respect to T when $T = 96^\circ\text{F}$:

$$g'(96) = \lim_{h \rightarrow 0} \frac{g(96 + h) - g(96)}{h} = \lim_{h \rightarrow 0} \frac{f(96 + h, 70) - f(96, 70)}{h}$$

Partial Derivatives

We can approximate $g'(96)$ using the values in Table 1 by taking $h = 2$ and -2 :

$$g'(96) \approx \frac{g(98) - g(96)}{2} = \frac{f(98, 70) - f(96, 70)}{2} = \frac{133 - 125}{2} = 4$$

$$g'(96) \approx \frac{g(94) - g(96)}{-2} = \frac{f(94, 70) - f(96, 70)}{-2} = \frac{118 - 125}{-2} = 3.5$$

Averaging these values, we can say that the derivative $g'(96)$ is approximately 3.75.

Partial Derivatives

This means that, when the actual temperature is 96°F and the relative humidity is 70%, the apparent temperature (heat index) rises by about 3.75°F for every degree that the actual temperature rises!

Partial Derivatives

Now let's look at the highlighted row in Table 1, which corresponds to a fixed temperature of $T = 96^\circ\text{F}$.

		Relative humidity (%)								
Actual temperature (°F)	$T \backslash H$	50	55	60	65	70	75	80	85	90
	90	96	98	100	103	106	109	112	115	119
	92	100	103	105	108	112	115	119	123	128
	94	104	107	111	114	118	122	127	132	137
	96	109	113	116	121	125	130	135	141	146
	98	114	118	123	127	133	138	144	150	157
	100	119	124	129	135	141	147	154	161	168

Heat index I as a function of
temperature and humidity

Table 1

Partial Derivatives

The numbers in this row are values of the function $G(H) = f(96, H)$, which describes how the heat index increases as the relative humidity H increases when the actual temperature is $T = 96^\circ\text{F}$.

The derivative of this function when $H = 70\%$ is the rate of change of I with respect to H when $H = 70\%$:

$$G'(70) = \lim_{h \rightarrow 0} \frac{G(70 + h) - G(70)}{h} = \lim_{h \rightarrow 0} \frac{f(96, 70 + h) - f(96, 70)}{h}$$

Partial Derivatives

By taking $h = 5$ and -5 , we approximate $G'(70)$ using the tabular values:

$$G'(70) \approx \frac{G(75) - G(70)}{5} = \frac{f(96, 75) - f(96, 70)}{5} = \frac{130 - 125}{5} = 1$$

$$G'(70) \approx \frac{G(65) - G(70)}{-5} = \frac{f(96, 65) - f(96, 70)}{-5} = \frac{121 - 125}{-5} = 0.8$$

By averaging these values we get the estimate $G'(70) \approx 0.9$. This says that, when the temperature is 96°F and the relative humidity is 70%, the heat index rises about 0.9°F for every percent that the relative humidity rises.

Partial Derivatives

In general, if f is a function of two variables x and y , suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant.

Then we are really considering a function of a single variable x , namely, $g(x) = f(x, b)$. If g has a derivative at a , then we call it the **partial derivative of f with respect to x at (a, b)** and denote it by $f_x(a, b)$. Thus

1

$$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$$

Partial Derivatives

By the definition of a derivative, we have

$$g'(a) = \lim_{h \rightarrow 0} \frac{g(a + h) - g(a)}{h}$$

and so Equation 1 becomes

2

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Partial Derivatives

Similarly, the **partial derivative of f with respect to y at (a, b)** , denoted by $f_y(a, b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

3

$$f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b + h) - f(a, b)}{h}$$

With this notation for partial derivatives, we can write the rates of change of the heat index I with respect to the actual temperature T and relative humidity H when $T = 96^\circ\text{F}$ and $H = 70\%$ as follows:

$$f_T(96, 70) \approx 3.75 \qquad f_H(96, 70) \approx 0.9$$

Partial Derivatives

If we now let the point (a, b) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

4 If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Partial Derivatives

There are many alternative notations for partial derivatives.

For instance, instead of f_x we can write f_1 or D_1f (to indicate differentiation with respect to the *first* variable) or $\partial f / \partial x$.

But here $\partial f / \partial x$ can't be interpreted as a ratio of differentials.

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1f = D_xf$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2f = D_yf$$

Partial Derivatives

To compute partial derivatives, all we have to do is remember from Equation 1 that the partial derivative with respect to x is just the *ordinary* derivative of the function g of a single variable that we get by keeping y fixed.

Thus we have the following rule.

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x , regard y as a constant and differentiate $f(x, y)$ with respect to x .
2. To find f_y , regard x as a constant and differentiate $f(x, y)$ with respect to y .

Example 1

If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Solution:

Holding y constant and differentiating with respect to x , we get

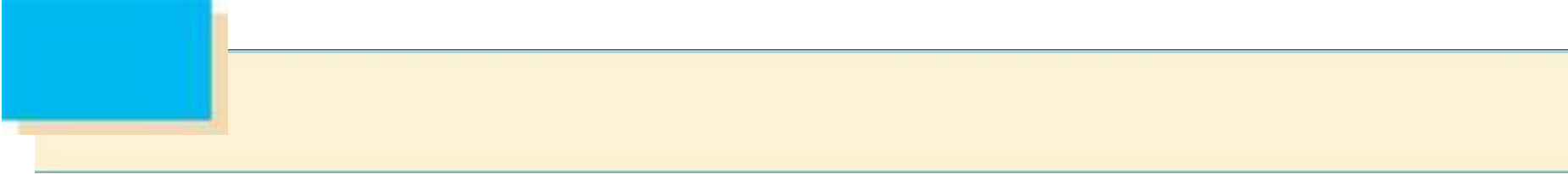
$$f_x(x, y) = 3x^2 + 2xy^3$$

and so
$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiating with respect to y , we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$



Interpretations of Partial Derivatives

Interpretations of Partial Derivatives

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S .

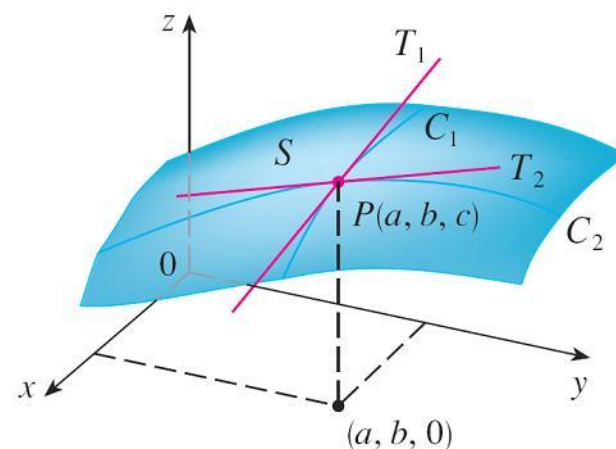
By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S . (In other words, C_1 is the trace of S in the plane $y = b$.)

Interpretations of Partial Derivatives

Likewise, the vertical plane $x = a$ intersects S in a curve C_2 . Both of the curves C_1 and C_2 pass through the point P . (See Figure 1.)

Notice that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$.

The curve C_2 is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.



The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2 .

Figure 1

Interpretations of Partial Derivatives

Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

As we have seen in the case of the heat index function, partial derivatives can also be interpreted as *rates of change*.

If $z = f(x, y)$, then $\partial z / \partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z / \partial y$ represents the rate of change of z with respect to y when x is fixed.

Example 2

If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

Solution:

We have

$$f_x(x, y) = -2x$$

$$f_y(x, y) = -4y$$

$$f_x(1, 1) = -2$$

$$f_y(1, 1) = -4$$

Example 2 – Solution

cont'd

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$. (As in the preceding discussion, we label it C_1 in Figure 2.)

The slope of the tangent line to this parabola at the point $(1, 1, 1)$ is $f_x(1, 1) = -2$.

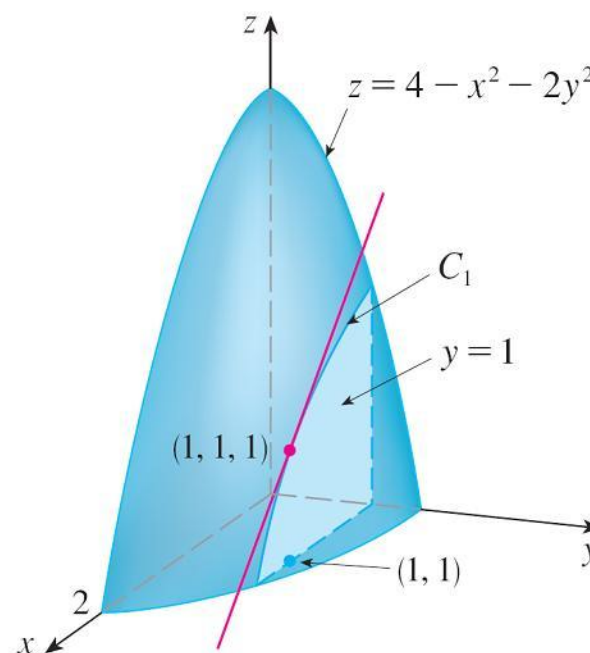


Figure 2

Example 2 – Solution

cont'd

Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$, $x = 1$, and the slope of the tangent line at $(1, 1, 1)$ is $f_y(1, 1) = -4$. (See Figure 3.)

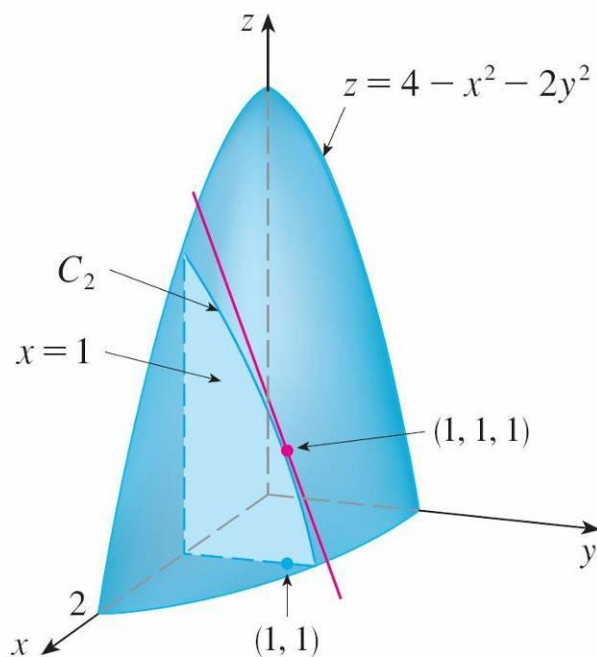
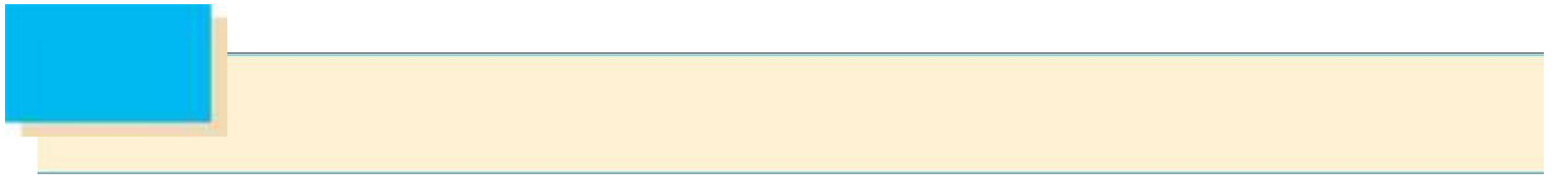


Figure 3



Functions of More Than Two Variables

Functions of More Than Two Variables

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x , y , and z , then its partial derivative with respect to x is defined as

$$f_x(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding y and z as constants and differentiating $f(x, y, z)$ with respect to x .

Functions of More Than Two Variables

If $w = f(x, y, z)$, then $f_x = \partial w / \partial x$ can be interpreted as the rate of change of w with respect to x when y and z are held fixed. But we can't interpret it geometrically because the graph of f lies in four-dimensional space.

In general, if u is a function of n variables, $u = f(x_1, x_2, \dots, x_n)$, its partial derivative with respect to the i th variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

Example 5

Find f_x , f_y , and f_z if $f(x, y, z) = e^{xy} \ln z$.

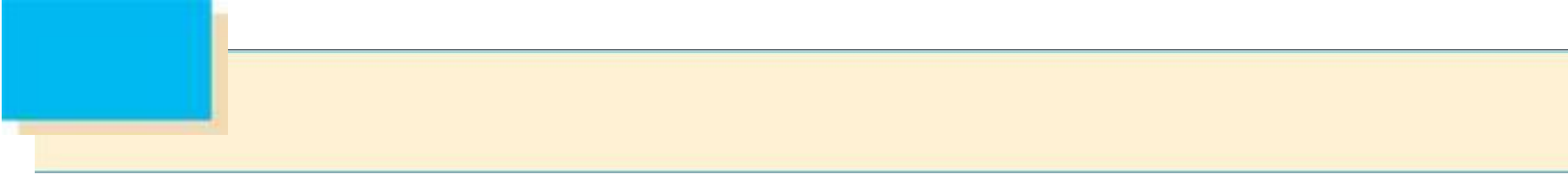
Solution:

Holding y and z constant and differentiating with respect to x , we have

$$f_x = ye^{xy} \ln z$$

Similarly,

$$f_y = xe^{xy} \ln z \quad \text{and} \quad f_z = \frac{e^{xy}}{z}$$



Higher Derivatives

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f .

If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

Higher Derivatives

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\partial^2 f / \partial y \partial x$) means that we first differentiate with respect to x and then with respect to y , whereas in computing f_{yx} the order is reversed.

Example 6

Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

Solution:

In Example 1 we found that

$$f_x(x, y) = 3x^2 + 2xy^3 \quad f_y(x, y) = 3x^2y^2 - 4y$$

Therefore

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x} (3x^2 + 2xy^3) \\ &= 6x + 2y^3 \end{aligned}$$

Example 6 – *Solution*

cont'd

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y} (3x^2 + 2xy^3) \\ &= 6xy^2 \end{aligned}$$

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial x} (3x^2y^2 - 4y) \\ &= 6xy^2 \end{aligned}$$

$$\begin{aligned} f_{yy} &= \frac{\partial}{\partial y} (3x^2y^2 - 4y) \\ &= 6x^2y - 4 \end{aligned}$$

Higher Derivatives

Notice that $f_{xy} = f_{yx}$ in Example 6. This is not just a coincidence.

It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most functions that one meets in practice.

The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b) . If the functions f_{xy} and f_{yx} are both continuous on D , then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

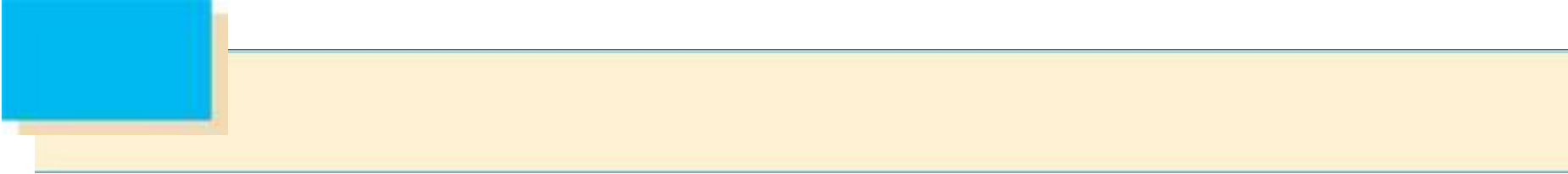
Higher Derivatives

Partial derivatives of order 3 or higher can also be defined.
For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y} \left(\frac{\partial^2 f}{\partial y \partial x} \right) = \frac{\partial^3 f}{\partial y^2 \partial x}$$

and using Clairaut's Theorem it can be shown that

$f_{xyy} = f_{yx y} = f_{yyx}$ if these functions are continuous.



Partial Differential Equations

Partial Differential Equations

Partial derivatives occur in *partial differential equations* that express certain physical laws.

For instance, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called **Laplace's equation** after Pierre Laplace (1749–1827).

Solutions of this equation are called **harmonic functions**; they play a role in problems of heat conduction, fluid flow, and electric potential.

Example 8

Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

Solution:

We first compute the needed second-order partial derivatives:

$$u_x = e^x \sin y$$

$$u_y = e^x \cos y$$

$$u_{xx} = e^x \sin y$$

$$u_{yy} = -e^x \sin y$$

So
$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

Therefore u satisfies Laplace's equation.

Partial Differential Equations

The wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string.

Partial Differential Equations

For instance, if $u(x, t)$ represents the displacement of a vibrating violin string at time t and at a distance x from one end of the string (as in Figure 8), then $u(x, t)$ satisfies the wave equation.

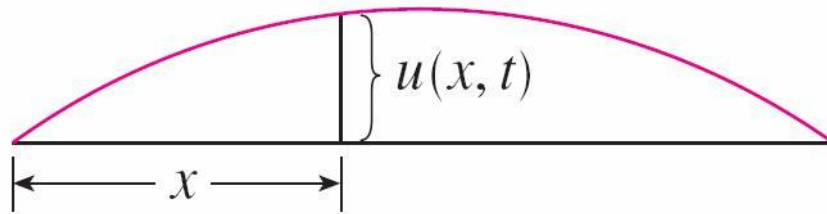


Figure 8

Here the constant a depends on the density of the string and on the tension in the string.

Partial Differential Equations

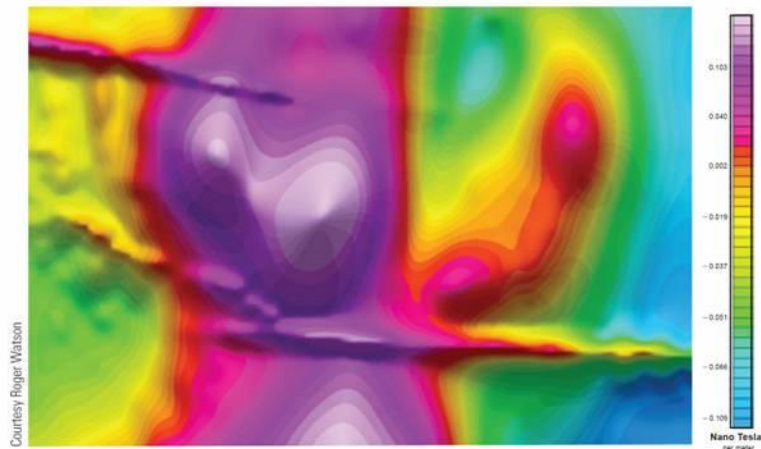
Partial differential equations involving functions of three variables are also very important in science and engineering. The three-dimensional Laplace equation is

$$\boxed{5} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

and one place it occurs is in geophysics. If $u(x, y, z)$ represents magnetic field strength at position (x, y, z) , then it satisfies Equation 5. The strength of the magnetic field indicates the distribution of iron-rich minerals and reflects different rock types and the location of faults.

Partial Differential Equations

Figure 9 shows a contour map of the earth's magnetic field as recorded from an aircraft carrying a magnetometer and flying 200 m above the surface of the ground. The contour map is enhanced by color-coding of the regions between the level curves.

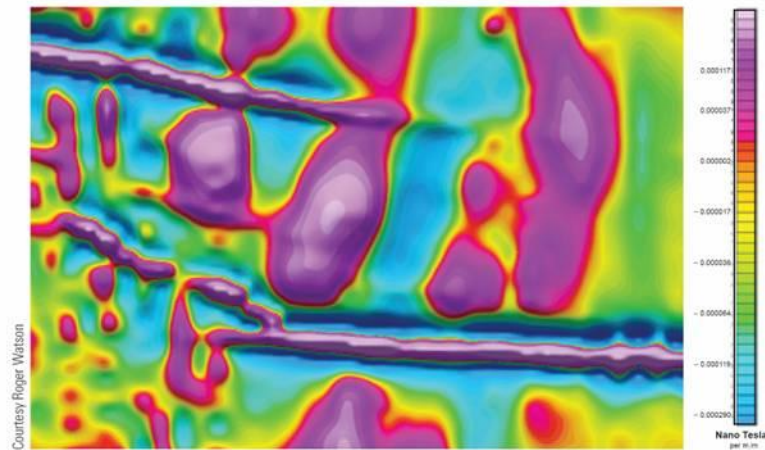


Magnetic field strength of the earth

Figure 9

Partial Differential Equations

Figure 10 shows a contour map for the second-order partial derivative of u in the vertical direction, that is, u_{zz} . It turns out that the values of the partial derivatives u_{xx} and u_{yy} are relatively easily measured from a map of the magnetic field. Then values of u_{zz} can be calculated from Laplace's equation [5].



Second vertical derivative of the magnetic field

Figure 10



The Cobb-Douglas Production Function

The Cobb-Douglas Production Function

We have described the work of Cobb and Douglas in modeling the total production P of an economic system as a function of the amount of labor L and the capital investment K .

Here we use partial derivatives to show how the particular form of their model follows from certain assumptions they made about the economy.

If the production function is denoted by $P = P(L, K)$, then the partial derivative $\partial P / \partial L$ is the rate at which production changes with respect to the amount of labor. Economists call it the marginal production with respect to labor or the ***marginal productivity of labor***.

The Cobb-Douglas Production Function

Likewise, the partial derivative $\partial P/\partial K$ is the rate of change of production with respect to capital and is called the ***marginal productivity of capital***.

In these terms, the assumptions made by Cobb and Douglas can be stated as follows.

- (i) If either labor or capital vanishes, then so will production.
- (ii) The marginal productivity of labor is proportional to the amount of production per unit of labor.
- (iii) The marginal productivity of capital is proportional to the amount of production per unit of capital.

The Cobb-Douglas Production Function

Because the production per unit of labor is P/L , assumption (ii) says that

$$\frac{\partial P}{\partial L} = \alpha \frac{P}{L}$$

for some constant α .

If we keep K constant ($K = K_0$), then this partial differential equation becomes an ordinary differential equation:

6

$$\frac{dP}{dL} = \alpha \frac{P}{L}$$

The Cobb-Douglas Production Function

If we solve this separable differential equation, we get

$$7 \quad P(L, K_0) = C_1(K_0)L^\alpha$$

Notice that we have written the constant C_1 as a function of K_0 because it could depend on the value of K_0 .

Similarly, assumption (iii) says that

$$\frac{\partial P}{\partial K} = \beta \frac{P}{K}$$

and we can solve this differential equation to get

$$8 \quad P(L_0, K) = C_2(L_0)K^\beta$$

The Cobb-Douglas Production Function

Comparing Equations 7 and 8, we have

$$\boxed{9} \quad P(L, K) = bL^{\alpha}K^{\beta}$$

where b is a constant that is independent of both L and K . Assumption (i) shows that $\alpha > 0$ and $\beta > 0$.

Notice from Equation 8 that if labor and capital are both increased by a factor m , then

$$P(mL, mK) = b(mL)^{\alpha}(mK)^{\beta} = m^{\alpha+\beta}bL^{\alpha}K^{\beta} = m^{\alpha+\beta}P(L, K)$$

The Cobb-Douglas Production Function

If $\alpha + \beta = 1$, then $P(mL, mK) = mP(L, K)$, which means that production is also increased by a factor of m . That is why Cobb and Douglas assumed that $\alpha + \beta = 1$ and therefore

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

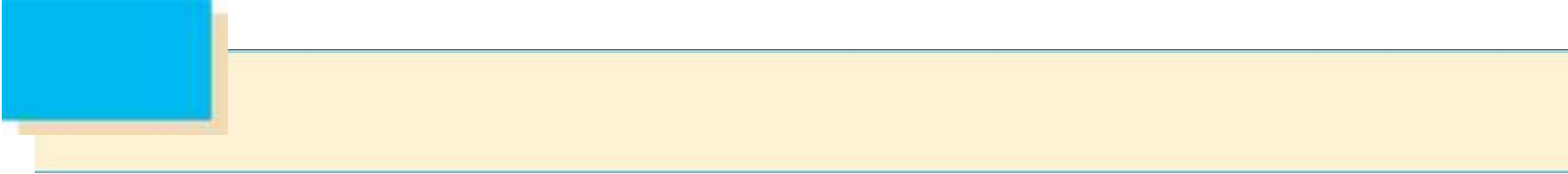
14

Partial Derivatives



14.4

Tangent Planes and Linear Approximations



Tangent Planes

Tangent Planes

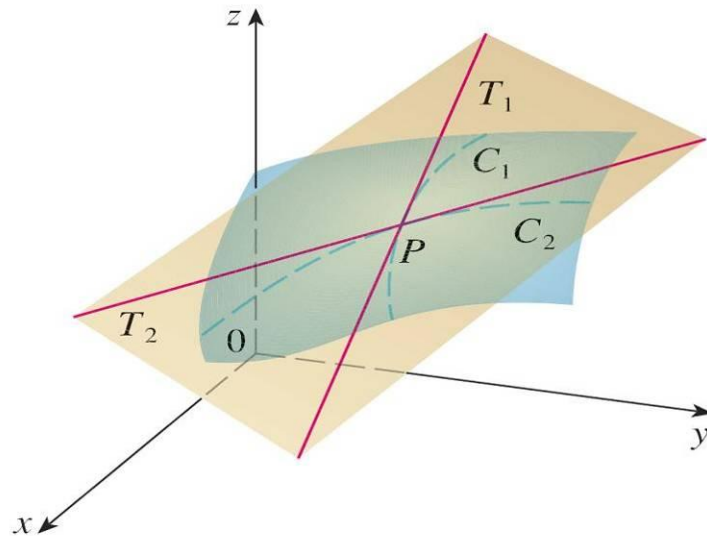
Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S .

Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S . Then the point P lies on both C_1 and C_2 .

Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P .

Tangent Planes

Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2 . (See Figure 1.)



The tangent plane contains the tangent lines T_1 and T_2 .

Figure 1

Tangent Planes

If C is any other curve that lies on the surface S and passes through P , then its tangent line at P also lies in the tangent plane.

Therefore you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P . The tangent plane at P is the plane that most closely approximates the surface S near the point P . We know that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Tangent Planes

By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

$$\boxed{1} \quad z - z_0 = a(x - x_0) + b(y - y_0)$$

If Equation 1 represents the tangent plane at P , then its intersection with the plane $y = y_0$ must be the tangent line T_1 . Setting $y = y_0$ in Equation 1 gives

$$z - z_0 = a(x - x_0) \quad \text{where } y = y_0$$

and we recognize this as the equation (in point-slope form) of a line with slope a .

Tangent Planes

But we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$.

Therefore $a = f_x(x_0, y_0)$.

Similarly, putting $x = x_0$ in Equation 1, we get $z - z_0 = b(y - y_0)$, which must represent the tangent line T_2 , so $b = f_y(x_0, y_0)$.

2 Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution:

Let $f(x, y) = 2x^2 + y^2$.

Then

$$f_x(x, y) = 4x$$

$$f_y(x, y) = 2y$$

$$f_x(1, 1) = 4$$

$$f_y(1, 1) = 2$$

Then $\square 2$ gives the equation of the tangent plane at $(1, 1, 3)$ as

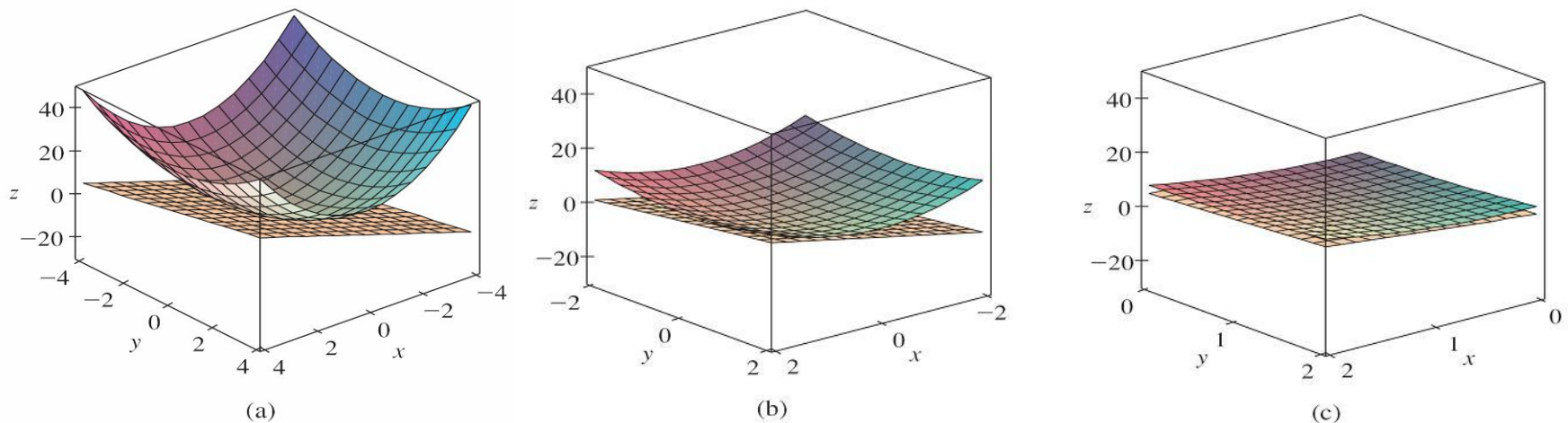
$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$

Tangent Planes

Figure 2(a) shows the elliptic paraboloid and its tangent plane at $(1, 1, 3)$ that we found in Example 1. In parts (b) and (c) we zoom in toward the point $(1, 1, 3)$ by restricting the domain of the function $f(x, y) = 2x^2 + y^2$.

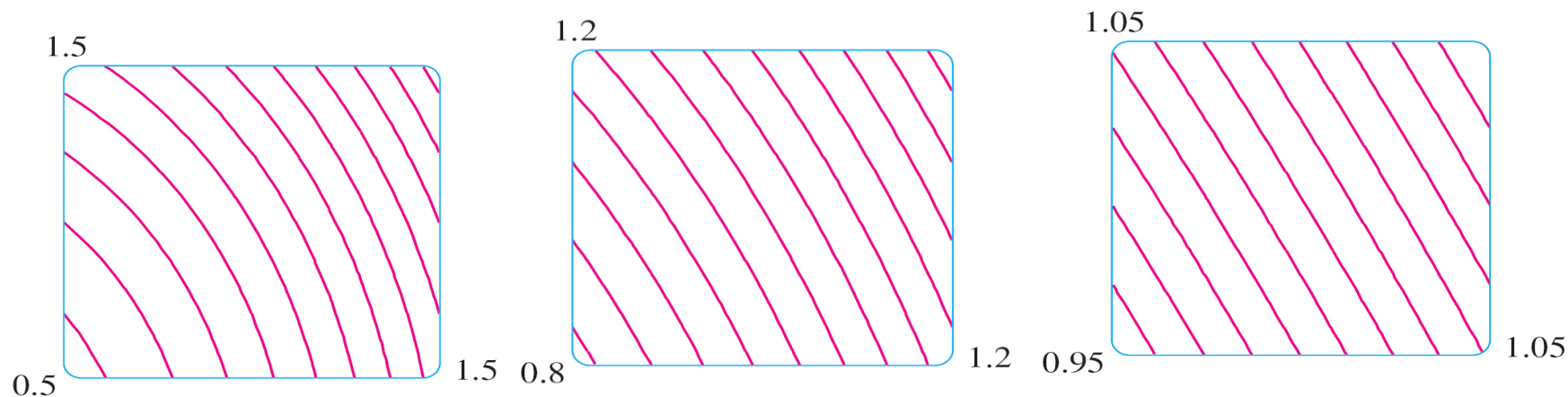


The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward $(1, 1, 3)$.

Figure 2

Tangent Planes

Notice that the more we zoom in, the flatter the graph appears and the more it resembles its tangent plane. In Figure 3 we corroborate this impression by zooming in toward the point $(1, 1)$ on a contour map of the function $f(x, y) = 2x^2 + y^2$.



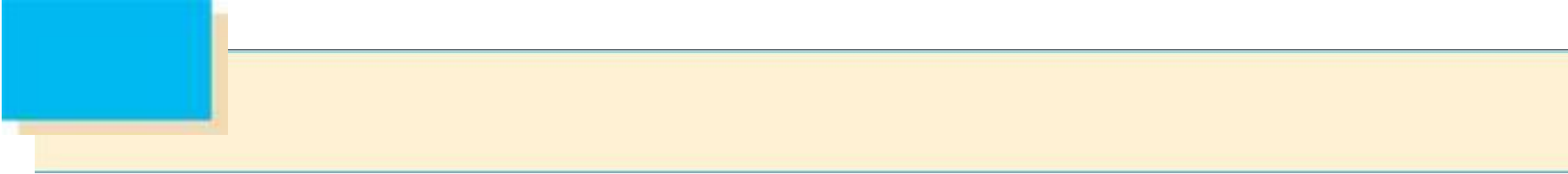
Zooming in toward $(1, 1)$ on a contour map of $f(x, y) = 2x^2 + y^2$

Figure 3



Tangent Planes

Notice that the more we zoom in, the more the level curves look like equally spaced parallel lines, which is characteristic of a plane.



Linear Approximations

Linear Approximations

In Example 1 we found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point $(1, 1, 3)$ is $z = 4x + 2y - 3$. Therefore, the linear function of two variables

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when (x, y) is near $(1, 1)$. The function L is called the *linearization* of f at $(1, 1)$ and the approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the *linear approximation* or *tangent plane approximation* of f at $(1, 1)$.

Linear Approximations

For instance, at the point (1.1, 0.95) the linear approximation gives

$$f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = 3.3$$

which is quite close to the true value of

$$f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225.$$

But if we take a point farther away from (1, 1), such as (2, 3), we no longer get a good approximation.

In fact, $L(2, 3) = 11$ whereas $f(2, 3) = 17$.

Linear Approximations

In general, we know from [2] that an equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane, namely

[3]
$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b)

Linear Approximations

The approximation

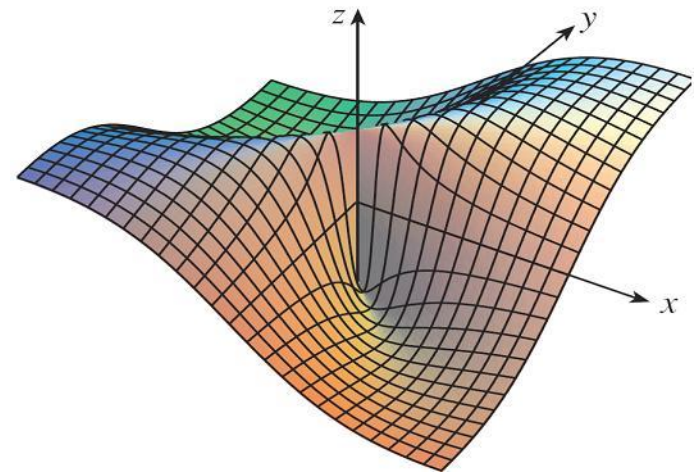
$$\boxed{4} \quad f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b) .

Linear Approximations

We have defined tangent planes for surfaces $z = f(x, y)$, where f has continuous first partial derivatives. What happens if f_x and f_y are not continuous? Figure 4 pictures such a function; its equation is

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$



$$f(x, y) = \frac{xy}{x^2 + y^2} \text{ if } (x, y) \neq (0, 0), \\ f(0, 0) = 0$$

Figure 4

You can verify that its partial derivatives exist at the origin and, in fact, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, but f_x and f_y are not continuous.

Linear Approximations

The linear approximation would be $f(x, y) \approx 0$, but $f(x, y) = \frac{1}{2}$ at all points on the line $y = x$.

So a function of two variables can behave badly even though both of its partial derivatives exist.

To rule out such behavior, we formulate the idea of a differentiable function of two variables.

Recall that for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

Linear Approximations

If f is differentiable at a , then

5

$$\Delta y = f'(a) \Delta x + \varepsilon \Delta x \quad \text{where } \varepsilon \rightarrow 0 \text{ as } \Delta x \rightarrow 0$$

Now consider a function of two variables, $z = f(x, y)$, and suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding **increment** of z is

6

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Thus the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

Linear Approximations

By analogy with [5] we define the differentiability of a function of two variables as follows.

7 Definition If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b) \Delta x + f_y(a, b) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

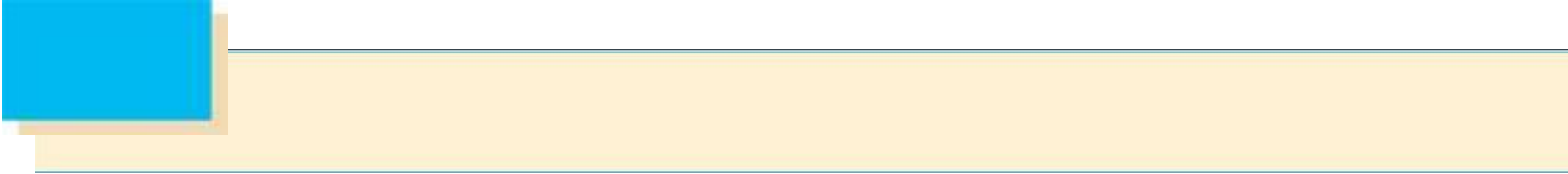
where ε_1 and $\varepsilon_2 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$.

Definition 7 says that a differentiable function is one for which the linear approximation [4] is a good approximation when (x, y) is near (a, b) . In other words, the tangent plane approximates the graph of f well near the point of tangency.

Linear Approximations

It's sometimes hard to use Definition 7 directly to check the differentiability of a function, but the next theorem provides a convenient sufficient condition for differentiability.

8 Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b) , then f is differentiable at (a, b) .



Differentials

Differentials

For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable; that is, dx can be given the value of any real number.

The differential of y is then defined as

$$\boxed{9} \quad dy = f'(x) dx$$

Differentials

Figure 6 shows the relationship between the increment Δy and the differential dy : Δy represents the change in height of the curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.

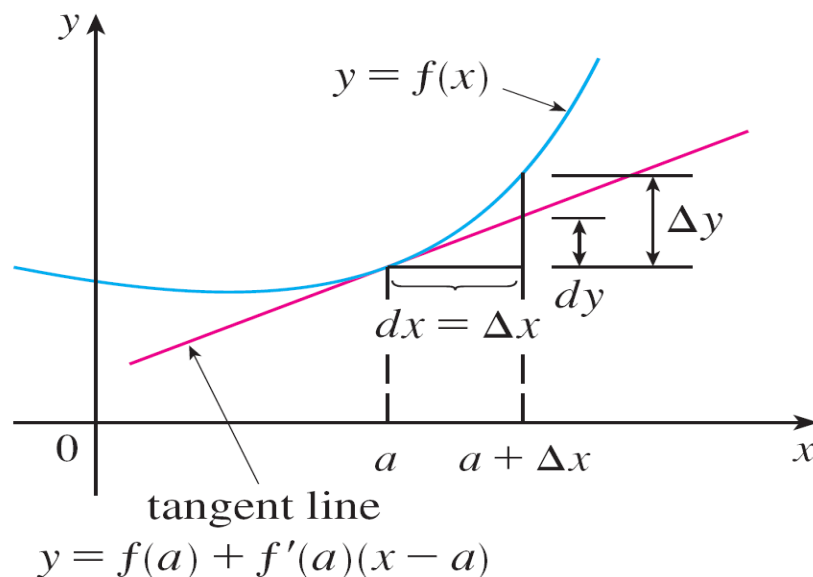


Figure 6

Differentials

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz , also called the **total differential**, is defined by

10

$$dz = f_x(x, y) dx + f_y(x, y) dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Sometimes the notation df is used in place of dz .

Differentials

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in Equation 10, then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

So, in the notation of differentials, the linear approximation [4] can be written as

$$f(x, y) \approx f(a, b) + dz$$

Differentials

Figure 7 is the three-dimensional counterpart of Figure 6 and shows the geometric interpretation of the differential dz and the increment Δz : dz represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

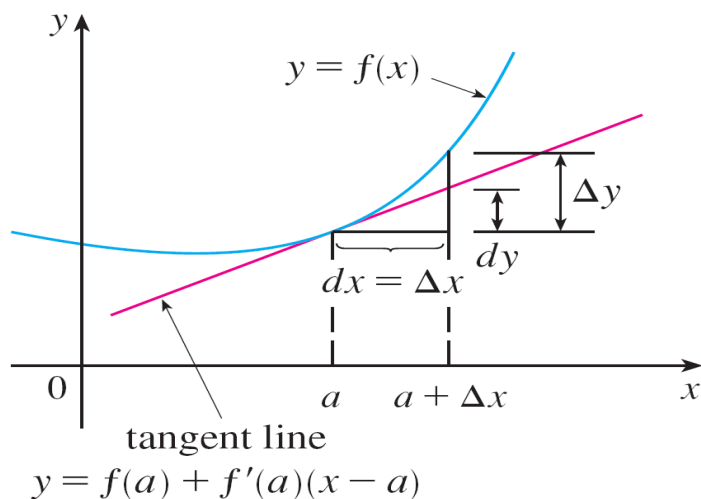


Figure 6

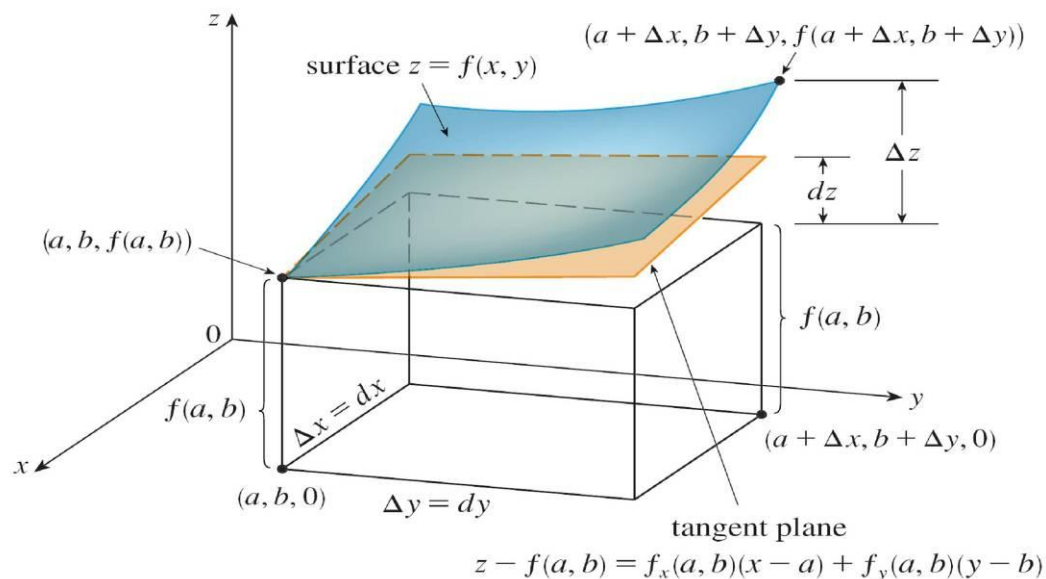


Figure 7

Example 4

- (a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz .
- (b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz .

Solution:

(a) Definition 10 gives

$$\begin{aligned} dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\ &= (2x + 3y) dx + (3x - 2y) dy \end{aligned}$$

Example 4 – *Solution*

cont'd

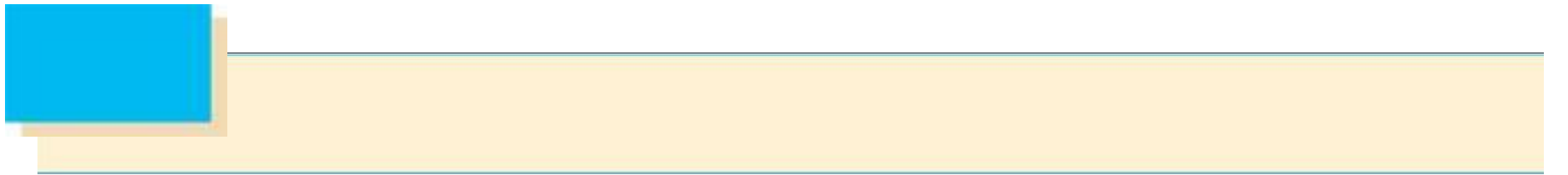
(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2, 3) \\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] \\ &\quad - [2^2 + 3(2)(3) - 3^2] \\ &= 0.6449\end{aligned}$$

Notice that $\Delta z \approx dz$ but dz is easier to compute.



Functions of Three or More Variables

Functions of Three or More Variables

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables. A differentiable function is defined by an expression similar to the one in Definition 7. For such functions the **linear approximation** is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization $L(x, y, z)$ is the right side of this expression.

Functions of Three or More Variables

If $w = f(x, y, z)$, then the **increment** of w is

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

The **differential** dw is defined in terms of the differentials dx , dy , and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

Example 6

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

Solution:

If the dimensions of the box are x , y , and z , its volume is $V = xyz$ and so

$$\begin{aligned} dV &= \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \\ &= yz dx + xz dy + xy dz \end{aligned}$$

Example 6 – *Solution*

cont'd

We are given that $|\Delta x| \leq 0.2$, $|\Delta y| \leq 0.2$, and $|\Delta z| \leq 0.2$.

To estimate the largest error in the volume, we therefore use $dx = 0.2$, $dy = 0.2$, and $dz = 0.2$ together with $x = 75$, $y = 60$, and $z = 40$:

$$\begin{aligned}\Delta V \approx dV &= (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) \\ &= 1980\end{aligned}$$

Example 6 – *Solution*

cont'd

Thus an error of only 0.2 cm in measuring each dimension could lead to an error of approximately 1980 cm³ in the calculated volume! This may seem like a large error, but it's only about 1% of the volume of the box.

14

Partial Derivatives



14.5

The Chain Rule

The Chain Rule

Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function:

If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\boxed{1} \quad \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function.

The Chain Rule

The first version (Theorem 2) deals with the case where $z = f(x, y)$ and each of the variables x and y is, in turn, a function of a variable t .

This means that z is indirectly a function of t , $z = f(g(t), h(t))$, and the Chain Rule gives a formula for differentiating z as a function of t . We assume that f is differentiable.

The Chain Rule

Recall that this is the case when f_x and f_y are continuous.

2 The Chain Rule (Case 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t . Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Since we often write $\partial z / \partial x$ in place of $\partial f / \partial x$, we can rewrite the Chain Rule in the form

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

Example 1

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

Solution:

The Chain Rule gives

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= (2xy + 3y^4)(2 \cos 2t) + (x^2 + 12xy^3)(-\sin t)\end{aligned}$$

It's not necessary to substitute the expressions for x and y in terms of t .

Example 1 – *Solution*

cont'd

We simply observe that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$.

Therefore

$$\left. \frac{dz}{dt} \right|_{t=0} = (0 + 3)(2 \cos 0) + (0 + 0)(-\sin 0) = 6$$

The Chain Rule

We now consider the situation where $z = f(x, y)$ but each of x and y is a function of two variables s and t :
 $x = g(s, t)$, $y = h(s, t)$.

Then z is indirectly a function of s and t and we wish to find $\partial z / \partial s$ and $\partial z / \partial t$.

Recall that in computing $\partial z / \partial t$ we hold s fixed and compute the ordinary derivative of z with respect to t .

Therefore we can apply Theorem 2 to obtain

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

The Chain Rule

A similar argument holds for $\partial z / \partial s$ and so we have proved the following version of the Chain Rule.

3 The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t . Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

Case 2 of the Chain Rule contains three types of variables: s and t are **independent** variables, x and y are called **intermediate** variables, and z is the **dependent** variable.

The Chain Rule

Notice that Theorem 3 has one term for each intermediate variable and each of these terms resembles the one-dimensional Chain Rule in Equation 1.

To remember the Chain Rule, it's helpful to draw the **tree diagram** in Figure 2.

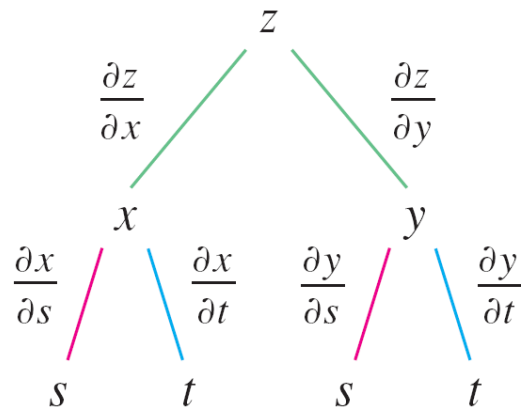


Figure 2

The Chain Rule

We draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y . Then we draw branches from x and y to the independent variables s and t .

On each branch we write the corresponding partial derivative. To find $\partial z / \partial s$, we find the product of the partial derivatives along each path from z to s and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

The Chain Rule

Similarly, we find $\partial z / \partial t$ by using the paths from z to t .

Now we consider the general situation in which a dependent variable u is a function of n intermediate variables $x_1, \dots, x_n$, each of which is, in turn, a function of m independent variables $t_1, \dots, t_m$.

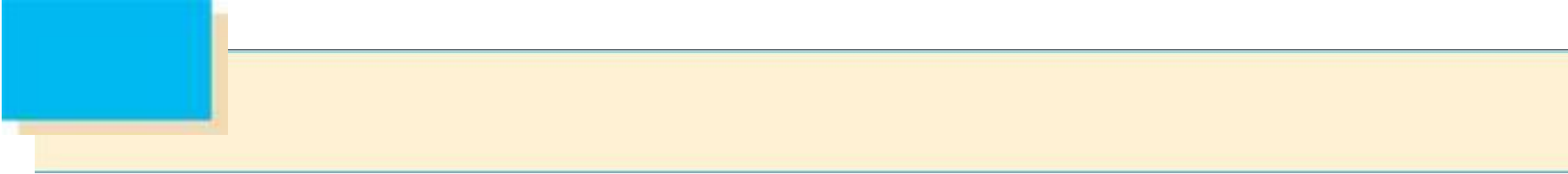
Notice that there are n terms, one for each intermediate variable. The proof is similar to that of Case 1.

The Chain Rule

4 The Chain Rule (General Version) Suppose that u is a differentiable function of the n variables $x_1, x_2, \dots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \dots, t_m$. Then u is a function of $t_1, t_2, \dots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \dots, m$.



Implicit Differentiation

Implicit Differentiation

The Chain Rule can be used to give a more complete description of the process of implicit differentiation.

We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x , that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f .

If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x .

Since both x and y are functions of x , we obtain

$$\frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$$

Implicit Differentiation

But $dx/dx = 1$, so if $\partial F/\partial x \neq 0$ we solve for dy/dx and obtain

6

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

To derive this equation we assumed that $F(x, y) = 0$ defines y implicitly as a function of x .

Implicit Differentiation

The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid:

It states that if F is defined on a disk containing (a, b) , where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by Equation 6.

Example 8

Find y' if $x^3 + y^3 = 6xy$.

Solution:

The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

so Equation 6 gives

$$\begin{aligned}\frac{dy}{dx} &= -\frac{F_x}{F_y} \\ &= -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}\end{aligned}$$

Implicit Differentiation

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$.

This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f . If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0$$

Implicit Differentiation

But $\frac{\partial}{\partial x}(x) = 1$ and $\frac{\partial}{\partial x}(y) = 0$

so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0$$

If $\partial F/\partial z \neq 0$, we solve for $\partial z/\partial x$ and obtain the first formula in Equations 7.

The formula for $\partial z/\partial y$ is obtained in a similar manner.

Implicit Differentiation

7

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Again, a version of the **Implicit Function Theorem** stipulates conditions under which our assumption is valid:

If F is defined within a sphere containing (a, b, c) , where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x , F_y , and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by 7.

14

Partial Derivatives



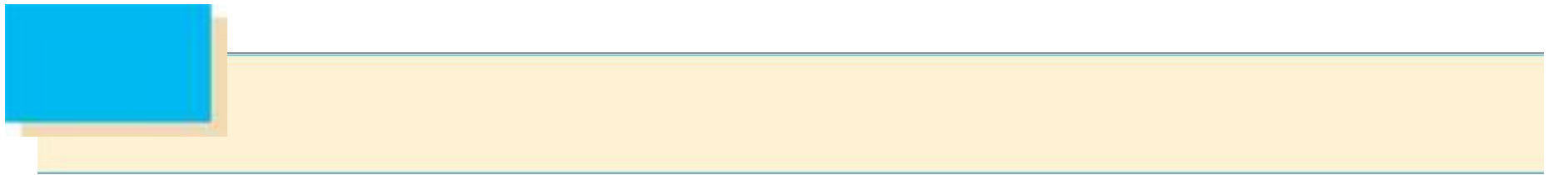
14.6

Directional Derivatives and the Gradient Vector



Directional Derivatives and the Gradient Vector

In this section we introduce a type of derivative, called a *directional derivative*, that enables us to find the rate of change of a function of two or more variables in any direction.



Directional Derivatives

Directional Derivatives

Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$f_x(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

1

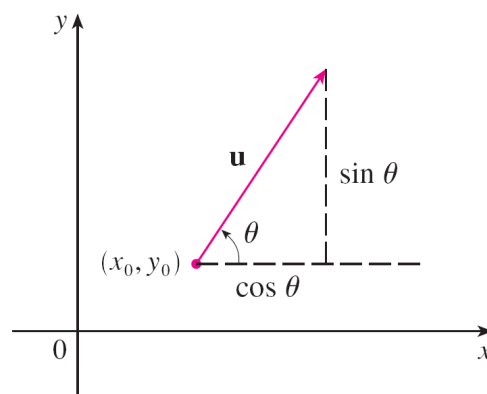
$$f_y(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

and represent the rates of change of z in the x - and y -directions, that is, in the directions of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

Directional Derivatives

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$. (See Figure 2.)

To do this we consider the surface S with the equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S .



A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos \theta, \sin \theta \rangle$

Figure 2

Directional Derivatives

The vertical plane that passes through P in the direction of $\mathbf{u}$ intersects S in a curve C . (See Figure 3.)

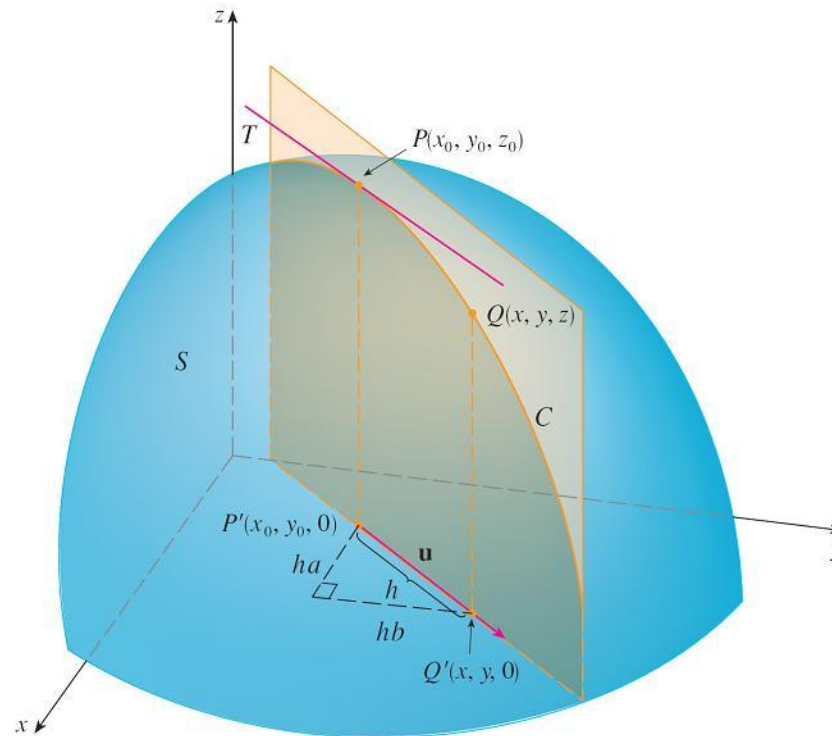


Figure 3

Directional Derivatives

The slope of the tangent line T to C at the point P is the rate of change of z in the direction of $\mathbf{u}$. If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto the xy -plane, then the vector $\overrightarrow{P'Q'}$ is parallel to $\mathbf{u}$ and so

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$$

for some scalar h . Therefore $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Directional Derivatives

If we take the limit as $h \rightarrow 0$, we obtain the rate of change of z (with respect to distance) in the direction of $\mathbf{u}$, which is called the directional derivative of f in the direction of $\mathbf{u}$.

2 Definition The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Directional Derivatives

By comparing Definition 2 with Equations 1, we see that if $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}}f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}}f = f_y$.

In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

Example 1

Use the weather map in Figure 1 to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.



Figure 1

Example 1 – *Solution*

The unit vector directed toward the southeast is $\mathbf{u} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$, but we won't need to use this expression.

We start by drawing a line through Reno toward the southeast (see Figure 4).

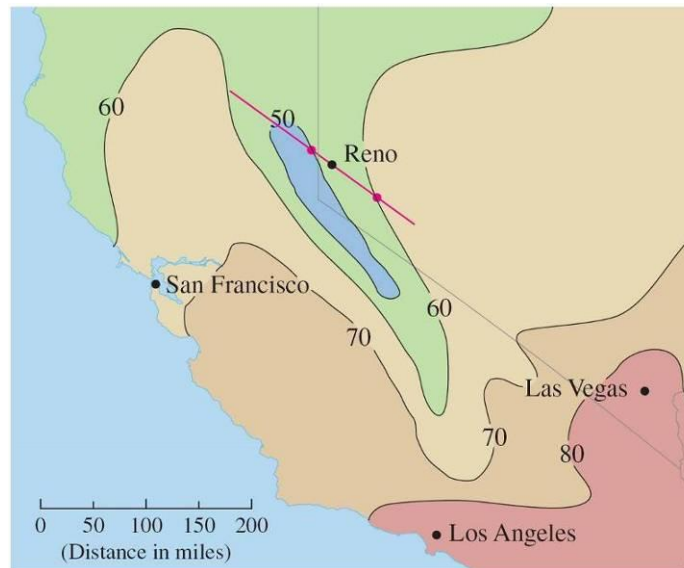


Figure 4

Example 1 – *Solution*

cont'd

We approximate the directional derivative $D_{\mathbf{u}}T$ by the average rate of change of the temperature between the points where this line intersects the isothermals $T = 50$ and $T = 60$.

The temperature at the point southeast of Reno is $T = 60^\circ\text{F}$ and the temperature at the point northwest of Reno is $T = 50^\circ\text{F}$.

The distance between these points looks to be about 75 miles. So the rate of change of the temperature in the southeasterly direction is

$$D_{\mathbf{u}}T \approx \frac{60 - 50}{75} = \frac{10}{75} \approx 0.13^\circ\text{F}/\text{mi}$$

Directional Derivatives

When we compute the directional derivative of a function defined by a formula, we generally use the following theorem.

3 Theorem If f is a differentiable function of x and y , then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$

Directional Derivatives

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x -axis (as in Figure 2), then we can write $\mathbf{u} = \langle \cos \theta, \sin \theta \rangle$ and the formula in Theorem 3 becomes

$$\boxed{6} \quad D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

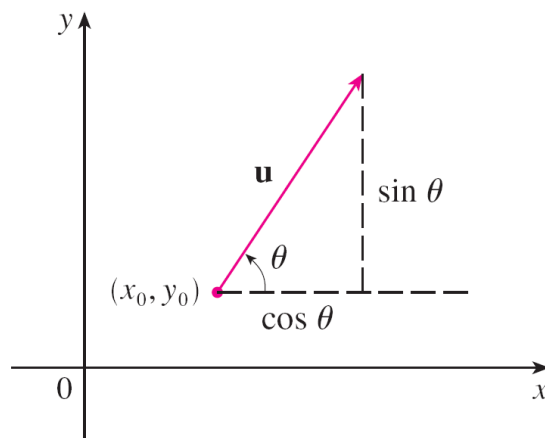
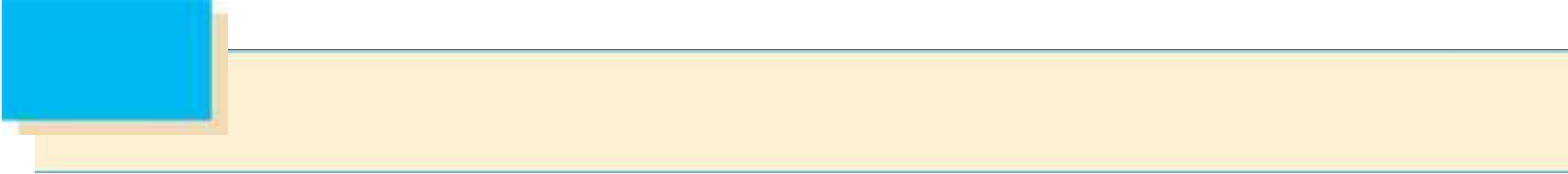


Figure 2

A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos \theta, \sin \theta \rangle$



The Gradient Vectors

The Gradient Vectors

Notice from Theorem 3 that the directional derivative of a differentiable function can be written as the dot product of two vectors:

$$\begin{aligned} \boxed{7} \quad D_{\mathbf{u}}f(x, y) &= f_x(x, y)a + f_y(x, y)b \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u} \end{aligned}$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well.

So we give it a special name (the *gradient* of f) and a special notation (**grad** f or ∇f , which is read “del f ”).

The Gradient Vectors

8 Definition If f is a function of two variables x and y , then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

Example 3

If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = \langle f_x, f_y \rangle$$

$$= \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

and $\nabla f(0, 1) = \langle 2, 0 \rangle$

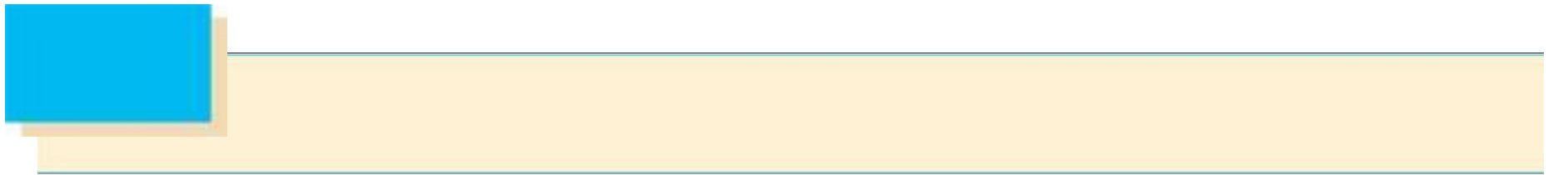
The Gradient Vectors

With this notation for the gradient vector, we can rewrite the expression (7) for the directional derivative of a differentiable function as

9

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

This expresses the directional derivative in the direction of $\mathbf{u}$ as the scalar projection of the gradient vector onto $\mathbf{u}$.



Functions of Three Variables

Functions of Three Variables

For functions of three variables we can define directional derivatives in a similar manner.

Again $D_{\mathbf{u}}f(x, y, z)$ can be interpreted as the rate of change of the function in the direction of a unit vector $\mathbf{u}$.

10 Definition The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

Functions of Three Variables

If we use vector notation, then we can write both definitions (2 and 10) of the directional derivative in the compact form

11

$$D_{\mathbf{u}}f(\mathbf{x}_0) = \lim_{h \rightarrow 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where $\mathbf{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$ and $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$.

This is reasonable because the vector equation of the line through $\mathbf{x}_0$ in the direction of the vector $\mathbf{u}$ is given by $\mathbf{x} = \mathbf{x}_0 + t\mathbf{u}$ and so $f(\mathbf{x}_0 + h\mathbf{u})$ represents the value of f at a point on this line.

Functions of Three Variables

If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c \rangle$, then

$$\boxed{12} \quad D_{\mathbf{u}}f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

For a function f of three variables, the **gradient vector**, denoted by ∇f or **grad** f , is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

or, for short,

$$\boxed{13} \quad \nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

Functions of Three Variables

Then, just as with functions of two variables, Formula 12 for the directional derivative can be rewritten as

14

$$D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

Example 5

If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution:

(a) The gradient of f is

$$\begin{aligned}\nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= \langle \sin yz, xz \cos yz, xy \cos yz \rangle\end{aligned}$$

Example 5 – Solution

cont'd

(b) At $(1, 3, 0)$ we have $\nabla f(1, 3, 0) = \langle 0, 0, 3 \rangle$.

The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

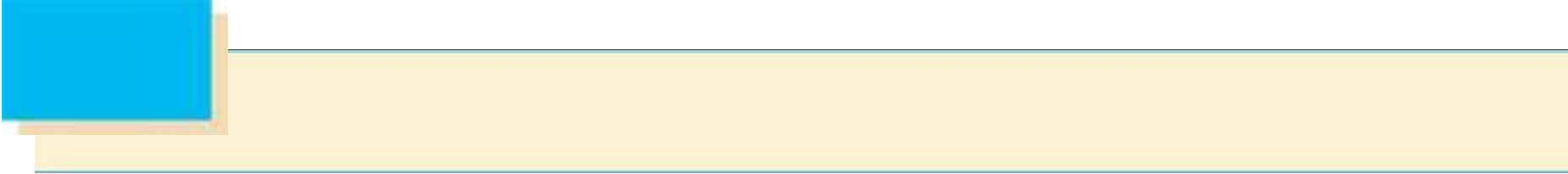
$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

Therefore Equation 14 gives

$$D_{\mathbf{u}}f(1, 3, 0) = \nabla f(1, 3, 0) \cdot \mathbf{u}$$

$$= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} \right)$$

$$= 3\left(-\frac{1}{\sqrt{6}}\right) = -\sqrt{\frac{3}{2}}$$



Maximizing the Directional Derivatives

Maximizing the Directional Derivatives

Suppose we have a function f of two or three variables and we consider all possible directional derivatives of f at a given point.

These give the rates of change of f in all possible directions.

We can then ask the questions: In which of these directions does f change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

15 Theorem Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Example 6

- (a)** If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.
- (b)** In what direction does f have the maximum rate of change? What is this maximum rate of change?

Solution:

- (a)** We first compute the gradient vector:

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$

$$\nabla f(2, 0) = \langle 1, 2 \rangle$$

Example 6 – *Solution*

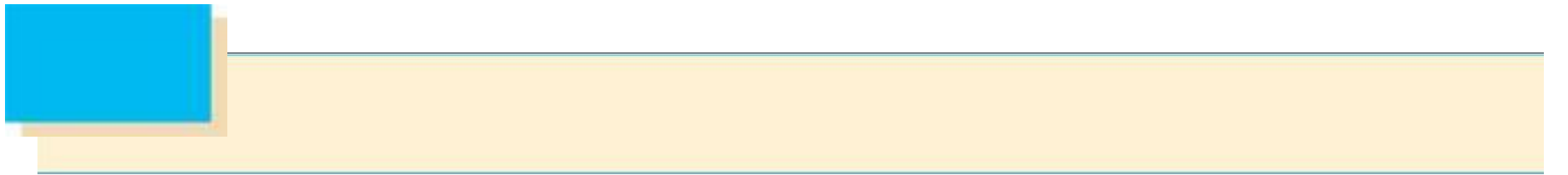
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The unit vector in the direction of $\overrightarrow{PQ} = \langle -1.5, 2 \rangle$ is $\mathbf{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$, so the rate of change of f in the direction from P to Q is

$$\begin{aligned} D_{\mathbf{u}}f(2, 0) &= \nabla f(2, 0) \cdot \mathbf{u} \\ &= \langle 1, 2 \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle \\ &= 1(-\frac{3}{5}) + 2(\frac{4}{5}) = 1 \end{aligned}$$

(b) According to Theorem 15, f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$.
The maximum rate of change is

$$|\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5}$$



Tangent Planes to Level Surfaces

Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S .

Let C be any curve that lies on the surface S and passes through the point P . Recall that the curve C is described by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Let t_0 be the parameter value corresponding to P ; that is, $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. Since C lies on S , any point $(x(t), y(t), z(t))$ must satisfy the equation of S , that is,

$$F(x(t), y(t), z(t)) = k$$

Tangent Planes to Level Surfaces

If x , y , and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation 16 as follows:

$$\boxed{17} \quad \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0$$

But, since $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$, Equation 17 can be written in terms of a dot product as

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

Tangent Planes to Level Surfaces

In particular, when $t = t_0$ we have $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$, so

18

$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$$

Equation 18 says that *the gradient vector at P , $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P . (See Figure 9.)*

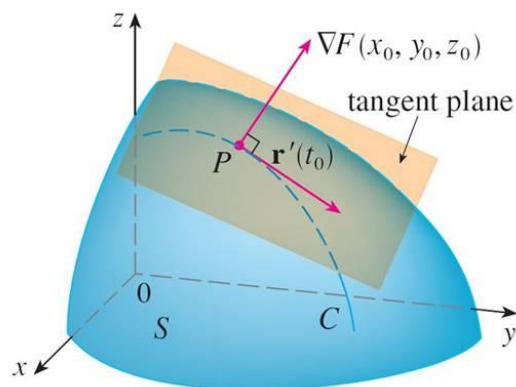


Figure 9

Tangent Planes to Level Surfaces

If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$** as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.

Using the standard equation of a plane, we can write the equation of this tangent plane as

$$19 \quad F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Tangent Planes to Level Surfaces

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane.

The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so, its symmetric equations are

20

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Tangent Planes to Level Surfaces

In the special case in which the equation of a surface S is of the form $z = f(x, y)$ (that is, S is the graph of a function f of two variables), we can rewrite the equation as

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface (with $k = 0$) of F . Then

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0)$$

$$F_y(x_0, y_0, z_0) = f_y(x_0, y_0)$$

$$F_z(x_0, y_0, z_0) = -1$$

so Equation 19 becomes

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Example 8

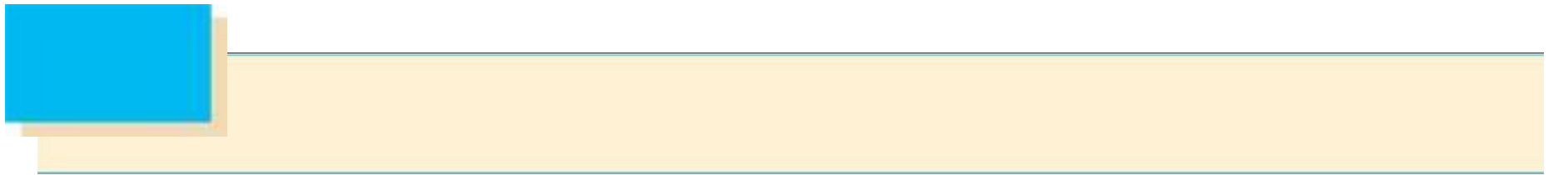
Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Solution:

The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$



Significance of the Gradient Vectors

Example 8 – *Solution*

cont'd

Therefore we have

$$F_x(x, y, z) = \frac{x}{2} \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = \frac{2z}{9}$$

$$F_x(-2, 1, -3) = -1 \quad F_y(-2, 1, -3) = 2 \quad F_z(-2, 1, -3) = -\frac{2}{3}$$

Then Equation 19 gives the equation of the tangent plane at $(-2, 1, -3)$ as

$$-1(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0$$

which simplifies to $3x - 6y + 2z + 18 = 0$.

By Equation 20, symmetric equations of the normal line are

$$\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-\frac{2}{3}}$$

Significance of the Gradient Vectors

We now summarize the ways in which the gradient vector is significant.

We first consider a function f of three variables and a point $P(x_0, y_0, z_0)$ in its domain.

On the one hand, we know from Theorem 15 that the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of f .

On the other hand, we know that $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface S of f through P . (Refer to Figure 9.)

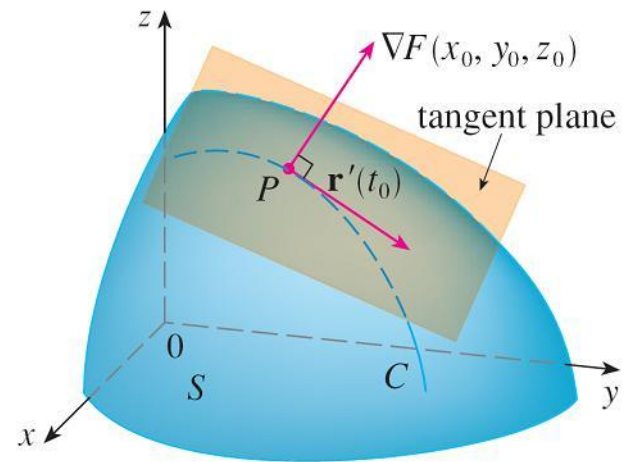


Figure 9

Significance of the Gradient Vectors

These two properties are quite compatible intuitively because as we move away from P on the level surface S , the value of f does not change at all.

So it seems reasonable that if we move in the perpendicular direction, we get the maximum increase.

In like manner we consider a function f of two variables and a point $P(x_0, y_0)$ in its domain.

Again the gradient vector $\nabla f(x_0, y_0)$ gives the direction of fastest increase of f . Also, by considerations similar to our discussion of tangent planes, it can be shown that $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ that passes through P .

Significance of the Gradient Vectors

Again this is intuitively plausible because the values of f remain constant as we move along the curve.
(See Figure 11.)

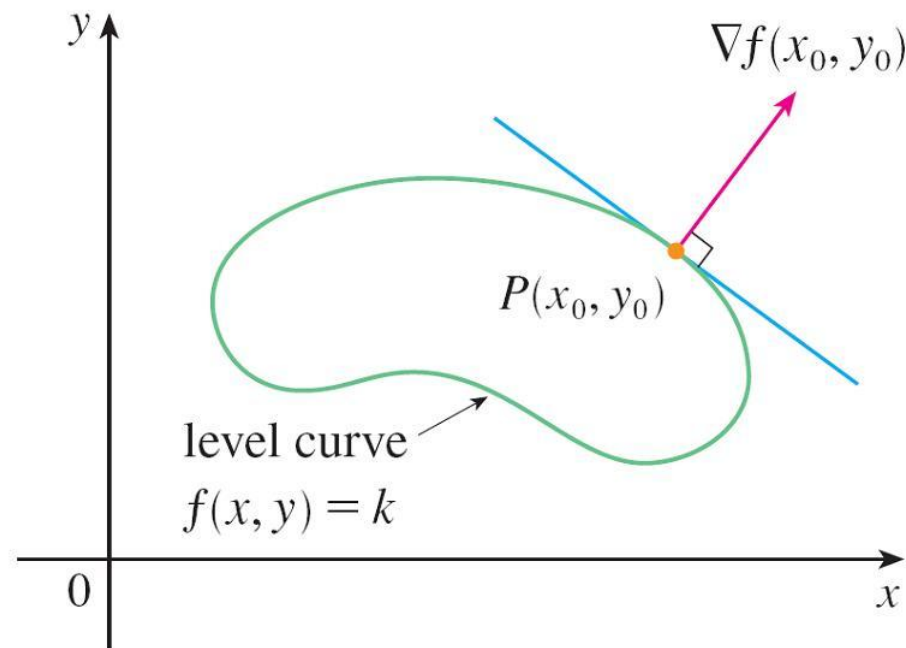


Figure 11

Significance of the Gradient Vectors

If we consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y) , then a curve of steepest ascent can be drawn as in Figure 12 by making it perpendicular to all of the contour lines.

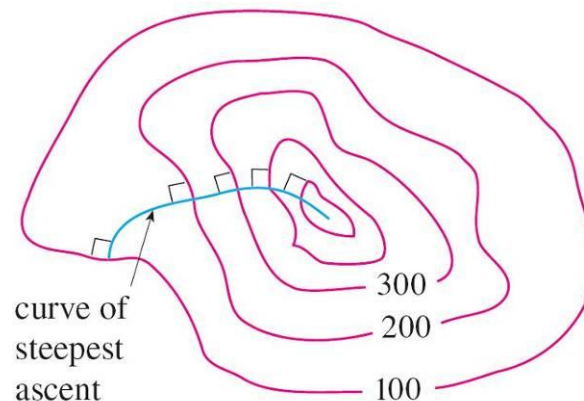


Figure 12

Significance of the Gradient Vectors

Computer algebra systems have commands that plot sample gradient vectors.

Each gradient vector $\nabla f(a, b)$ is plotted starting at the point (a, b) . Figure 13 shows such a plot (called a *gradient vector field*) for the function $f(x, y) = x^2 - y^2$ superimposed on a contour map of f .

As expected, the gradient vectors point “uphill” and are perpendicular to the level curves.

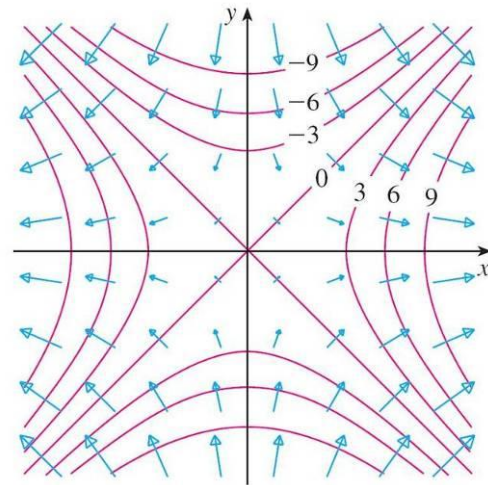


Figure 13

14

Partial Derivatives



14.7 Maximum and Minimum Values

Maximum and Minimum Values

In this section we see how to use partial derivatives to locate maxima and minima of functions of two variables.

Look at the hills and valleys in the graph of f shown in Figure 1.

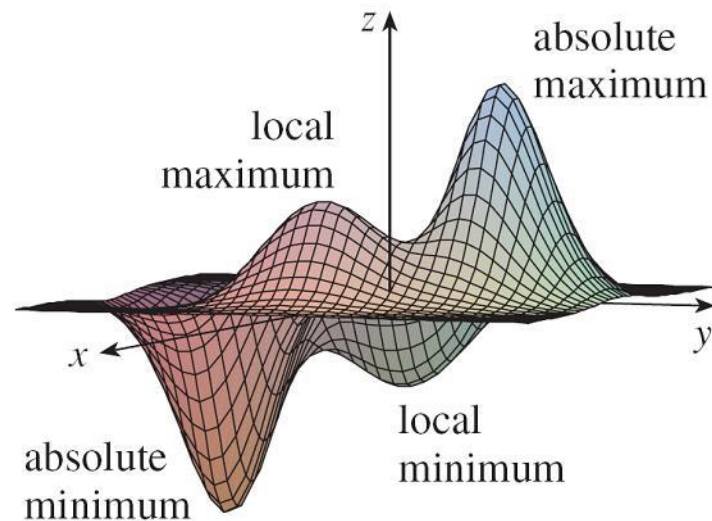


Figure 1

Maximum and Minimum Values

There are two points (a, b) where f has a *local maximum*, that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$.

The larger of these two values is the *absolute maximum*.

Likewise, f has two *local minima*, where $f(a, b)$ is smaller than nearby values.

The smaller of these two values is the *absolute minimum*.

Maximum and Minimum Values

1 Definition A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b) . [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b) .] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b) , then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

If the inequalities in Definition 1 hold for *all* points (x, y) in the domain of f , then f has an **absolute maximum** (or **absolute minimum**) at (a, b) .

2 Fermat's Theorem for Functions of Two Variables If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Maximum and Minimum Values

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

Theorem 2 says that if f has a local maximum or minimum at (a, b) , then (a, b) is a critical point of f .

However, as in single-variable calculus, not all critical points give rise to maxima or minima.

At a critical point, a function could have a local maximum or a local minimum or neither.

Example 1

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

Then

$$f_x(x, y) = 2x - 2 \quad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$.

By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Example 1

cont'd

Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x and y .

Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f .

This can be confirmed geometrically from the graph of f , which is the elliptic paraboloid with vertex $(1, 3, 4)$ shown in Figure 2.

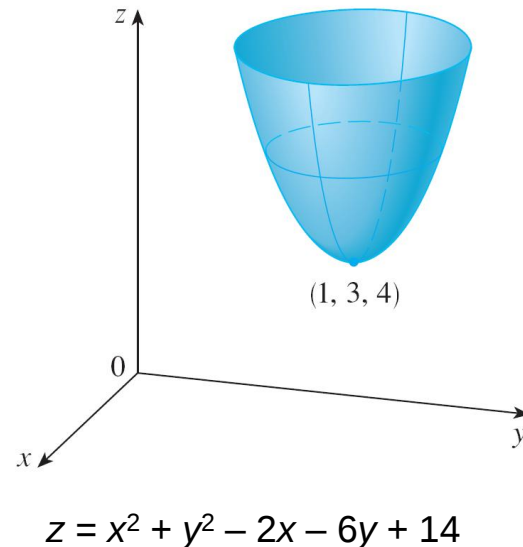


Figure 2

Maximum and Minimum Values

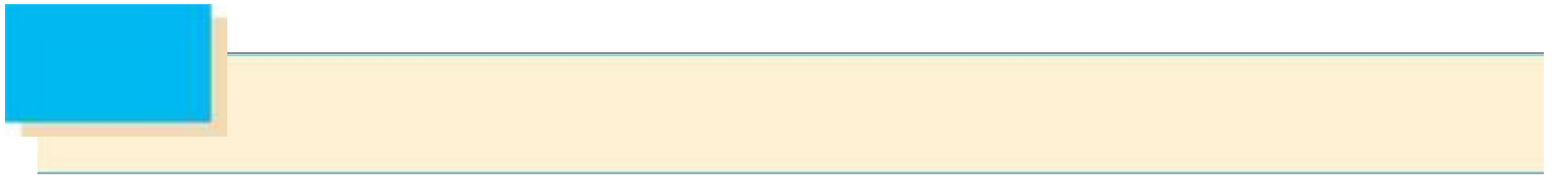
The following test, is analogous to the Second Derivative Test for functions of one variable.

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b) , and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- (c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b) .



Absolute Maximum and Minimum Values

Absolute Maximum and Minimum Values

For a function f of one variable, the Extreme Value Theorem says that if f is continuous on a closed interval $[a, b]$, then f has an absolute minimum value and an absolute maximum value.

According to the Closed Interval Method, we found these by evaluating f not only at the critical numbers but also at the endpoints a and b .

There is a similar situation for functions of two variables. Just as a closed interval contains its endpoints, a **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points.

Absolute Maximum and Minimum Values

[A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D .]

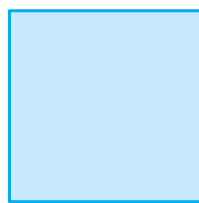
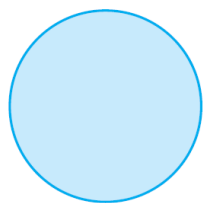
For instance, the disk

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

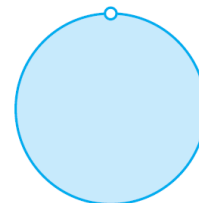
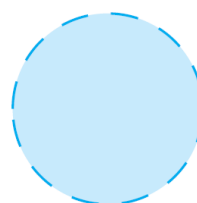
which consists of all points on and inside the circle $x^2 + y^2 = 1$, is a closed set because it contains all of its boundary points (which are the points on the circle $x^2 + y^2 = 1$).

Absolute Maximum and Minimum Values

But if even one point on the boundary curve were omitted, the set would not be closed. (See Figure 11.)



Closed sets
Figure 11(a)



Sets that are not closed
Figure 11(b)

A **bounded set** in $\mathbb{R}^2$ is one that is contained within some disk.

Absolute Maximum and Minimum Values

In other words, it is finite in extent.

Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem in two dimensions.

8 Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D .

Absolute Maximum and Minimum Values

To find the extreme values guaranteed by Theorem 8, we note that, by Theorem 2, if f has an extreme value at (x_1, y_1) , then (x_1, y_1) is either a critical point of f or a boundary point of D .

Thus we have the following extension of the Closed Interval Method.

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D :

1. Find the values of f at the critical points of f in D .
2. Find the extreme values of f on the boundary of D .
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 7

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) | 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

Solution:

Since f is a polynomial, it is continuous on the closed, bounded rectangle D , so Theorem 8 tells us there is both an absolute maximum and an absolute minimum.

According to step 1 in [9], we first find the critical points. These occur when

$$f_x = 2x - 2y = 0$$

$$f_y = -2x + 2 = 0$$

Example 7 – Solution

cont'd

so the only critical point is $(1, 1)$, and the value of f there is $f(1, 1) = 1$.

In step 2 we look at the values of f on the boundary of D , which consists of the four line segments L_1, L_2, L_3, L_4 shown in Figure 12.

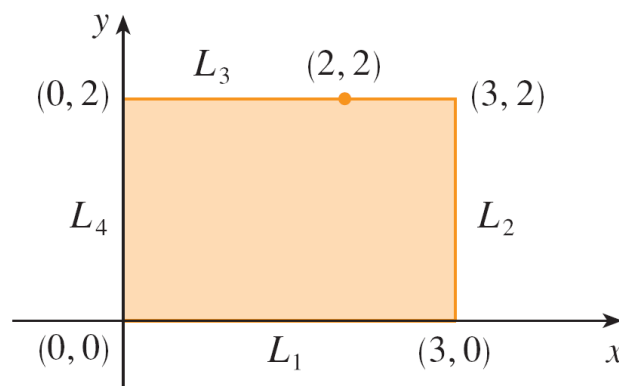


Figure 12

Example 7 – *Solution*

cont'd

On L_1 we have $y = 0$ and

$$f(x, 0) = x^2 \qquad 0 \leq x \leq 3$$

This is an increasing function of x , so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$.

On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \qquad 0 \leq y \leq 2$$

This is a decreasing function of y , so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$.

Example 7 – *Solution*

cont'd

On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \qquad 0 \leq x \leq 3$$

Simply by observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$.

Example 7 – *Solution*

cont'd

Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \quad 0 \leq y \leq 2$$

with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$.

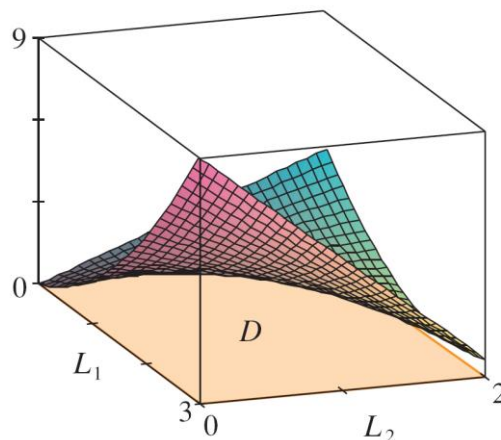
Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

Example 7 – Solution

cont'd

In step 3 we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$.

Figure 13 shows the graph of f .



$$f(x, y) = x^2 - 2xy + 2y$$

Figure 13

14

Partial Derivatives



14.8

Lagrange Multipliers

Lagrange Multipliers

In this section we present Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$.

It's easier to explain the geometric basis of Lagrange's method for functions of two variables. So we start by trying to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$.

In other words, we seek the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$.

Lagrange Multipliers

Figure 1 shows this curve together with several level curves of f .

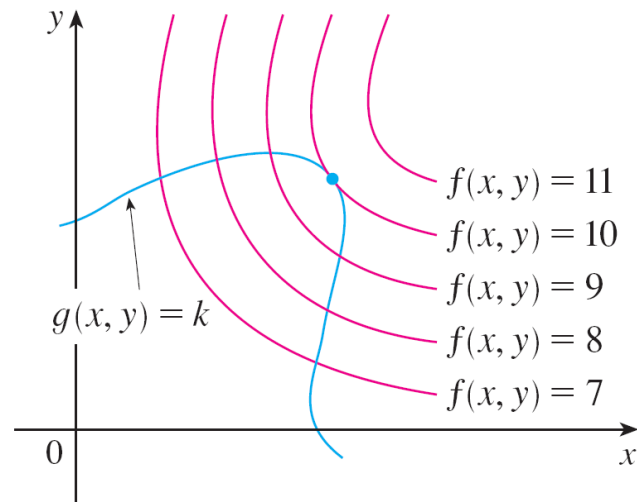


Figure 1

These have the equations $f(x, y) = c$, where $c = 7, 8, 9, 10, 11$.

Lagrange Multipliers

To maximize $f(x, y)$ subject to $g(x, y) = k$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$.

It appears from Figure 1 that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.)

Lagrange Multipliers

This means that the normal lines at the point (x_0, y_0) where they touch are identical. So the gradient vectors are parallel; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ .

This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$.

Thus the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = k$.

Lagrange Multipliers

Instead of the level curves in Figure 1, we consider the level surfaces $f(x, y, z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = k$ and so the corresponding gradient vectors are parallel.

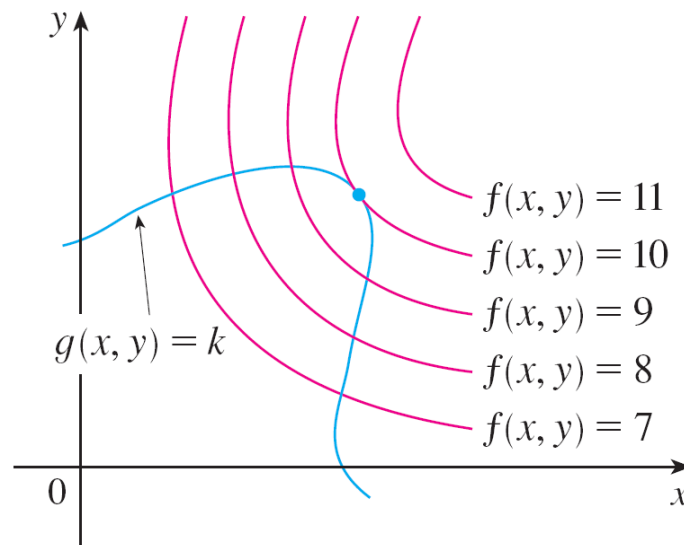


Figure 1

Lagrange Multipliers

This intuitive argument can be made precise as follows. Suppose that a function f has an extreme value at a point $P(x_0, y_0, z_0)$ on the surface S and let C be a curve with vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ that lies on S and passes through P .

If t_0 is the parameter value corresponding to the point P , then $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$.

The composite function $h(t) = f(x(t), y(t), z(t))$ represents the values that f takes on the curve C .

Lagrange Multipliers

Since f has an extreme value at (x_0, y_0, z_0) , it follows that h has an extreme value at t_0 , so $h'(t_0) = 0$. But if f is differentiable, we can use the Chain Rule to write

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) \end{aligned}$$

This shows that the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ to every such curve C . But we already know that the gradient vector of g , $\nabla g(x_0, y_0, z_0)$, is also orthogonal to $\mathbf{r}'(t_0)$ for every such curve.

Lagrange Multipliers

This means that the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ must be parallel. Therefore, if $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$, there is a number λ such that

1

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

The number λ in Equation 1 is called a **Lagrange multiplier**.

Lagrange Multipliers

The procedure based on Equation 1 is as follows.

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z , and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f ; the smallest is the minimum value of f .

Lagrange Multipliers

If we write the vector equation $\nabla f = \lambda \nabla g$ in terms of components, then the equations in step (a) become

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad f_z = \lambda g_z \quad g(x, y, z) = k$$

This is a system of four equations in the four unknowns x , y , z , and λ , but it is not necessary to find explicit values for λ .

For functions of two variables the method of Lagrange multipliers is similar to the method just described.

Lagrange Multipliers

To find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$, we look for values of x , y , and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = k$$

This amounts to solving three equations in three unknowns:

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = k$$

Example 1 – *Maximizing a volume using Lagrange multipliers*

A rectangular box without a lid is to be made from 12 m² of cardboard. Find the maximum volume of such a box.

Solution:

Let x , y , and z be the length, width, and height, respectively, of the box in meters.

Then we wish to maximize

$$V = xyz$$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Example 1 – *Solution*

cont'd

Using the method of Lagrange multipliers, we look for values of x , y , z , and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$.

This gives the equations

$$V_x = \lambda g_x$$

$$V_y = \lambda g_y$$

$$V_z = \lambda g_z$$

$$2xz + 2yz + xy = 12$$

Example 1 – *Solution*

cont'd

Which become

$$\boxed{2} \quad yz = \lambda(2z + y)$$

$$\boxed{3} \quad xz = \lambda(2z + x)$$

$$\boxed{4} \quad xy = \lambda(2x + 2y)$$

$$\boxed{5} \quad 2xz + 2yz + xy = 12$$

Example 1 – *Solution*

cont'd

There are no general rules for solving systems of equations. Sometimes some ingenuity is required.

In the present example you might notice that if we multiply [2] by x , [3] by y , and [4] by z , then the left sides of these equations will be identical.

Doing this, we have

[6]

$$xyz = \lambda(2xz + xy)$$

[7]

$$xyz = \lambda(2yz + xy)$$

[8]

$$xyz = \lambda(2xz + 2yz)$$

Example 1 – *Solution*

cont'd

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from [2], [3], and [4] and this would contradict [5].

Therefore, from [6] and [7], we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$.

But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$.

Example 1 – *Solution*

cont'd

From $\boxed{7}$ and $\boxed{8}$ we have

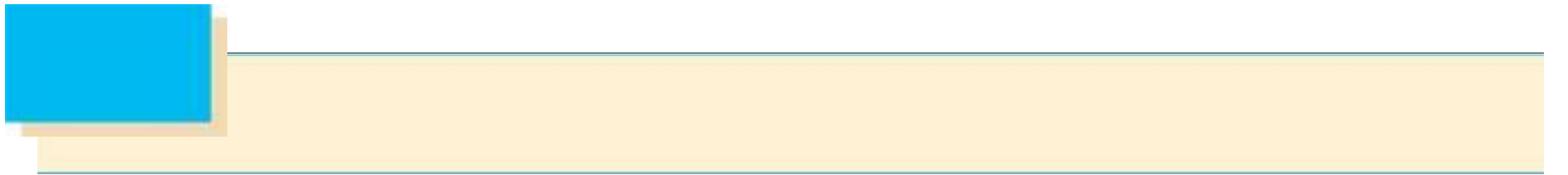
$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$.

If we now put $x = y = 2z$ in $\boxed{5}$, we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since x , y , and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$.



Two Constraints

Two Constraints

Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$.

Geometrically, this means that we are looking for the extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$. (See Figure 5.)

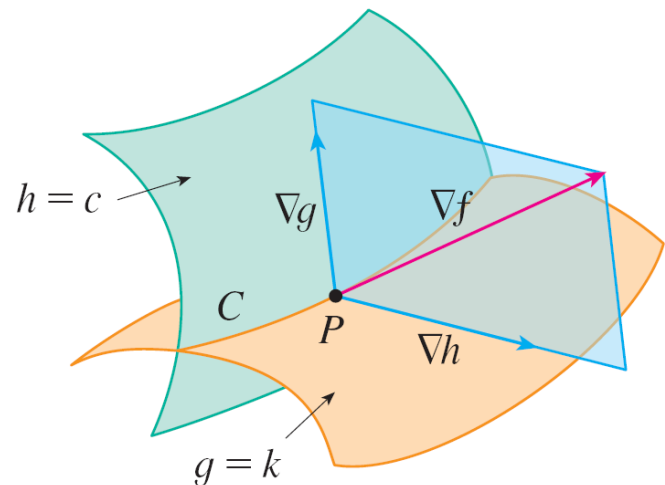


Figure 5

Two Constraints

Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$. We know from the beginning of this section that ∇f is orthogonal to C at P .

But we also know that ∇g is orthogonal to $g(x, y, z) = k$ and ∇h is orthogonal to $h(x, y, z) = c$, so ∇g and ∇h are both orthogonal to C .

This means that the gradient vector $\nabla f(x_0, y_0, z_0)$ is in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$. (We assume that these gradient vectors are not zero and not parallel.)

Two Constraints

So there are numbers λ and μ (called Lagrange multipliers) such that

16

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

In this case Lagrange's method is to look for extreme values by solving five equations in the five unknowns x , y , z , λ , and μ .

Two Constraints

These equations are obtained by writing Equation 16 in terms of its components and using the constraint equations:

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

$$g(x, y, z) = k$$

$$h(x, y, z) = c$$

Example 5 – *A maximum problem with two constraints*

Find the maximum value of the function

$f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Solution:

We maximize the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$.

Example 5 – Solution

cont'd

The Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$, so we solve the equations

$$\boxed{17} \quad 1 = \lambda + 2x\mu$$

$$\boxed{18} \quad 2 = -\lambda + 2y\mu$$

$$\boxed{19} \quad 3 = \lambda$$

$$\boxed{20} \quad x - y + z = 1$$

$$\boxed{21} \quad x^2 + y^2 = 1$$

Putting $\lambda = 3$ [from $\boxed{19}$] in $\boxed{17}$, we get $2x\mu = -2$, so $x = -1/\mu$. Similarly, $\boxed{18}$ gives $y = 5/(2\mu)$.

Example 5 – *Solution*

cont'd

Substitution in [21] then gives

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

and so $\mu^2 = \frac{29}{4}$, $\mu = \pm\sqrt{29}/2$.

Then $x = \mp 2/\sqrt{29}$, $y = \pm 5/\sqrt{29}$, and, from [20],
 $z = 1 - x + y = 1 \pm 7/\sqrt{29}$.

Example 5 – *Solution*

cont'd

The corresponding values of f are

$$\mp \frac{2}{\sqrt{29}} + 2\left(\pm \frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}$$

Therefore the maximum value of f on the given curve is $3 + \sqrt{29}$.