15 Multiple Integrals

15.1 Double Integrals over Rectangles

Review of the Definite Integrals

Review of the Definite Integrals

First let's recall the basic facts concerning definite integrals of functions of a single variable.

If $f(x)$ is defined for $a \le x \le b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$ and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

1
$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

and take the limit of such sums as $n \to \infty$ to obtain the definite integral of f from a to b:

2
$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Review of the Definite Integrals

In the special case where $f(x) \geq 0$, the Riemann sum can be interpreted as the sum of the areas of the approximating rectangles in Figure 1, and $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b.

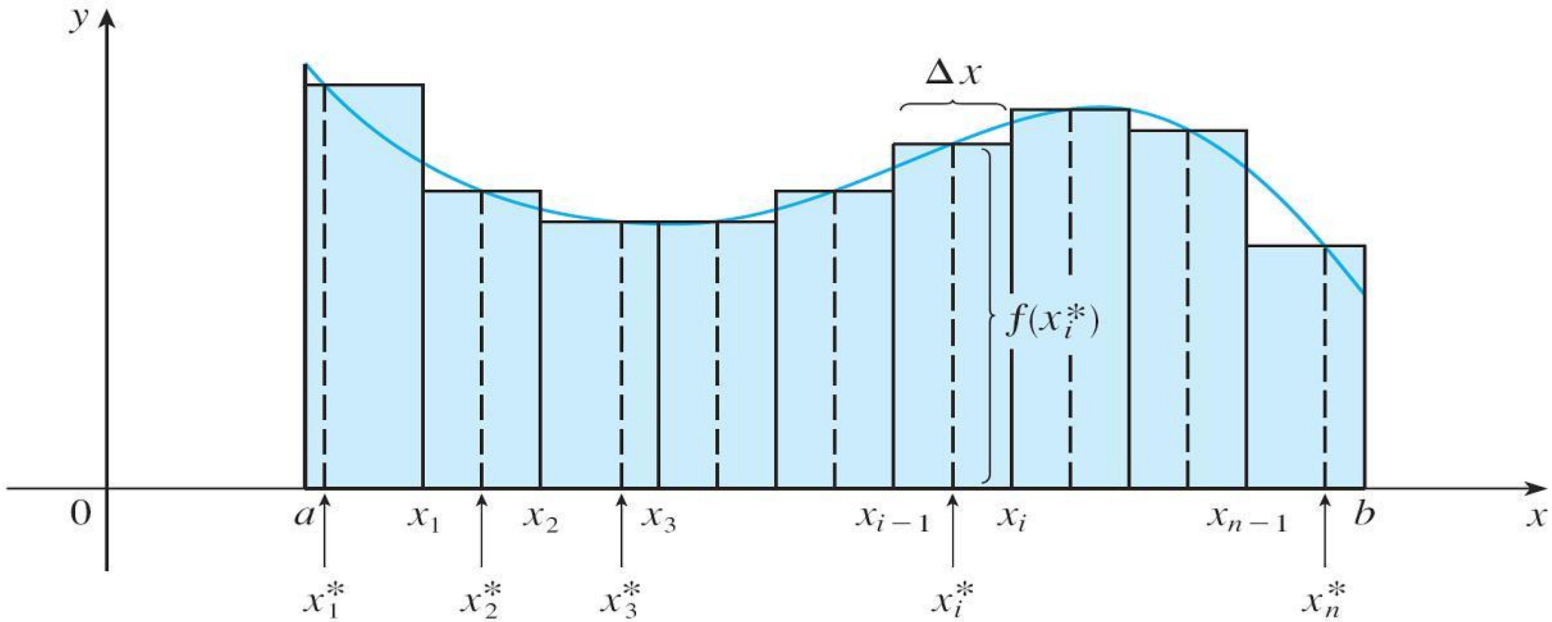


Figure 1

Volumes and Double Integrals

Volumes and Double Integrals

In a similar manner we consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$$

and we first suppose that $f(x, y) \ge 0$.

The graph of f is a surface with equation $z = f(x, y)$.

Let S be the solid that lies above R and under the graph of f, that is,

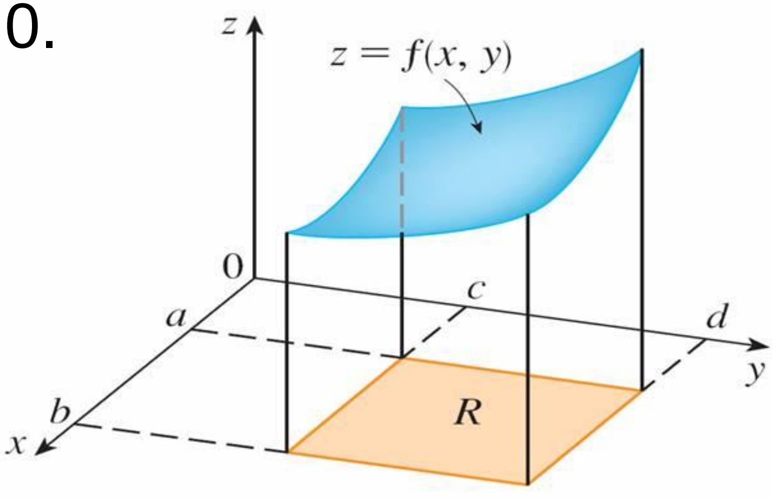


Figure 2

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le z \le f(x, y),\ (x, y) \in R\}$$

(See Figure 2.)

Volumes and Double Integrals

Our goal is to find the volume of S.

The first step is to divide the rectangle R into subrectangles.

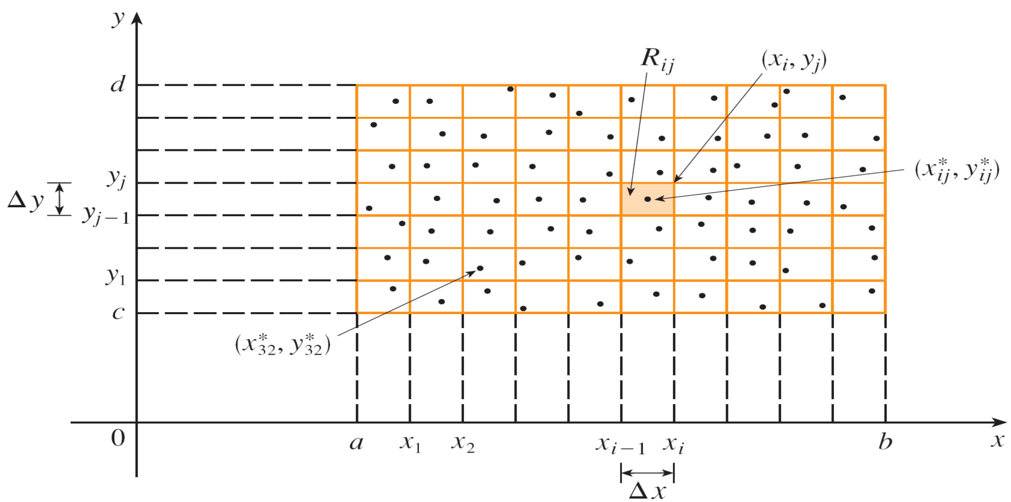
We accomplish this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$.

Volumes and Double Integrals

By drawing lines parallel to the coordinate axes through the endpoints of these subintervals, as in Figure 3, we form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \le x \le x_i,\ y_{j-1} \le y \le y_j\}$$

each with area $\Delta A = \Delta x\, \Delta y$.



Dividing *R* into subrectangles

Figure 3

Volumes and Double Integrals

If we choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ as shown in Figure 4.

The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

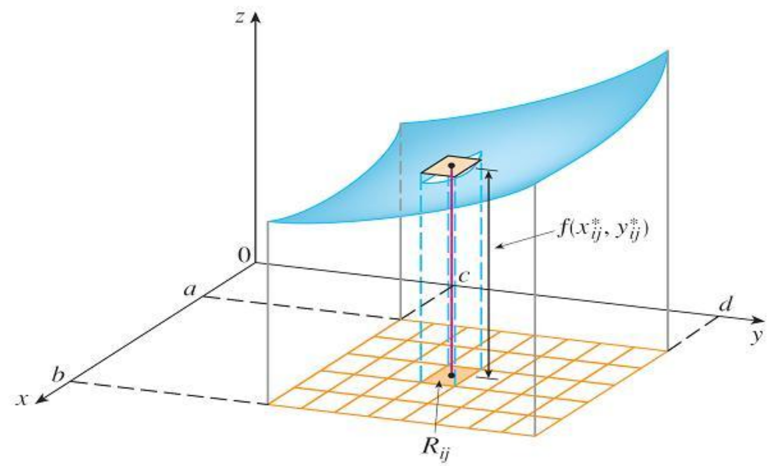


Figure 4

Volumes and Double Integrals

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of *S*:

3 $$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*})\, \Delta A$$

(See Figure 5.) This double sum means that for each subrectangle we evaluate *f* at the chosen point and multiply by the area of the subrectangle, and then we add the results.

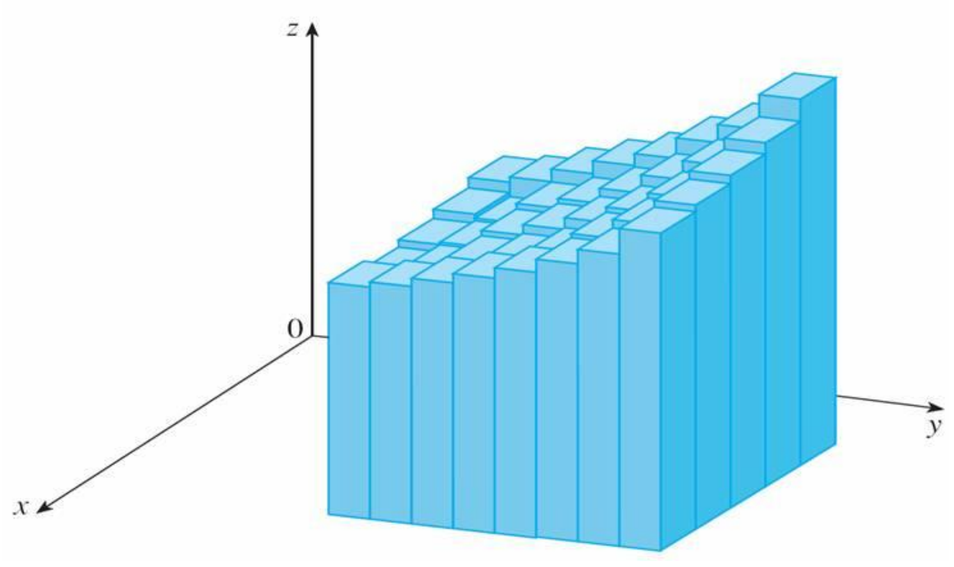


Figure 5

Volumes and Double Integrals

Our intuition tells us that the approximation given in (3) becomes better as *m* and *n* become larger and so we would expect that

$$\boxed{4} \qquad V = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*})\, \Delta A$$

We use the expression in Equation 4 to define the **volume** of the solid *S* that lies under the graph of *f* and above the rectangle *R*.

Volumes and Double Integrals

Limits of the type that appear in Equation 4 occur frequently, not just in finding volumes but in a variety of other situations even when *f* is not a positive function. So we make the following definition.

$\boxed{5}$ **Definition** The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists.

Volumes and Double Integrals

The precise meaning of the limit in Definition 5 is that for every number $\varepsilon > 0$ there is an integer N such that

$$\left| \iint_R f(x, y)\, dA - \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A \right| < \varepsilon$$

for all integers m and n greater than N and for any choice of sample points (x_{ij}^*, y_{ij}^*) in R_{ij}.

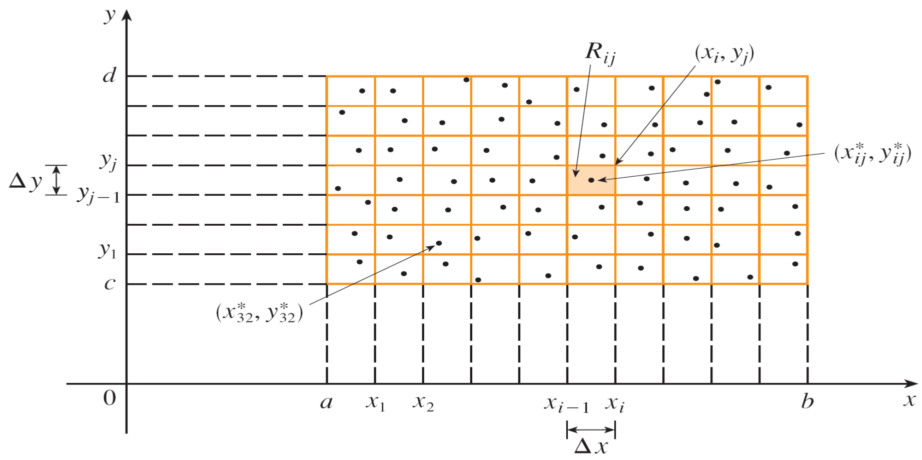
A function f is called **integrable** if the limit in Definition 5 exists.

Volumes and Double Integrals

It is shown in courses on advanced calculus that all continuous functions are integrable. In fact, the double integral of f exists provided that f is “not too discontinuous.”

In particular, if f is bounded [that is, there is a constant M such that $|f(x, y)| \leq M$ for all (x, y) in R], and f is continuous there, except on a finite number of smooth curves, then f is integrable over R.

Volumes and Double Integrals



Dividing R into subrectangles

Figure 3

The sample point (x_{ij}^*, y_{ij}^*) can be chosen to be any point in the subrectangle R_{ij}, but if we choose it to be the upper right-hand corner of R_{ij} [namely (x_i, y_j), see Figure 3], then the expression for the double integral looks simpler:

6

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

Volumes and Double Integrals

By comparing Definitions 4 and 5, we see that a volume can be written as a double integral:

If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA$$

Volumes and Double Integrals

The sum in Definition 5,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*})\, \Delta A$$

is called a **double Riemann sum** and is used as an approximation to the value of the double integral. [Notice how similar it is to the Riemann sum in $\boxed{1}$ for a function of a single variable.]

Volumes and Double Integrals

If *f* happens to be a *positive* function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 5, and is an approximation to the volume under the graph of *f*.

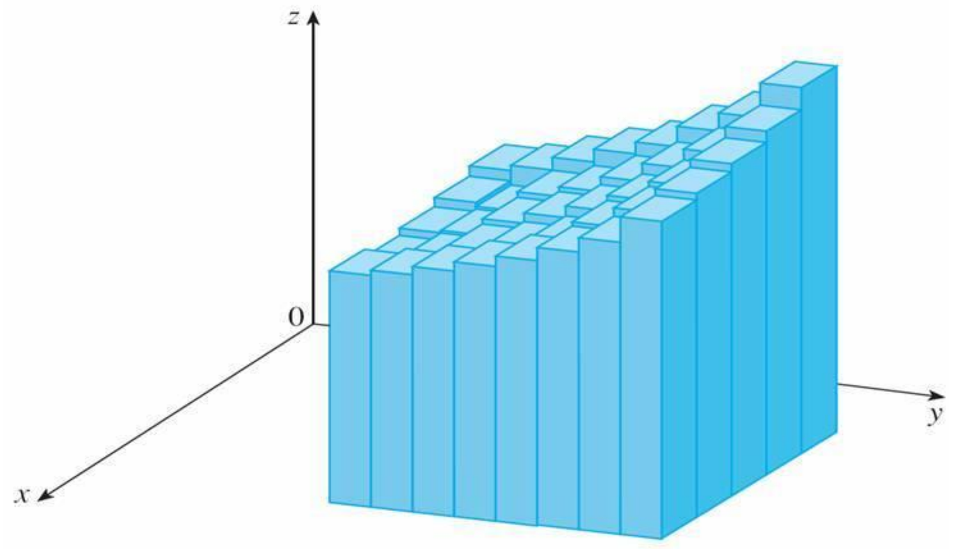


Figure 5

Example 1

Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

Example 1 – *Solution*

The squares are shown in Figure 6.

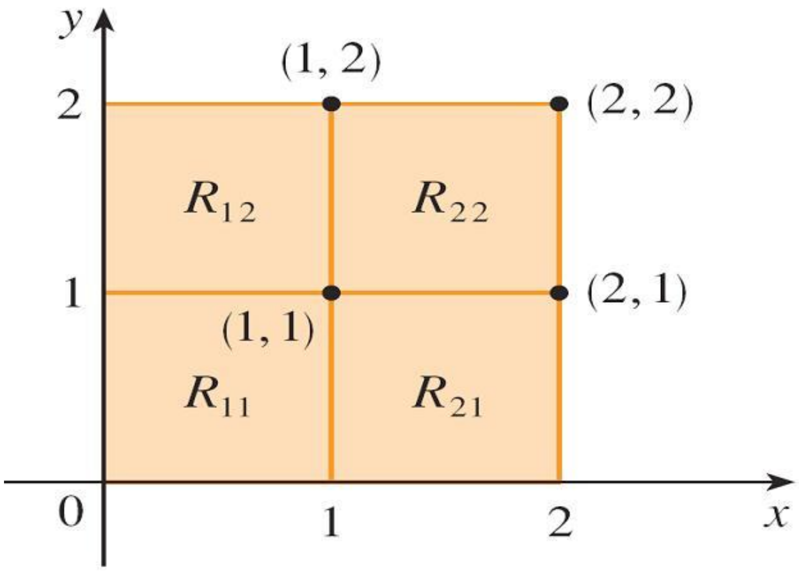


Figure 6

The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$.

Example 1 – *Solution*

cont'd

Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A$$

$$= f(1, 1)\, \Delta A + f(1, 2)\, \Delta A + f(2, 1)\, \Delta A + f(2, 2)\, \Delta A$$

$$= 13(1) + 7(1) + 10(1) + 4(1)$$

$$= 34$$

Example 1 – *Solution*

cont'd

This is the volume of the approximating rectangular boxes shown in Figure 7.

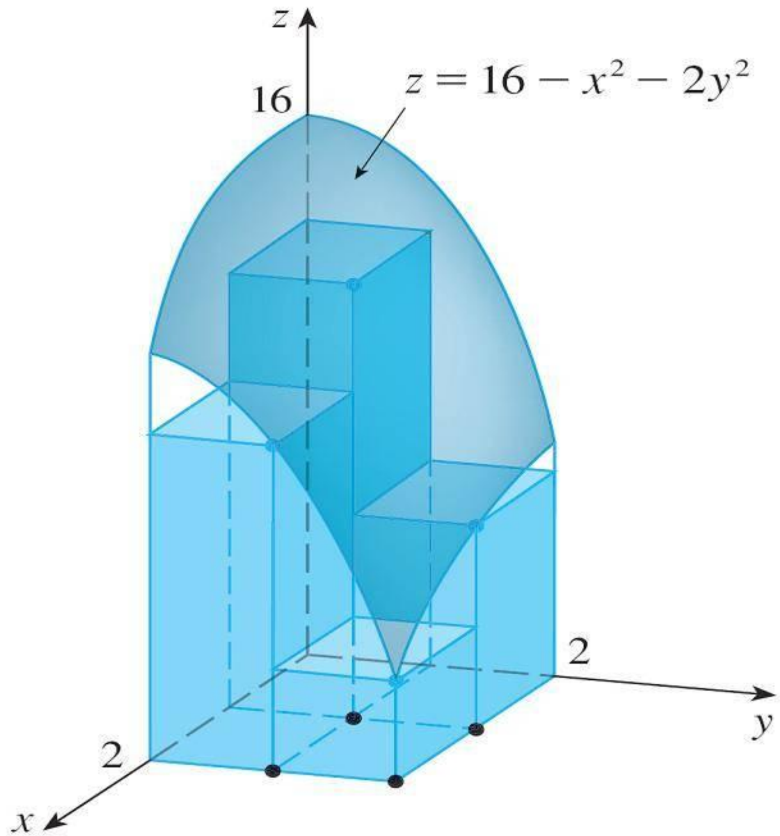


Figure 7

The Midpoint Rule

The Midpoint Rule

The methods that we used for approximating single integrals (the Midpoint Rule, the Trapezoidal Rule, Simpson's Rule) all have counterparts for double integrals. Here we consider only the Midpoint Rule for double integrals.

This means that we use a double Riemann sum to approximate the double integral, where the sample point (x_{ij}^*, y_{ij}^*) in R_{ij} is chosen to be the center $(\overline{x}_i, \overline{y}_j)$ of R_{ij}. In other words, $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

The Midpoint Rule

Midpoint Rule for Double Integrals

$$\iint\limits_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\, \Delta A$$

where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Example 3

Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x - 3y^2)\, dA$, where

$$R = \{(x, y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}.$$

Solution:

In using the Midpoint Rule with $m = n = 2$, we evaluate $f(x, y) = x - 3y^2$ at the centers of the four subrectangles shown in Figure 10.

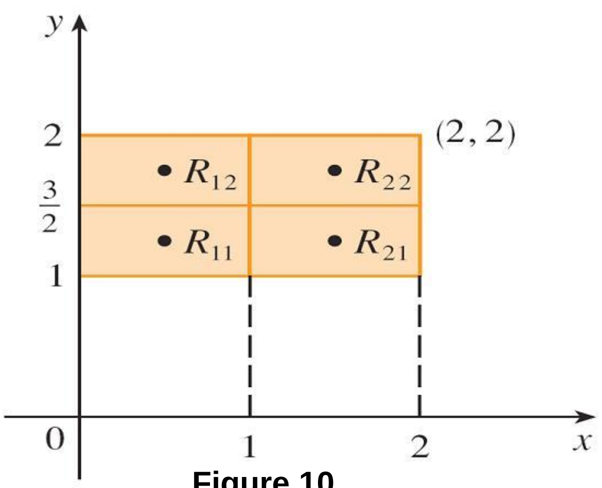


Figure 10

Example 3 – *Solution*

cont'd

So $\bar{x}_1 = \frac{1}{2}$, $\bar{x}_2 = \frac{3}{2}$, $\bar{y}_1 = \frac{5}{4}$, and $\bar{y}_2 = \frac{7}{4}$.

The area of each subrectangle is $\Delta A = \frac{1}{2}$.

Thus

$$\iint_R (x - 3y^2)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

$$= f(\bar{x}_1, \bar{y}_1)\, \Delta A + f(\bar{x}_1, \bar{y}_2)\, \Delta A + f(\bar{x}_2, \bar{y}_1)\, \Delta A + f(\bar{x}_2, \bar{y}_2)\, \Delta A$$

$$= f\left(\tfrac{1}{2}, \tfrac{5}{4}\right) \Delta A + f\left(\tfrac{1}{2}, \tfrac{7}{4}\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{5}{4}\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{7}{4}\right) \Delta A$$

Example 3 – *Solution*

cont'd

$$= \left(-\frac{67}{16}\right)\frac{1}{2} + \left(-\frac{139}{16}\right)\frac{1}{2} + \left(-\frac{51}{16}\right)\frac{1}{2} + \left(-\frac{123}{16}\right)\frac{1}{2}$$

$$= -\frac{95}{8}$$

$$= -11.875$$

Thus we have

$$\iint_R (x - 3y^2)\, dA \approx -11.875$$

Average Values

Average Values

Recall that the average value of a function *f* of one variable defined on an interval [*a*, *b*] is

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

In a similar fashion we define the **average value** of a function *f* of two variables defined on a rectangle *R* to be

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x, y)\,dA$$

where *A*(*R*) is the area of *R*.

Average Values

If $f(x, y) \geq 0$, the equation

$$A(R) \times f_{\text{ave}} = \iint_R f(x, y)\, dA$$

says that the box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f.

[If $z = f(x, y)$ describes a mountainous region and you chop off the tops of the mountains at height f_{ave}, then you can use them to fill in the valleys so that the region becomes completely flat. See Figure 11.]

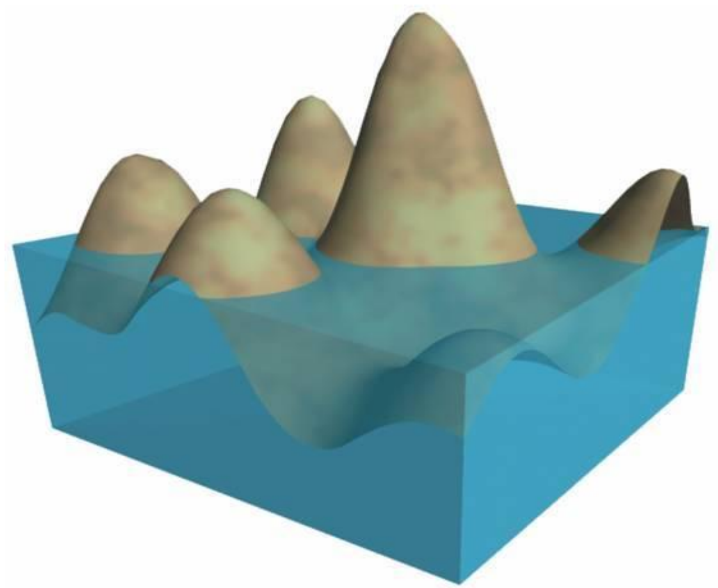


Figure 11

Example 4

The contour map in Figure 12 shows the snowfall, in inches, that fell on the state of Colorado on December 20 and 21, 2006. (The state is in the shape of a rectangle that measures 388 mi west to east and 276 mi south to north.) Use the contour map to estimate the average snowfall for the entire state of Colorado on those days.

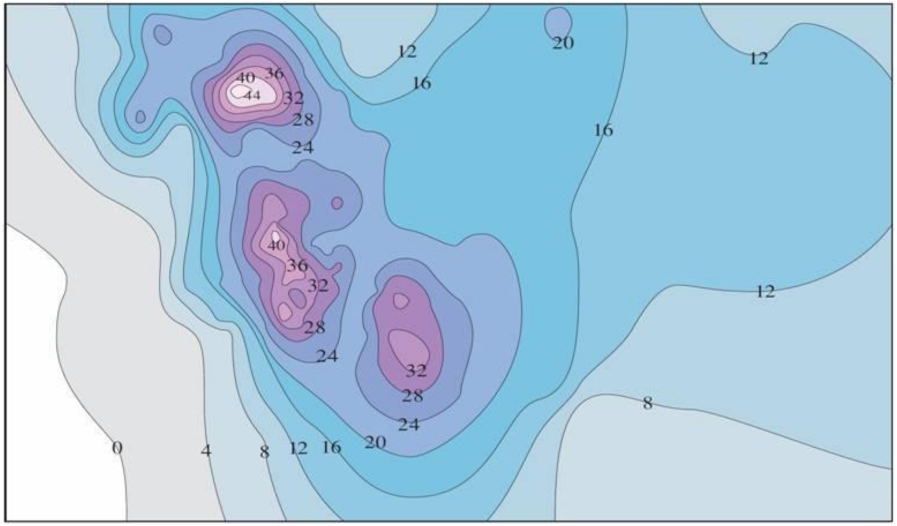


Figure 12

Example 4 – *Solution*

Let's place the origin at the southwest corner of the state.

Then $0 \le x \le 388$, $0 \le y \le 276$, and $f(x, y)$ is the snowfall, in inches, at a location x miles to the east and y miles to the north of the origin.

If R is the rectangle that represents Colorado, then the average snowfall for the state on December 20–21 was

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

where $A(R) = 388 \cdot 276$.

Example 4 – *Solution*

cont'd

To estimate the value of this double integral, let's use the Midpoint Rule with $m = n = 4$. In other words, we divide R into 16 subrectangles of equal size, as in Figure 13.

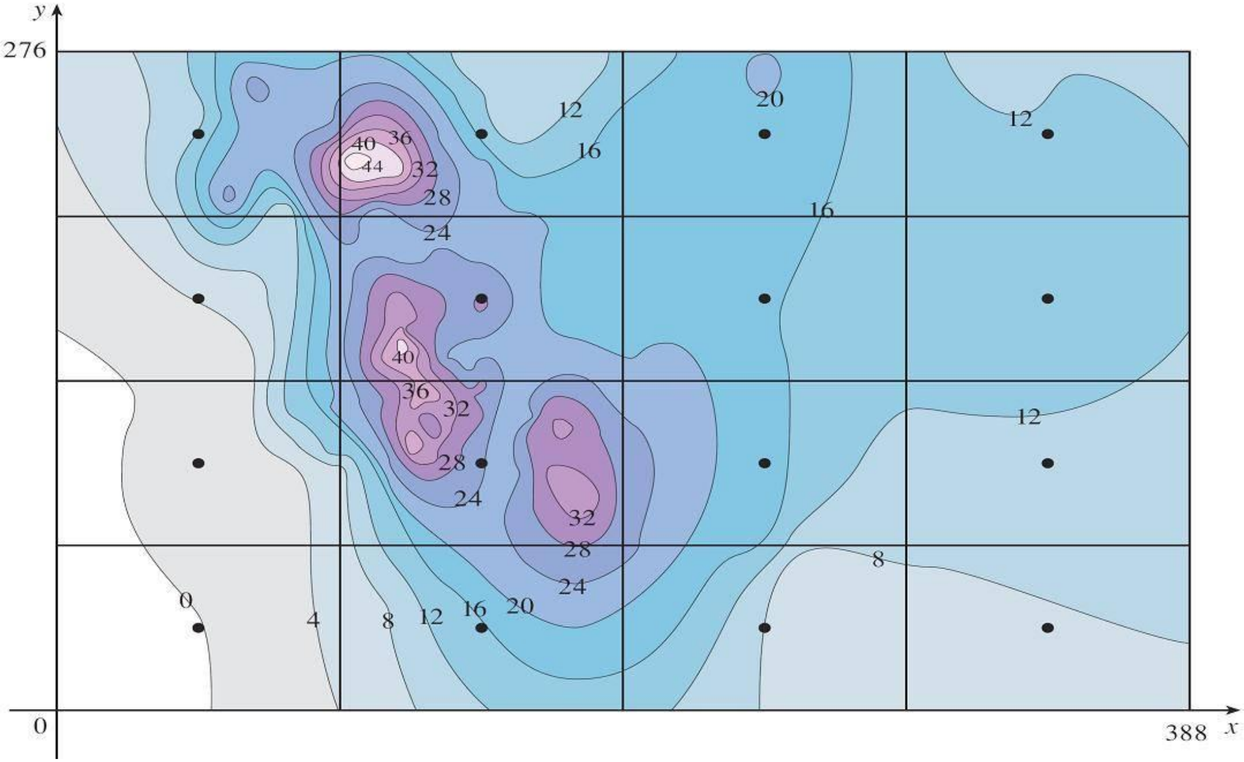


Figure 13

Example 4 – *Solution*

cont'd

The area of each subrectangle is

$$\Delta A = \tfrac{1}{16}(388)(276)$$

$$= 6693 \text{ mi}^2$$

Using the contour map to estimate the value of *f* at the center of each subrectangle, we get

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{4} \sum_{j=1}^{4} f(\overline{x}_i, \overline{y}_j)\, \Delta A$$

$$\approx \Delta A[0 + 15 + 8 + 7 + 2 + 25 + 18.5$$
$$+ 11 + 4.5 + 28 + 17 + 13.5 + 12$$
$$+ 15 + 17.5 + 13]$$

$$= (6693)(207)$$

Example 4 – *Solution*

cont'd

Therefore

$$f_{\text{ave}} \approx \frac{(6693)(207)}{(388)(276)}$$

$$\approx 12.9$$

On December 20–21, 2006, Colorado received an average of approximately 13 inches of snow.

Properties of Double Integrals

Properties of Double Integrals

We list here three properties of double integrals. We assume that all of the integrals exist. Properties 7 and 8 are referred to as the *linearity* of the integral.

7 $$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

8 $$\iint_R c\, f(x, y)\, dA = c \iint_R f(x, y)\, dA \quad \text{where } c \text{ is a constant}$$

If $f(x, y) \geq g(x, y)$ for all (x, y) in R, then

9 $$\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$$

15

Multiple Integrals

15.2 Iterated Integrals

Iterated Integrals

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

We use the notation $\int_c^d f(x, y)\, dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called *partial integration with respect to y*. (Notice its similarity to partial differentiation.)

Now $\int_c^d f(x, y)\, dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x, y)\, dy$$

Iterated Integrals

If we now integrate the function *A* with respect to *x* from $x = a$ to $x = b$, we get

$$\boxed{1} \qquad \int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy \right] dx$$

The integral on the right side of Equation 1 is called an **iterated integral**. Usually the brackets are omitted. Thus

$$\boxed{2} \qquad \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy \right] dx$$

means that we first integrate with respect to *y* from *c* to *d* and then with respect to *x* from *a* to *b*.

Iterated Integrals

Similarly, the iterated integral

$$\boxed{3} \qquad \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

means that we first integrate with respect to *x* (holding *y* fixed) from $x = a$ to $x = b$ and then we integrate the resulting function of *y* with respect to *y* from $y = c$ to $y = d$.

Notice that in both Equations 2 and 3 we work *from the inside out*.

Example 1

Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$ (b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

Solution:

(a) Regarding *x* as a constant, we obtain

$$\int_1^2 x^2 y \, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2}$$

$$= x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right)$$

$$= \tfrac{3}{2} x^2$$

Example 1 – *Solution*

cont'd

Thus the function *A* in the preceding discussion is given by $A(x) = \frac{3}{2}x^2$ in this example.

We now integrate this function of *x* from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y\, dy\, dx = \int_0^3 \left[\int_1^2 x^2 y\, dy \right] dx$$

$$= \int_0^3 \tfrac{3}{2} x^2\, dx$$

$$= \frac{x^3}{2} \Big]_0^3$$

$$= \frac{27}{2}$$

Example 1 – *Solution*

cont'd

(b) Here we first integrate with respect to x:

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy$$

$$= \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy$$

$$= \int_1^2 9y \, dy$$

$$= 9 \frac{y^2}{2} \bigg]_1^2 = \frac{27}{2}$$

Iterated Integrals

Notice that in Example 1 we obtained the same answer whether we integrated with respect to y or x first.

In general, it turns out (see Theorem 4) that the two iterated integrals in Equations 2 and 3 are always equal; that is, the order of integration does not matter. (This is similar to Clairaut's Theorem on the equality of the mixed partial derivatives.)

Iterated Integrals

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

$\boxed{4}$ **Fubini's Theorem** If f is continuous on the rectangle $R = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Iterated Integrals

In the special case where $f(x, y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form.

To be specific, suppose that $f(x, y) = g(x)h(y)$ and $R = [a, b] \times [c, d]$.

Then Fubini's Theorem gives

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d \left[\int_a^b g(x)h(y)\, dx \right] dy$$

Iterated Integrals

In the inner integral, y is a constant, so $h(y)$ is a constant and we can write

$$\int_c^d \left[\int_a^b g(x)\,h(y)\,dx \right] dy = \int_c^d \left[h(y) \left(\int_a^b g(x)\,dx \right) \right] dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

since $\int_a^b g(x)\,dx$ is a constant.

Therefore, in this case, the double integral of f can be written as the product of two single integrals:

5 $$\iint_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy \qquad \text{where } R = [a, b] \times [c, d]$$

15 Multiple Integrals

15.3 Double Integrals over General Regions

Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval.

But for double integrals, we want to be able to integrate a function *f* not just over rectangles but also over regions *D* of more general shape, such as the one illustrated in Figure 1.

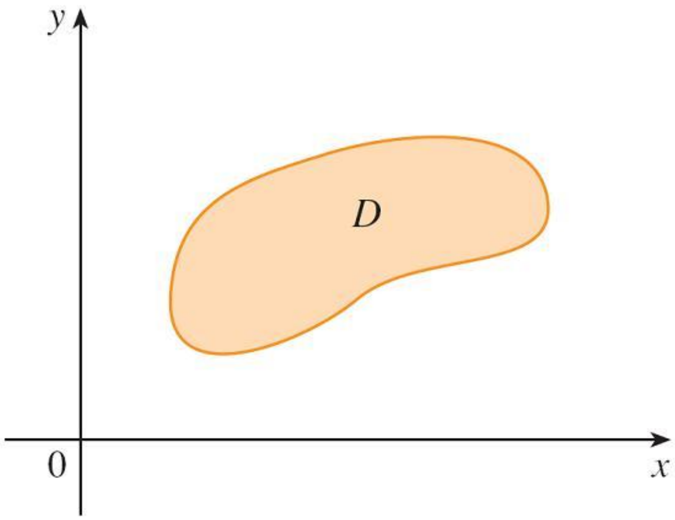


Figure 1

Double Integrals over General Regions

We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 2.

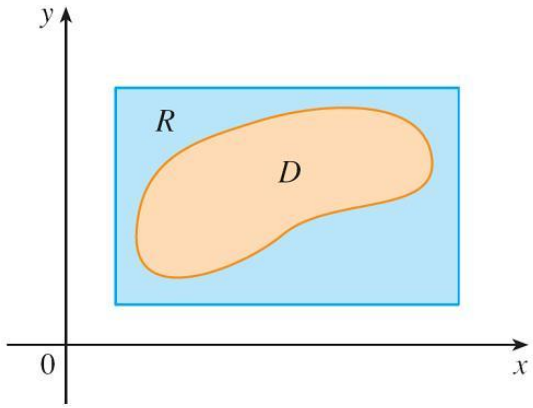


Figure 2

Then we define a new function F with domain R by

1
$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

Double Integrals over General Regions

If *F* is integrable over *R*, then we define the **double integral of *f* over *D*** by

$$\boxed{2} \qquad \iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \qquad \text{where } F \text{ is given by Equation 1}$$

Definition 2 makes sense because *R* is a rectangle and so $\iint_R F(x, y)\, dA$ has been previously defined.

Double Integrals over General Regions

The procedure that we have used is reasonable because the values of $F(x, y)$ are 0 when (x, y) lies outside D and so they contribute nothing to the integral.

This means that it doesn't matter what rectangle R we use as long as it contains D.

In the case where $f(x, y) \geq 0$, we can still interpret $\iint_D f(x, y)\, dA$ as the volume of the solid that lies above D and under the surface $z = f(x, y)$ (the graph of f).

Double Integrals over General Regions

You can see that this is reasonable by comparing the graphs of *f* and *F* in Figures 3 and 4 and remembering that $\iint_R F(x, y)\, dA$ is the volume under the graph of *F*.

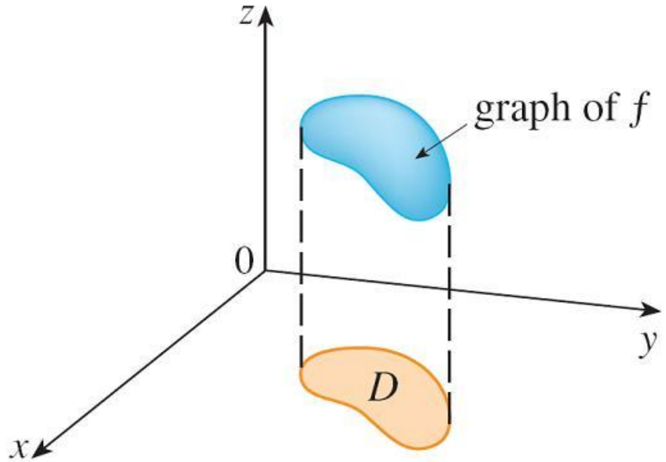


Figure 3

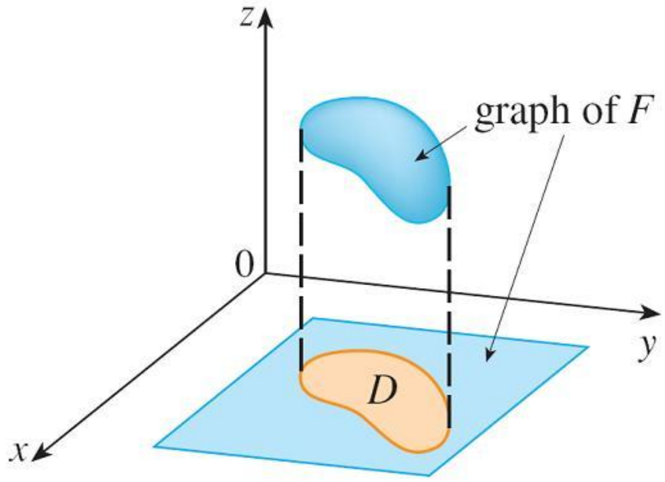


Figure 4

Double Integrals over General Regions

Figure 4 also shows that F is likely to have discontinuities at the boundary points of D.

Nonetheless, if f is continuous on D and the boundary curve of D is "well behaved", then it can be shown that $\iint_R F(x, y)\, dA$ exists and therefore $\iint_D f(x, y)\, dA$ exists.

In particular, this is the case for **type I** and **type II** regions.

Double Integrals over General Regions

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.

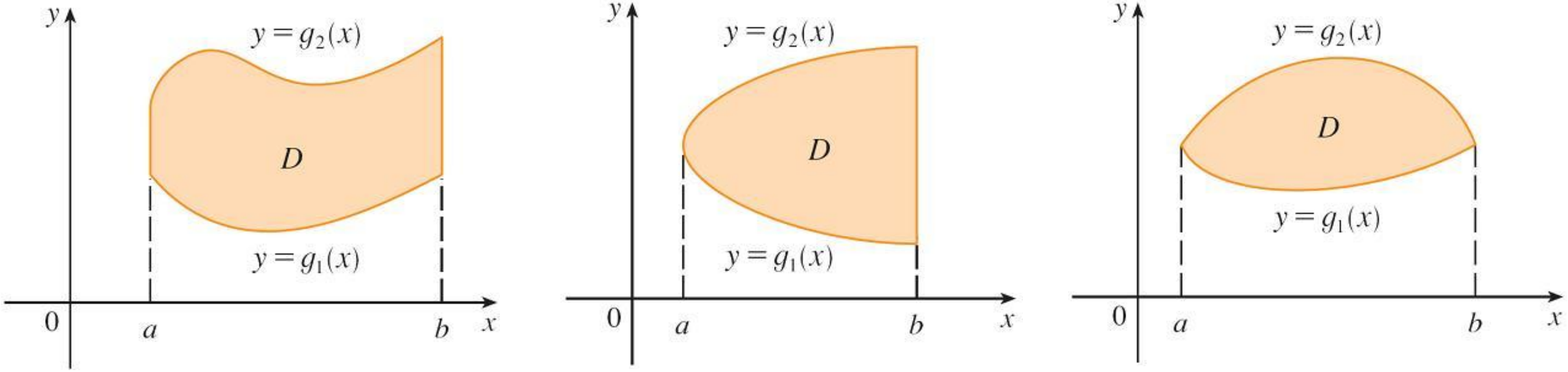


Figure 5

Some type I regions

Double Integrals over General Regions

In order to evaluate $\iint_D f(x, y)\, dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D, as in Figure 6, and we let F be the function given by Equation 1; that is, F agrees with f on D and F is 0 outside D.

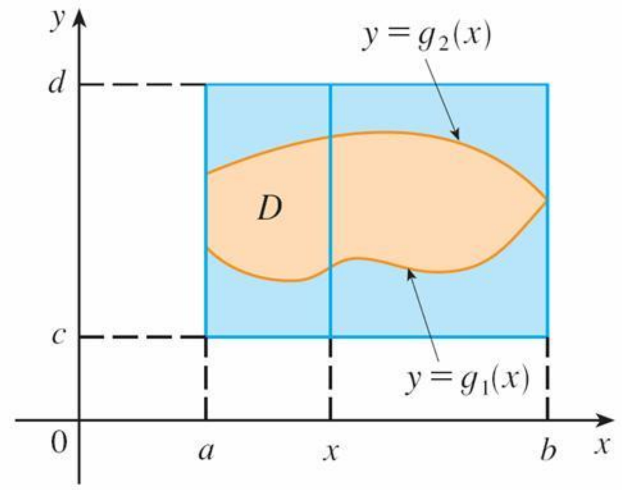


Figure 6

Double Integrals over General Regions

Then, by Fubini's Theorem,

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

Observe that $F(x, y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$ because (x, y) then lies outside D. Therefore

$$\int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

because $F(x, y) = f(x, y)$ when $g_1(x) \le y \le g_2(x)$.

Double Integrals over General Regions

Thus we have the following formula that enables us to evaluate the double integral as an iterated integral.

3 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

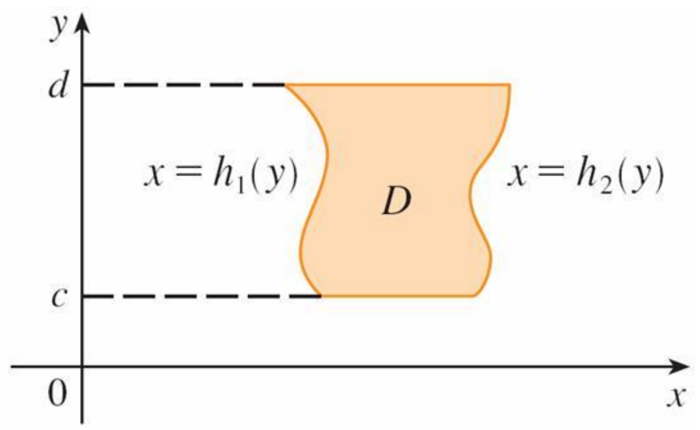
The integral on the right side of $\boxed{3}$ is an iterated integral, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

Double Integrals over General Regions

We also consider plane regions of **type II**, which can be expressed as

4 $$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous. Two such regions are illustrated in Figure 7.



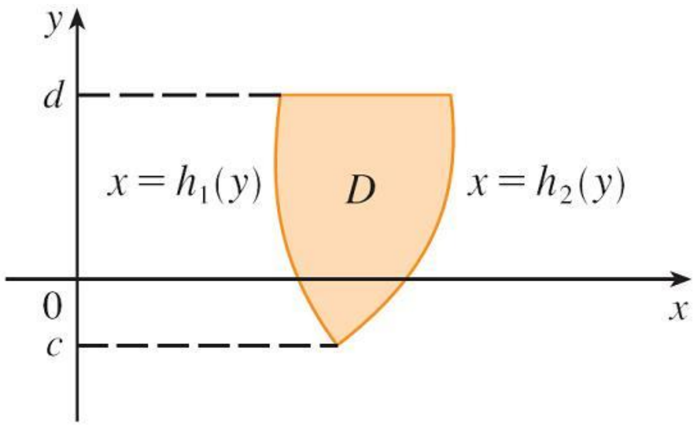


Figure 7
Some type II regions

Double Integrals over General Regions

Using the same methods that were used in establishing $\boxed{3}$, we can show that

$$\boxed{5} \qquad \iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where D is a type II region given by Equation 4.

Example 1

Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution:

The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$.

We note that the region D, sketched in Figure 8, is a type I region but not a type II region and we can write

$$D = \{(x, y) \mid -1 \leq x \leq 1,\ 2x^2 \leq y \leq 1 + x^2\}$$

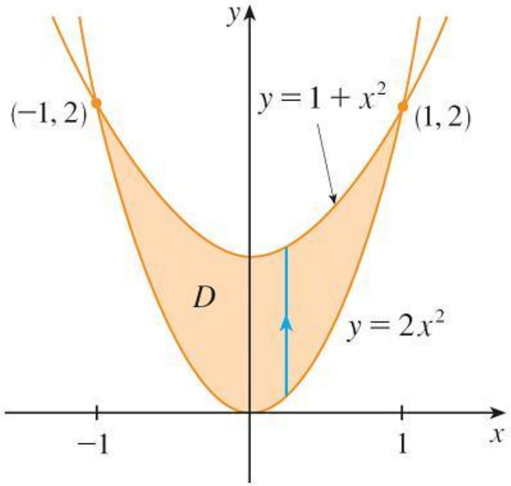


Figure 8

Example 1 – *Solution*

cont'd

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 3 gives

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \Big[xy + y^2\Big]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\, dx$$

Example 1 – *Solution*

cont'd

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \Bigg]_{-1}^{1}$$

$$= \frac{32}{15}$$

Properties of Double Integrals

Properties of Double Integrals

We assume that all of the following integrals exist. The first three properties of double integrals over a region *D* follow immediately from Definition 2.

6 $$\iint_D [f(x, y) + g(x, y)]\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA$$

7 $$\iint_D c f(x, y)\, dA = c \iint_D f(x, y)\, dA$$

If $f(x, y) \geq g(x, y)$ for all (x, y) in D, then

8 $$\iint_D f(x, y)\, dA \geqslant \iint_D g(x, y)\, dA$$

Properties of Double Integrals

The next property of double integrals is similar to the property of single integrals given by the equation

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

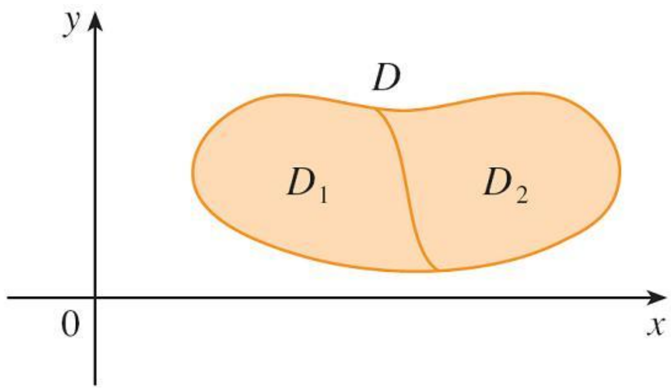


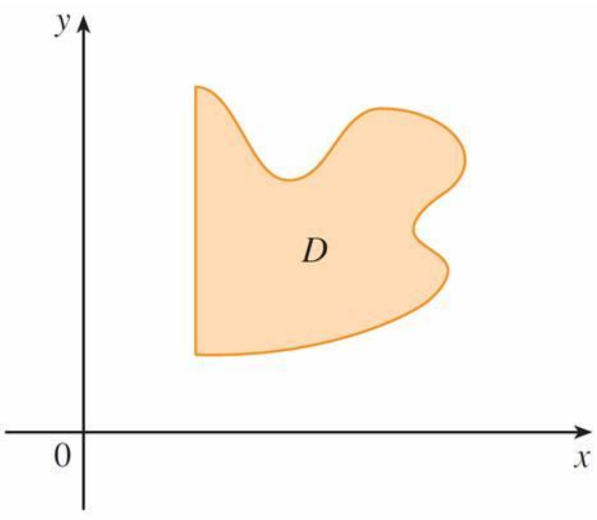
Figure 17

9

$$\boxed{\iint_D f(x, y)\,dA = \iint_{D_1} f(x, y)\,dA + \iint_{D_2} f(x, y)\,dA}$$

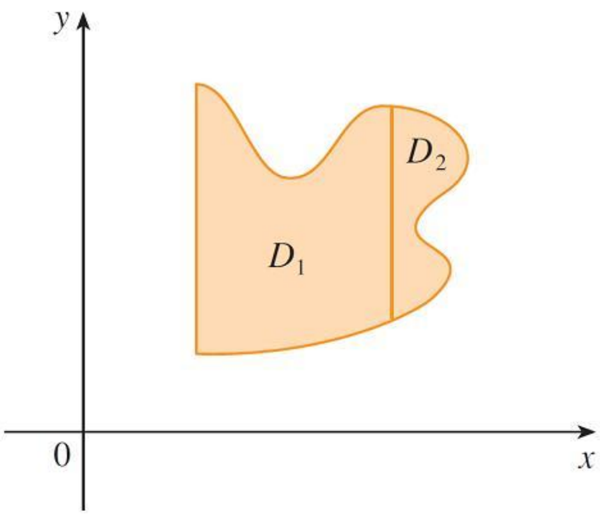
Properties of Double Integrals

Property 9 can be used to evaluate double integrals over regions *D* that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure.



D is neither type I nor type II.

Figure 18(a)



$D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

Figure 18(b)

Properties of Double Integrals

The next property of integrals says that if we integrate the constant function *f*(*x*, *y*) = 1 over a region *D*, we get the area of *D*:

10

$$\boxed{\iint_D 1\, dA = A(D)}$$

Properties of Double Integrals

Figure 19 illustrates why Equation 10 is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1\, dA$.

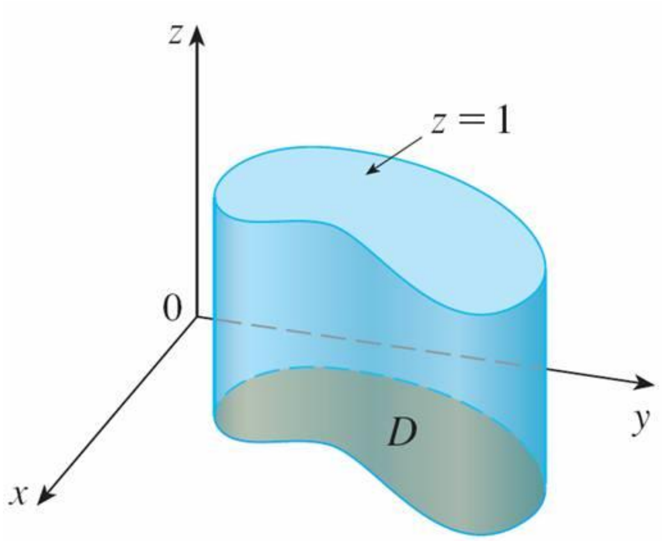


Figure 19

Cylinder with base *D* and height 1

Properties of Double Integrals

Finally, we can combine Properties 7, 8, and 10 to prove the following property.

$\boxed{11}$ If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$mA(D) \leqslant \iint_D f(x, y)\, dA \leqslant MA(D)$$

Example 6

Use Property 11 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

Solution:

Since $-1 \le \sin x \le 1$ and $-1 \le \cos y \le 1$, we have $-1 \le \sin x \cos y \le 1$ and therefore

$$e^{-1} \le e^{\sin x \cos y} \le e^1 = e$$

Thus, using $m = e^{-1} = 1/e$, $M = e$, and $A(D) = \pi(2)^2$ in Property 11, we obtain

$$\frac{4\pi}{e} \leqslant \iint_D e^{\sin x \cos y}\, dA \leqslant 4\pi e$$

15 Multiple Integrals

15.4 Double Integrals in Polar Coordinates

Double Integrals in Polar Coordinates

Suppose that we want to evaluate a double integral $\iint_R f(x, y)\, dA$, where R is one of the regions shown in Figure 1. In either case the description of R in terms of rectangular coordinates is rather complicated, but R is easily described using polar coordinates.

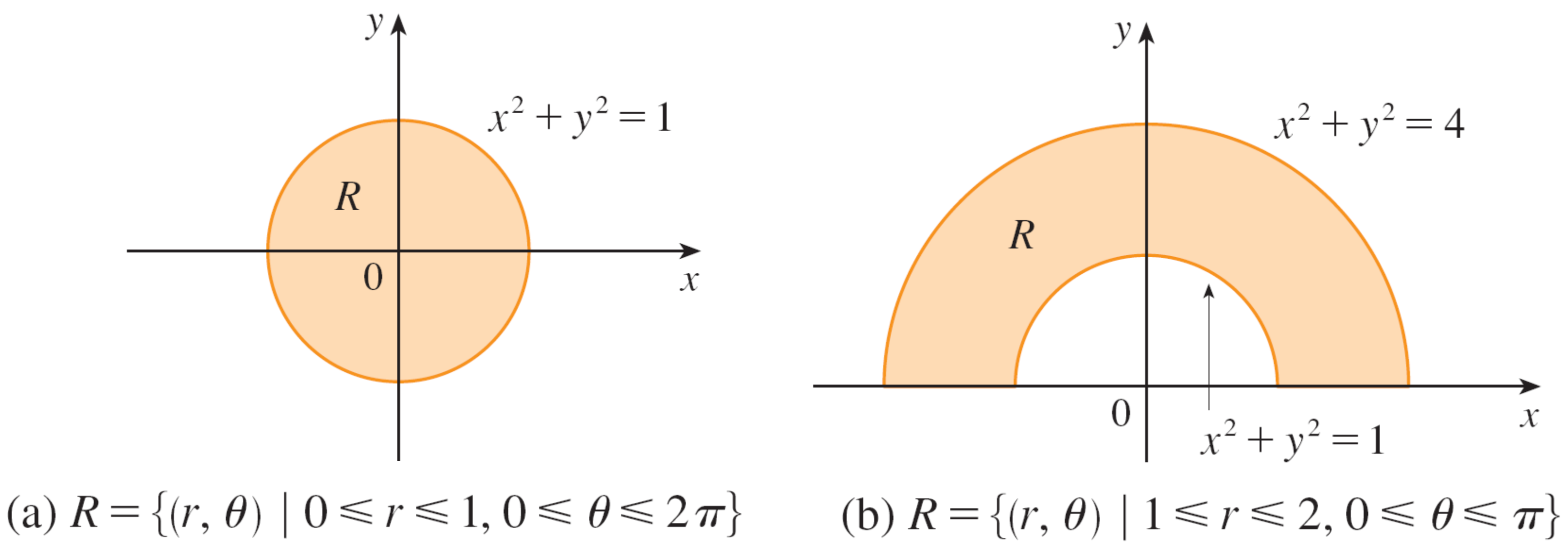


Figure 1

Double Integrals in Polar Coordinates

Recall from Figure 2 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

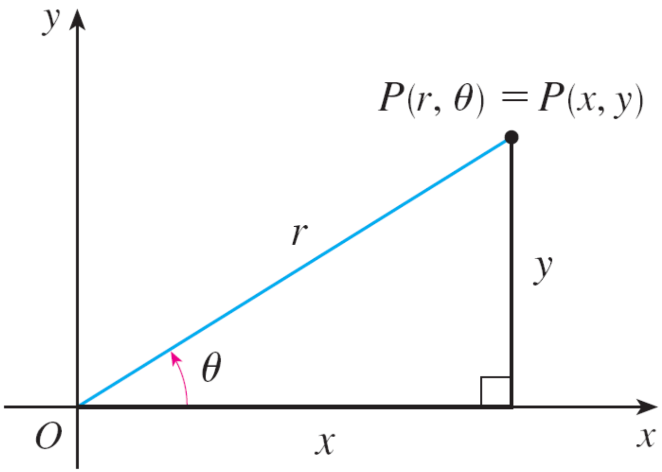


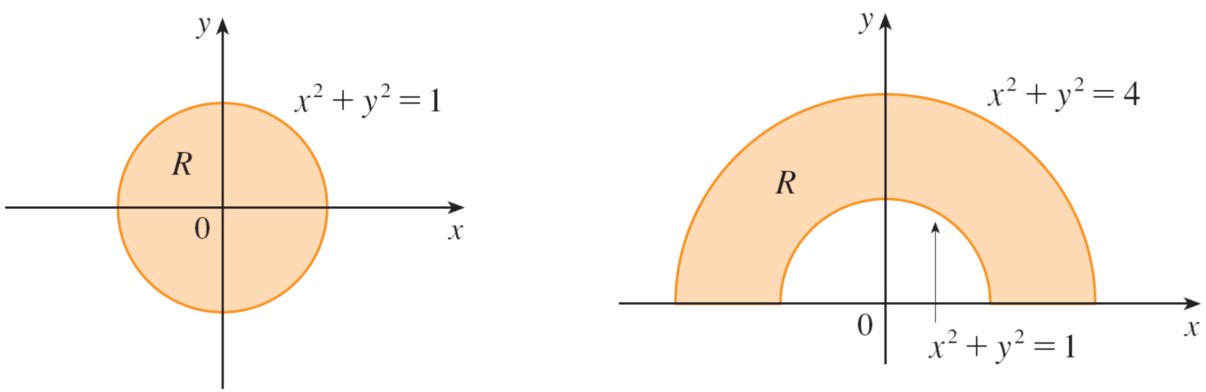
Figure 2

Double Integrals in Polar Coordinates

The regions in Figure 1 are special cases of a **polar rectangle**

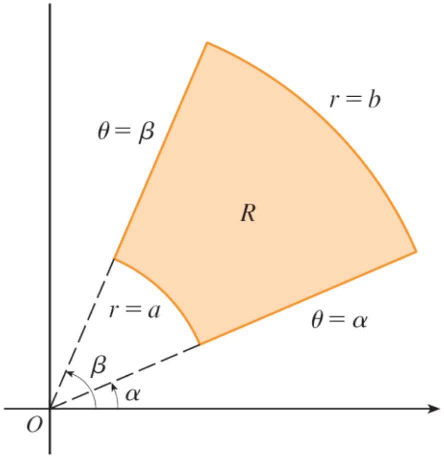
$$R = \{(r, \theta) \mid a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$$

which is shown in Figure 3.



(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

Figure 1



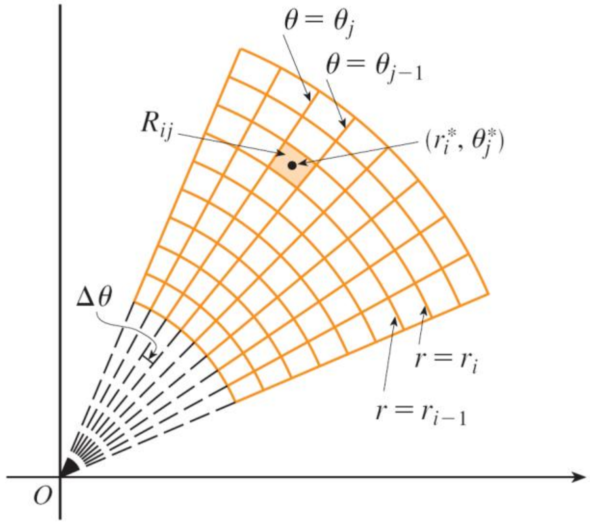
Polar rectangle

Figure 3

Double Integrals in Polar Coordinates

In order to compute the double integral $\iint_R f(x, y)\, dA$, where R is a polar rectangle, we divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = (b - a)/m$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{i-1}, \theta_j]$ of equal width $\Delta\theta = (\beta - \alpha)/n$.

Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles R_{ij} shown in Figure 4.



Dividing R into polar subrectangles

Figure 4

Double Integrals in Polar Coordinates

The "center" of the polar subrectangle

$$R_{ij} = \{(r, \theta) \mid r_{i-1} \leq r \leq r_i, \theta_{i-1} \leq \theta \leq \theta_j\}$$

has polar coordinates

$$r_i^* = \tfrac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \tfrac{1}{2}(\theta_{j-1} + \theta_j)$$

We compute the area of R_{ij} using the fact that the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$.

Double Integrals in Polar Coordinates

Subtracting the areas of two such sectors, each of which has central angle $\Delta\theta = \theta_j - \theta_{j-1}$, we find that the area of R_{ij} is

$$\Delta A_i = \tfrac{1}{2} r_i^2 \, \Delta\theta - \tfrac{1}{2} r_{i-1}^2 \, \Delta\theta = \tfrac{1}{2} (r_i^2 - r_{i-1}^2) \, \Delta\theta$$

$$= \tfrac{1}{2} (r_i + r_{i-1})(r_i - r_{i-1}) \, \Delta\theta = r_i^* \, \Delta r \, \Delta\theta$$

Although we have defined the double integral $\iint_R f(x, y)\, dA$ in terms of ordinary rectangles, it can be shown that, for continuous functions f, we always obtain the same answer using polar rectangles.

Double Integrals in Polar Coordinates

The rectangular coordinates of the center of R_{ij} are $(r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*)$, so a typical Riemann sum is

$$\boxed{1} \qquad \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*) \, \Delta A_i = \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*) \, r_i^* \, \Delta r \, \Delta\theta$$

If we write $g(r, \theta) = rf(r \cos \theta, r \sin \theta)$, then the Riemann sum in Equation 1 can be written as

$$\sum_{i=1}^{m} \sum_{j=1}^{n} g(r_i^*, \theta_j^*) \, \Delta r \, \Delta\theta$$

which is a Riemann sum for the double integral

$$\int_{\alpha}^{\beta} \int_{a}^{b} g(r, \theta) \, dr \, d\theta$$

Double Integrals in Polar Coordinates

Therefore we have

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos \theta_j^*, r_i^* \sin \theta_j^*)\, \Delta A_i$$

$$= \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} g(r_i^*, \theta_j^*)\, \Delta r\, \Delta \theta = \int_\alpha^\beta \int_a^b g(r, \theta)\, dr\, d\theta$$

$$= \int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta)\, r\, dr\, d\theta$$

2 Change to Polar Coordinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta)\, r\, dr\, d\theta$$

Double Integrals in Polar Coordinates

The formula in $\boxed{2}$ says that we convert from rectangular to polar coordinates in a double integral by writing $x = r \cos \theta$ and $y = r \sin \theta$, using the appropriate limits of integration for r and θ, and replacing dA by $r\,dr\,d\theta$.

Be careful not to forget the additional factor r on the right side of Formula 2.

A classical method for remembering this is shown in Figure 5, where the "infinitesimal" polar rectangle can be thought of as an ordinary rectangle with dimensions $r\,d\theta$ and dr and therefore has "area" $dA = r\,dr\,d\theta$.

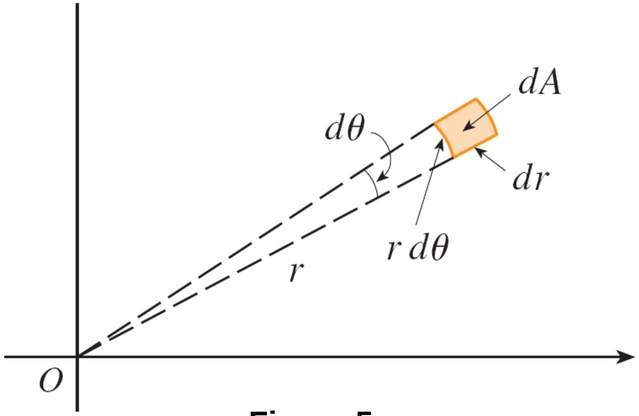


Figure 5

Example 1

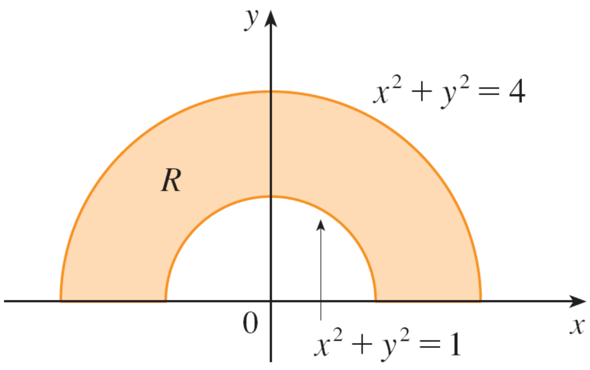
Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution:

The region R can be described as

$$R = \{(x, y) \mid y \geq 0,\ 1 \leq x^2 + y^2 \leq 4\}$$

It is the half-ring shown in Figure 1(b), and in polar coordinates it is given by $1 \leq r \leq 2,\ 0 \leq \theta \leq \pi$.



$R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant \pi\}$

Figure 1(b)

Example 1 – *Solution*

cont'd

Therefore, by Formula 2,

$$\iint_R (3x + 4y^2)\, dA = \int_0^{\pi} \int_1^2 (3r\cos\theta + 4r^2 \sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi} \int_1^2 (3r^2\cos\theta + 4r^3 \sin^2\theta)\, dr\, d\theta$$

$$= \int_0^{\pi} \Big[r^3\cos\theta + r^4 \sin^2\theta\Big]_{r=1}^{r=2}\, d\theta$$

$$= \int_0^{\pi} (7\cos\theta + 15 \sin^2\theta)\, d\theta$$

$$= \int_0^{\pi} \Big[7\cos\theta + \tfrac{15}{2}(1 - \cos 2\theta)\Big]\, d\theta$$

$$= 7\sin\theta + \frac{15\theta}{2} - \frac{15}{4}\sin 2\theta\Big]_0^{\pi} = \frac{15\pi}{2}$$

Double Integrals in Polar Coordinates

What we have done so far can be extended to the more complicated type of region shown in Figure 7. In fact, by combining Formula 2 with

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

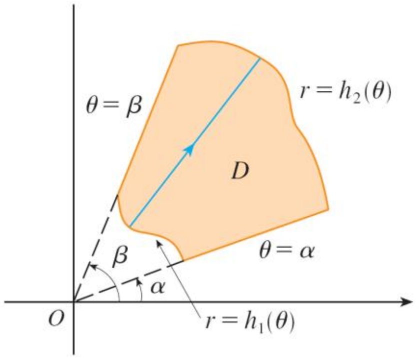


Figure 7

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

where *D* is a type II region, we obtain the following formula.

3 If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Double Integrals in Polar Coordinates

In particular, taking $f(x, y) = 1$, $h_1(\theta) = 0$, and $h_2(\theta) = h(\theta)$ in this formula, we see that the area of the region D bounded by $\theta = \alpha$, $\theta = \beta$, and $r = h(\theta)$ is

$$
\begin{aligned}
A(D) &= \iint_D 1 \, dA \\
&= \int_\alpha^\beta \int_0^{h(\theta)} r \, dr \, d\theta \\
&= \int_\alpha^\beta \left[\frac{r^2}{2} \right]_0^{h(\theta)} d\theta \\
&= \int_\alpha^\beta \tfrac{1}{2} [h(\theta)]^2 \, d\theta
\end{aligned}
$$

15 Multiple Integrals

15.5 Applications of Double Integrals

Density and Mass

Density and Mass

We were able to use single integrals to compute moments and the center of mass of a thin plate or lamina with constant density.

But now, equipped with the double integral, we can consider a lamina with variable density.

Suppose the lamina occupies a region D of the xy-plane and its **density** (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D.

Density and Mass

This means that

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$$

where Δm and ΔA are the mass and area of a small rectangle that contains (x, y) and the limit is taken as the dimensions of the rectangle approach 0. (See Figure 1.)

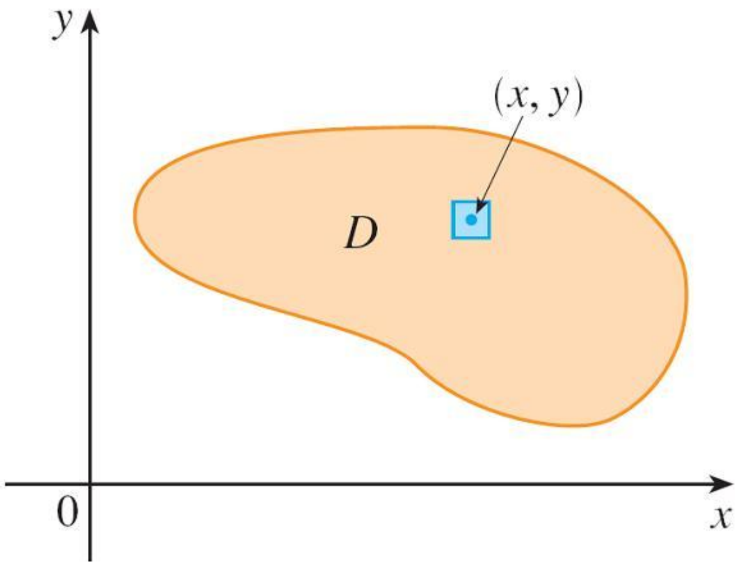


Figure 1

Density and Mass

To find the total mass m of the lamina we divide a rectangle R containing D into subrectangles R_{ij} of the same size (as in Figure 2) and consider $\rho(x, y)$ to be 0 outside D.

If we choose a point (x_{ij}^*, y_{ij}^*) in R_{ij}, then the mass of the part of the lamina that occupies R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\,\Delta A$, where ΔA is the area of R_{ij}.

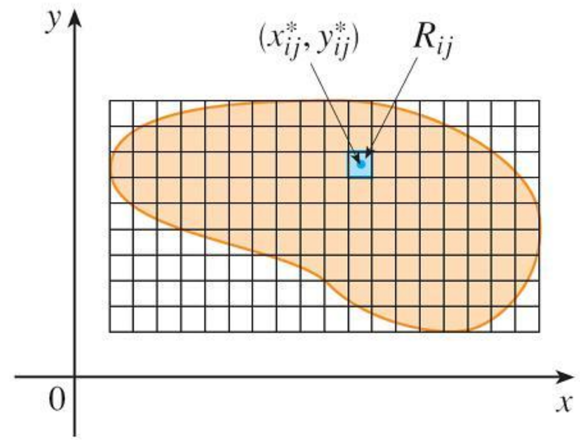


Figure 2

If we add all such masses, we get an approximation to the total mass:

$$m \approx \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\,\Delta A$$

Density and Mass

If we now increase the number of subrectangles, we obtain the total mass *m* of the lamina as the limiting value of the approximations:

$$\boxed{1} \qquad \boxed{m = \lim_{k,\,l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D \rho(x, y)\, dA}$$

Physicists also consider other types of density that can be treated in the same manner.

Density and Mass

For example, if an electric charge is distributed over a region D and the charge density (in units of charge per unit area) is given by $\sigma(x, y)$ at a point (x, y) in D, then the total charge Q is given by

$$\boxed{2} \qquad Q = \iint_D \sigma(x, y)\, dA$$

Example 1

Charge is distributed over the triangular region *D* in Figure 3 so that the charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.

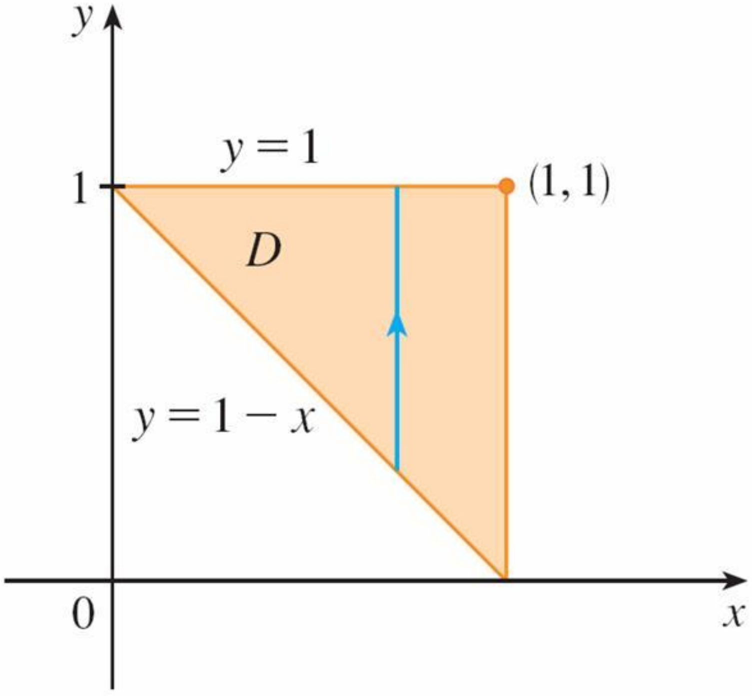


Figure 3

Example 1 – *Solution*

From Equation 2 and Figure 3 we have

$$Q = \iint_D \sigma(x, y)\, dA$$

$$= \int_0^1 \int_{1-x}^1 xy\, dy\, dx$$

$$= \int_0^1 \left[x\frac{y^2}{2} \right]_{y=1-x}^{y=1} dx$$

$$= \int_0^1 \frac{x}{2}\left[1^2 - (1 - x)^2\right] dx$$

Example 1 – *Solution*

cont'd

$$= \tfrac{1}{2}\int_0^1 (2x^2 - x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{5}{24}$$

Thus the total charge is $\frac{5}{24}$ C.

Moments and Centers of Mass

Moments and Centers of Mass

We have found the center of mass of a lamina with constant density; here we consider a lamina with variable density.

Suppose the lamina occupies a region D and has density function $\rho(x, y)$.

Recall that we defined the moment of a particle about an axis as the product of its mass and its directed distance from the axis.

We divide D into small rectangles.

Moments and Centers of Mass

Then the mass of R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\ \Delta A$, so we can approximate the moment of R_{ij} with respect to the *x*-axis by

$$[\rho(x_{ij}^*, y_{ij}^*)\ \Delta A]y_{ij}^*$$

If we now add these quantities and take the limit as the number of subrectangles becomes large, we obtain the **moment** of the entire lamina **about the *x*-axis**:

3 $$\boxed{M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^*\ \rho(x_{ij}^*, y_{ij}^*)\ \Delta A = \iint_D y\,\rho(x, y)\, dA}$$

Moments and Centers of Mass

Similarly, the **moment about the *y*-axis** is

$$\boxed{4} \qquad M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \, \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D x \, \rho(x, y) \, dA$$

As before, we define the center of mass $(\bar{x}, \bar{y})$ so that

$m\bar{x} = M_y$ and $m\bar{y} = M_x$.

The physical significance is that the lamina behaves as if its entire mass is concentrated at its center of mass.

Moments and Centers of Mass

Thus the lamina balances horizontally when supported at its center of mass (see Figure 4).

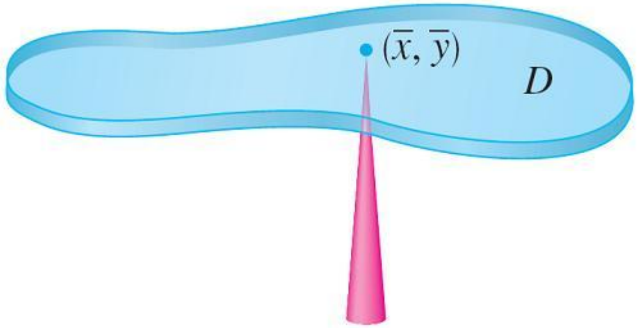


Figure 4

5 The coordinates $(\overline{x}, \overline{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\,\rho(x, y)\,dA \qquad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\,\rho(x, y)\,dA$$

where the mass m is given by

$$m = \iint_D \rho(x, y)\,dA$$

Moment of Inertia

Moment of Inertia

The **moment of inertia** (also called the **second moment**) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis.

We extend this concept to a lamina with density function $\rho(x, y)$ and occupying a region D by proceeding as we did for ordinary moments.

We divide D into small rectangles, approximate the moment of inertia of each subrectangle about the x-axis, and take the limit of the sum as the number of subrectangles becomes large.

Moment of Inertia

The result is the **moment of inertia** of the lamina **about the *x*-axis:**

6

$$I_x = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D y^2 \rho(x, y)\, dA$$

Similarly, the **moment of inertia about the *y*-axis** is

7

$$I_y = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D x^2 \rho(x, y)\, dA$$

Moment of Inertia

It is also of interest to consider the **moment of inertia about the origin**, also called the **polar moment of inertia:**

$$\boxed{8} \quad I_0 = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D (x^2 + y^2)\, \rho(x, y)\, dA$$

Note that $I_0 = I_x + I_y$.

Example 4

Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x, y) = \rho$, center the origin, and radius a.

Solution:

The boundary of D is the circle $x^2 + y^2 = a^2$ and in polar coordinates D is described by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq a$.

Let's compute I_0 first:

$$I_0 = \iint_D (x^2 + y^2)\rho \, dA$$

$$= \rho \int_0^{2\pi} \int_0^a r^2 \, r \, dr \, d\theta$$

Example 4 – *Solution*

cont'd

$$= \rho \int_0^{2\pi} d\theta \int_0^a r^3\, dr$$

$$= 2\pi\rho \left[\frac{r^4}{4} \right]_0^a$$

$$= \frac{\pi\rho a^4}{2}$$

Instead of computing I_x and I_y directly, we use the facts that $I_x + I_y = I_0$ and $I_x = I_y$ (from the symmetry of the problem).

Thus

$$I_x = I_y = \frac{I_0}{2} = \frac{\pi\rho a^4}{4}$$

Moment of Inertia

In Example 4 notice that the mass of the disk is

$$m = \text{density} \times \text{area} = \rho(\pi a^2)$$

so the moment of inertia of the disk about the origin (like a wheel about its axle) can be written as

$$I_0 = \frac{\pi\rho a^4}{2} = \tfrac{1}{2}(\rho\pi a^2)a^2 = \tfrac{1}{2}ma^2$$

Thus if we increase the mass or the radius of the disk, we thereby increase the moment of inertia.

Moment of Inertia

In general, the moment of inertia plays much the same role in rotational motion that mass plays in linear motion.

The moment of inertia of a wheel is what makes it difficult to start or stop the rotation of the wheel, just as the mass of a car is what makes it difficult to start or stop the motion of the car.

The **radius of gyration of a lamina about an axis** is the number R such that

$$\boxed{9} \qquad mR^2 = I$$

Moment of Inertia

Where m is the mass of the lamina I and is the moment of inertia about the given axis. Equation 9 says that if the mass of the lamina were concentrated at a distance R from the axis, then the moment of inertia of this "point mass" would be the same as the moment of inertia of the lamina.

Moment of Inertia

In particular, the radius of gyration $\bar{\bar{y}}$ with respect to the *x*-axis and the radius of gyration $\bar{\bar{x}}$ with respect to the *y*-axis are given by the equations

$$\boxed{10} \qquad m\bar{\bar{y}}^2 = I_x \qquad m\bar{\bar{x}}^2 = I_y$$

Thus $(\bar{\bar{x}}, \bar{\bar{y}})$ is the point at which the mass of the lamina can be concentrated without changing the moments of inertia with respect to the coordinate axes. (Note the analogy with the center of mass.)

Probability

Probability

We have considered the *probability density function f* of a continuous random variable *X*.

This means that $f(x) \geq 0$ for all x, $\int_{-\infty}^{\infty} f(x)\,dx = 1$, and the probability that *X* lies between *a* and *b* is found by integrating *f* from *a* to *b*:

$$P(a \leqslant X \leqslant b) = \int_a^b f(x)\,dx$$

Probability

Now we consider a pair of continuous random variables *X* and *Y*, such as the lifetimes of two components of a machine or the height and weight of an adult female chosen at random.

The **joint density function** of *X* and *Y* is a function *f* of two variables such that the probability that (*X*, *Y*) lies in a region *D* is

$$P\big((X, Y) \in D\big) = \iint_D f(x, y)\, dA$$

Probability

In particular, if the region is a rectangle, the probability that *X* lies between *a* and *b* and *Y* lies between *c* and *d* is

$$P(a \leqslant X \leqslant b,\ c \leqslant Y \leqslant d) = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

(See Figure 7.)

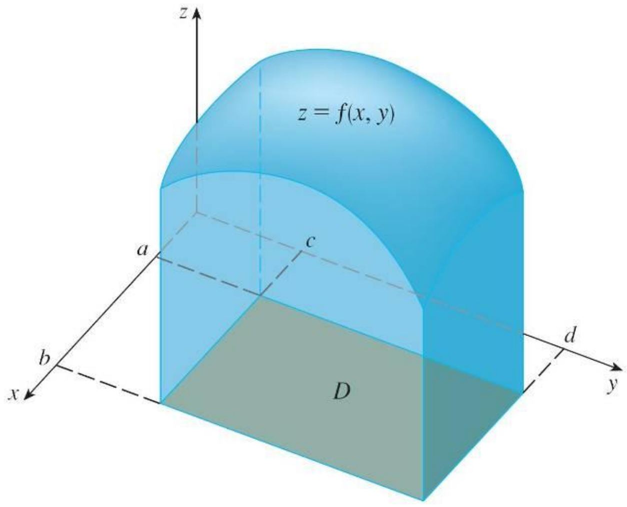


Figure 7

The probability that *X* lies between *a* and *b* and *Y* lies between *c* and *d* is the volume that lies above the rectangle *D* = [*a*, *b*] × [*c*, *d*] and below the graph of the joint density function.

Probability

Because probabilities aren't negative and are measured on a scale from 0 to 1, the joint density function has the following properties:

$$f(x, y) \geq 0 \qquad \iint_{\mathbb{R}^2} f(x, y)\, dA = 1$$

The double integral over $\mathbb{R}^2$ is an improper integral defined as the limit of double integrals over expanding circles or squares, and we can write

$$\iint_{\mathbb{R}^2} f(x, y)\, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$

Example 6

If the joint density function for X and Y is given by

$$f(x, y) = \begin{cases} C(x + 2y) & \text{if } 0 \le x \le 10,\ 0 \le y \le 10 \\ 0 & \text{otherwise} \end{cases}$$

find the value of the constant C. Then find $P(X \le 7, Y \ge 2)$.

Example 6 – *Solution*

We find the value of C by ensuring that the double integral of f is equal to 1. Because $f(x, y) = 0$ outside the rectangle $[0, 10] \times [0, 10]$, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dy\, dx = \int_{0}^{10} \int_{0}^{10} C(x + 2y)\, dy\, dx$$

$$= C \int_{0}^{10} \left[xy + y^2\right]_{y=0}^{y=10} dx$$

$$= C \int_{0}^{10} (10x + 100)\, dx$$

$$= 1500C$$

Therefore $1500C = 1$ and so $C = \frac{1}{1500}$.

Example 6 – *Solution*

cont'd

Now we can compute the probability that *X* is at most 7 and *Y* is at least 2:

$$P(X \leqslant 7, Y \geqslant 2) = \int_{-\infty}^{7} \int_{2}^{\infty} f(x, y)\, dy\, dx$$

$$= \int_{0}^{7} \int_{2}^{10} \frac{1}{1500}(x + 2y)\, dy\, dx$$

$$= \frac{1}{1500} \int_{0}^{7} \left[xy + y^2\right]_{y=2}^{y=10} dx$$

$$= \frac{1}{1500} \int_{0}^{7} (8x + 96)\, dx$$

$$= \frac{868}{1500} \approx 0.5787$$

Probability

Suppose X is a random variable with probability density function $f_1(x)$ and Y is a random variable with density function $f_2(y)$.

Then X and Y are called **independent random variables** if their joint density function is the product of their individual density functions:

$$f(x, y) = f_1(x)f_2(y)$$

We have modeled waiting times by using exponential density functions

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mu^{-1}e^{-t/\mu} & \text{if } t \geq 0 \end{cases}$$

where μ is the mean waiting time.

Expected Values

Expected Values

Recall that if *X* is a random variable with probability density function *f*, then its *mean* is

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

Now if *X* and *Y* are random variables with joint density function *f*, we define the ***X*-mean** and ***Y*-mean**, also called the **expected values** of *X* and *Y*, to be

$$\boxed{11} \qquad \mu_1 = \iint_{\mathbb{R}^2} x f(x, y)\, dA \qquad \mu_2 = \iint_{\mathbb{R}^2} y f(x, y)\, dA$$

Expected Values

Notice how closely the expressions for μ_1 and μ_2 in $\boxed{11}$ resemble the moments M_x and M_y of a lamina with density function ρ in Equations 3 and 4.

In fact, we can think of probability as being like continuously distributed mass. We calculate probability the way we calculate mass—by integrating a density function.

And because the total "probability mass" is 1, the expressions for $\bar{x}$ and $\bar{y}$ in $\boxed{5}$ show that we can think of the expected values of X and Y, μ_1 and μ_2, as the coordinates of the "center of mass" of the probability distribution.

Expected Values

In the next example we deal with normal distributions. A single random variable is *normally distributed* if its probability density function is of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

where μ is the mean and σ is the standard deviation.

Example 8

A factory produces (cylindrically shaped) roller bearings that are sold as having diameter 4.0 cm and length 6.0 cm. In fact, the diameters X are normally distributed with mean 4.0 cm and standard deviation 0.01 cm while the lengths Y are normally distributed with mean 6.0 cm and standard deviation 0.01 cm. Assuming that X and Y are independent, write the joint density function and graph it.

Find the probability that a bearing randomly chosen from the production line has either length or diameter that differs from the mean by more than 0.02 cm.

Example 8 – *Solution*

We are given that X and Y are normally distributed with $\mu_1 = 4.0$, $\mu_2 = 6.0$ and $\sigma_1 = \sigma_2 = 0.01$.

So the individual density functions for X and Y are

$$f_1(x) = \frac{1}{0.01\sqrt{2\pi}}\, e^{-(x-4)^2/0.0002} \qquad f_2(y) = \frac{1}{0.01\sqrt{2\pi}}\, e^{-(y-6)^2/0.0002}$$

Since X and Y are independent, the joint density function is the product:

$$f(x, y) = f_1(x)f_2(y)$$

$$= \frac{1}{0.0002\pi}\, e^{-(x-4)^2/0.0002} e^{-(y-6)^2/0.0002}$$

Example 8 – *Solution*

cont'd

$$= \frac{5000}{\pi} e^{-5000[(x-4)^2+(y-6)^2]}$$

A graph of this function is shown in Figure 9.

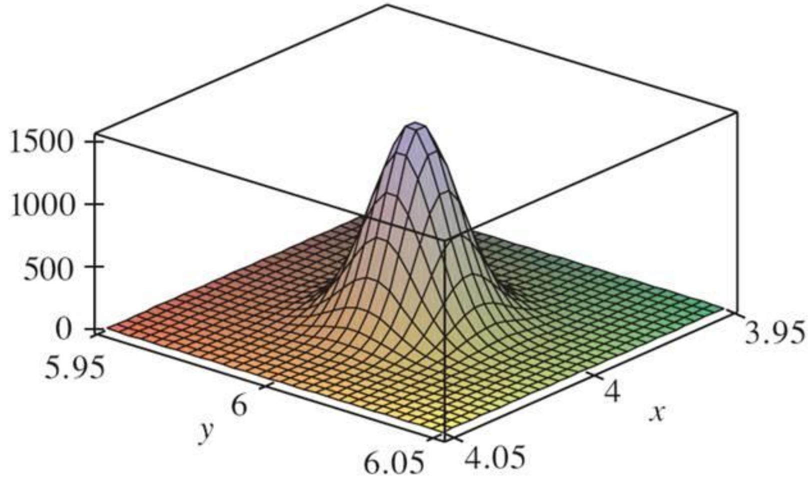


Figure 9

Graph of the bivariate normal joint density function in this example

Example 8 – *Solution*

cont'd

Let's first calculate the probability that both *X* and *Y* differ from their means by less than 0.02 cm. Using a calculator or computer to estimate the integral, we have

$$P(3.98 < X < 4.02,\ 5.98 < Y < 6.02)$$

$$= \int_{3.98}^{4.02} \int_{5.98}^{6.02} f(x, y)\, dy\, dx$$

$$= \frac{5000}{\pi} \int_{3.98}^{4.02} \int_{5.98}^{6.02} e^{-5000[(x-4)^2+(y-6)^2]}\, dy\, dx$$

$$\approx 0.91$$

Example 8 – *Solution*

cont'd

Then the probability that either X or Y differs from its mean by more than 0.02 cm is approximately

$$1 - 0.91 = 0.09$$

15 Multiple Integrals

15.6 Surface Area

Surface Area

In this section we apply double integrals to the problem of computing the area of a surface.

Here we compute the area of a surface with equation $z = f(x, y)$, the graph of a function of two variables.

Let S be a surface with equation $z = f(x, y)$, where f has continuous partial derivatives.

For simplicity in deriving the surface area formula, we assume that $f(x, y) \geq 0$ and the domain D of f is a rectangle.

We divide D into small rectangles R_{ij} with area $\Delta A = \Delta x \Delta y$.

Surface Area

If (x_i, y_j) is the corner of R_{ij} closest to the origin, let $P_{ij}(x_i, y_j, f(x_i, y_j))$ be the point on S directly above it (see Figure 1).

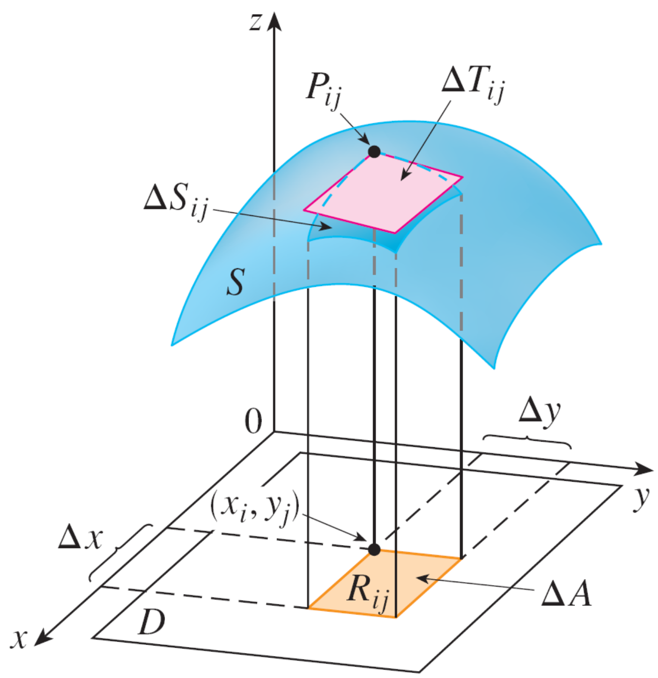


Figure 1

Surface Area

The tangent plane to S at P_{ij} is an approximation to S near P_{ij}. So the area ΔT_{ij} of the part of this tangent plane (a parallelogram) that lies directly above R_{ij} is an approximation to the area ΔS_{ij} of the part of S that lies directly above R_{ij}.

Thus the sum $\Sigma\Sigma\Delta T_{ij}$ is an approximation to the total area of S, and this approximation appears to improve as the number of rectangles increases.

Therefore we define the **surface area** of S to be

1

$$A(S) = \lim_{m,\,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

Surface Area

To find a formula that is more convenient than Equation 1 for computational purposes, we let **a** and **b** be the vectors that start at P_{ij} and lie along the sides of the parallelogram with area ΔT_{ij}. (See Figure 2.)

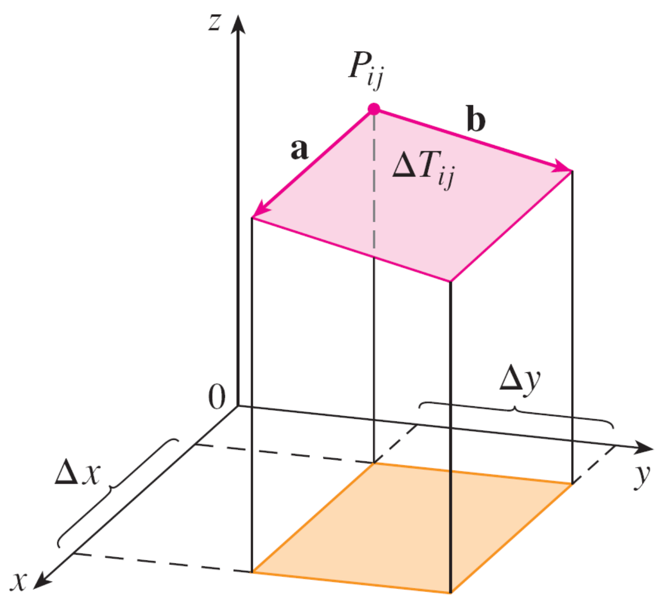


Figure 2

Surface Area

Then $\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|$. Recall that $f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij} in the directions of **a** and **b**.

Therefore

$$\mathbf{a} = \Delta x\mathbf{i} + f_x(x_i, y_j)\,\Delta x\mathbf{k}$$

$$\mathbf{b} = \Delta y\mathbf{j} + f_y(x_i, y_j)\,\Delta y\mathbf{k}$$

and

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_i, y_j)\,\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\,\Delta y \end{vmatrix}$$

Surface Area

$$= -f_x(x_i, y_j)\,\Delta x\,\Delta y\,\mathbf{i} - f_y(x_i, y_j)\,\Delta x\,\Delta y\,\mathbf{j} + \Delta x\,\Delta y\,\mathbf{k}$$

$$= [-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}]\,\Delta A$$

Thus

$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}| = \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\,\Delta A$$

From Definition 1 we then have

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

Surface Area

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\, \Delta A$$

and by the definition of a double integral we get the following formula.

2 The area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous, is

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

Surface Area

If we use the alternative notation for partial derivatives, we can rewrite Formula 2 as follows:

3

$$\boxed{A(s) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA}$$

Notice the similarity between the surface area formula in Equation 3 and the arc length formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example 1

Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vertices (0, 0), (1, 0), and (1, 1).

Solution:

The region T is shown in Figure 3 and is described by

$$T = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}$$

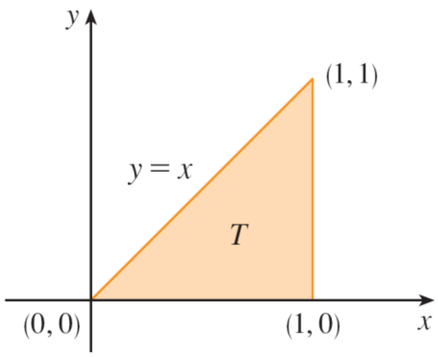


Figure 3

Example 1 – *Solution*

cont'd

Using Formula 2 with $f(x, y) = x^2 + 2y$, we get

$$A = \iint_T \sqrt{(2x)^2 + (2)^2 + 1}\, dA$$

$$= \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dy\, dx$$

$$= \int_0^1 x\sqrt{4x^2 + 5}\, dx$$

$$= \tfrac{1}{8} \cdot \tfrac{2}{3}(4x^2 + 5)^{3/2}\Big]_0^1$$

$$= \tfrac{1}{12}\left(27 - 5\sqrt{5}\right)$$

Example 1 – *Solution*

cont'd

Figure 4 shows the portion of the surface whose area we have just computed.

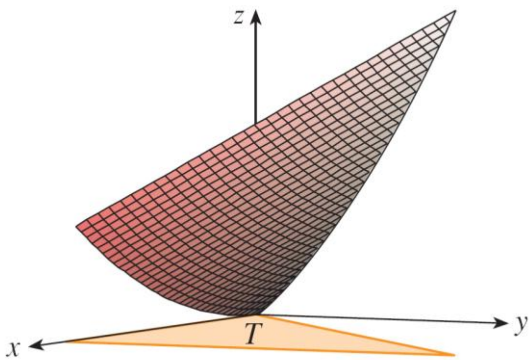


Figure 4

15 Multiple Integrals

15.7 Triple Integrals

Triple Integrals

We have defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables.

Let's first deal with the simplest case where f is defined on a rectangular box:

$$\boxed{1} \qquad B = \{(x, y, z) \mid a \leq x \leq b,\ c \leq y \leq d,\ r \leq z \leq s\}$$

The first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz.

Triple Integrals

The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box *B* into *lmn* sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

which are shown in Figure 1.

Each sub-box has volume

$\Delta V = \Delta x \, \Delta y \, \Delta z$.

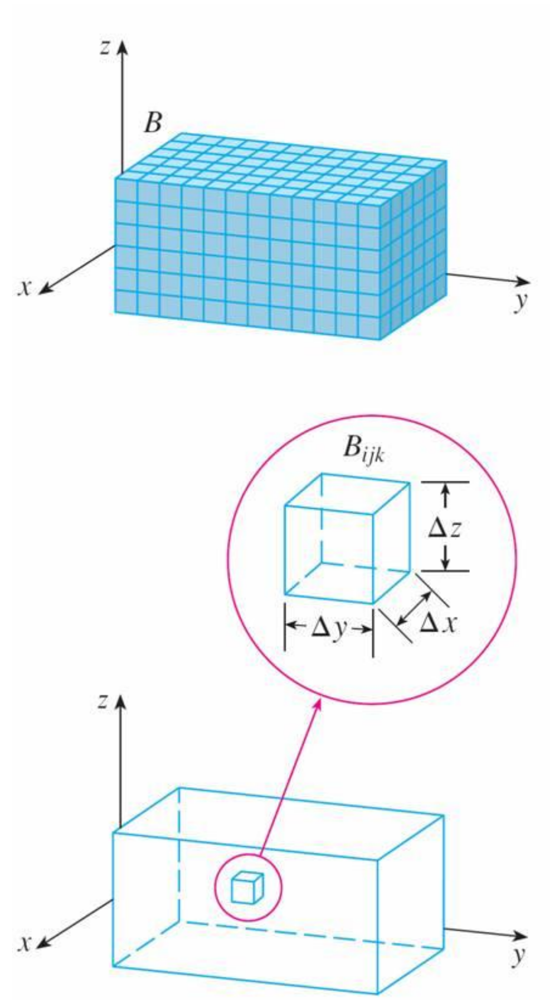


Figure 1

Triple Integrals

Then we form the **triple Riemann sum**

$$\boxed{2} \qquad \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

where the sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in B_{ijk}.

By analogy with the definition of a double integral, we define the triple integral as the limit of the triple Riemann sums in $\boxed{2}$.

Triple Integrals

3 Definition The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if this limit exists.

Again, the triple integral always exists if f is continuous. We can choose the sample point to be any point in the sub-box, but if we choose it to be the point (x_i, y_j, z_k) we get a simpler-looking expression for the triple integral:

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\, \Delta V$$

Triple Integrals

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

4 Fubini's Theorem for Triple Integrals If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally we integrate with respect to z.

Triple Integrals

There are five other possible orders in which we can integrate, all of which give the same value.

For instance, if we integrate with respect to *y*, then *z*, and then *x*, we have

$$\iiint_B f(x, y, z)\, dV = \int_a^b \int_r^s \int_c^d f(x, y, z)\, dy\, dz\, dx$$

Example 1

Evaluate the triple integral $\iiint_B xyz^2\, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$$

Solution:

We could use any of the six possible orders of integration.

If we choose to integrate with respect to x, then y, and then z, we obtain

$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz$$

Example 1 – *Solution*

cont'd

$$= \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2} \right]_{x=0}^{x=1} dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz$$

$$= \int_0^3 \left[\frac{y^2z^2}{4} \right]_{y=-1}^{y=2} dz$$

$$= \int_0^3 \frac{3z^2}{4}\, dz$$

$$= \frac{z^3}{4} \Big]_0^3 = \frac{27}{4}$$

Triple Integrals

Now we define the **triple integral over a general bounded region *E*** in three-dimensional space (a solid) by much the same procedure that we used for double integrals.

We enclose *E* in a box *B* of the type given by Equation 1. Then we define *F* so that it agrees with *f* on *E* but is 0 for points in *B* that are outside *E*.

By definition,

$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV$$

This integral exists if *f* is continuous and the boundary of *E* is "reasonably smooth."

Triple Integrals

The triple integral has essentially the same properties as the double integral.

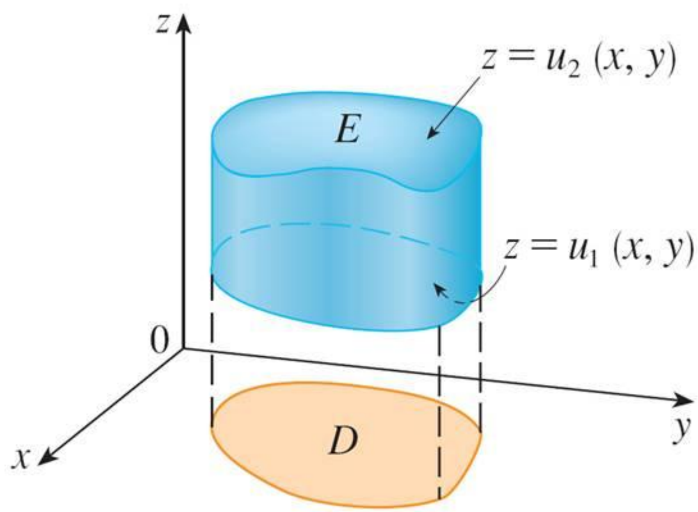
We restrict our attention to continuous functions *f* and to certain simple types of regions.

Triple Integrals

A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$\boxed{5} \qquad E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is the projection of E onto the xy-plane as shown in Figure 2.



A type 1 solid region

Figure 2

Triple Integrals

Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

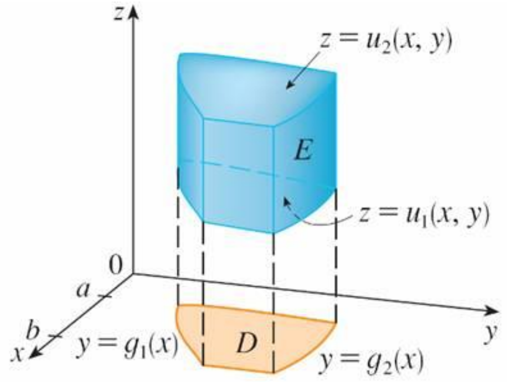
By the same sort of argument, it can be shown that if E is a type 1 region given by Equation 5, then

$$\boxed{6} \qquad \iiint\limits_E f(x, y, z)\, dV = \iint\limits_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

The meaning of the inner integral on the right side of Equation 6 is that x and y are held fixed, and therefore $u_1(x, y)$ and $u_2(x, y)$ are regarded as constants, while $f(x, y, z)$ is integrated with respect to z.

Triple Integrals

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure 3),



A type 1 solid region where the projection D is a type I plane region

Figure 3

Triple Integrals

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure 3), then

$E = \{(x, y, z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}$

and Equation 6 becomes

$$\boxed{7} \qquad \iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx$$

Triple Integrals

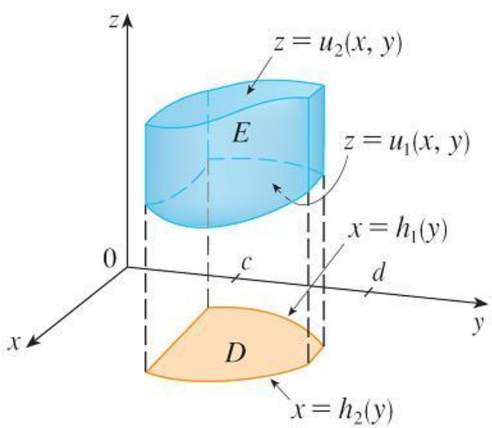
If, on the other hand, D is a type II plane region (as in Figure 4), then

$E = \{(x, y, z) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y), u_1(x, y) \leq z \leq u_2(x, y)\}$

and Equation 6 becomes

8

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy$$



A type 1 solid region with a type II projection

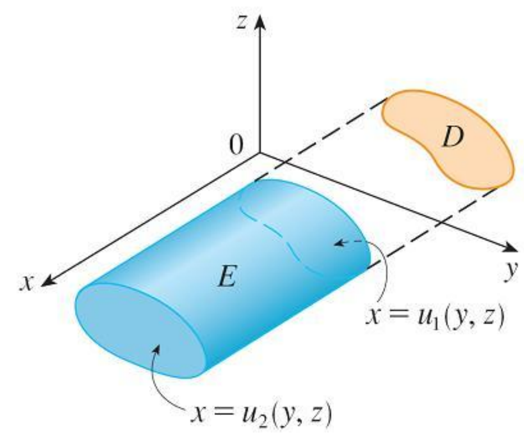
Figure 4

Triple Integrals

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \le x \le u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz-plane (see Figure 7).



A type 2 region

Figure 7

The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

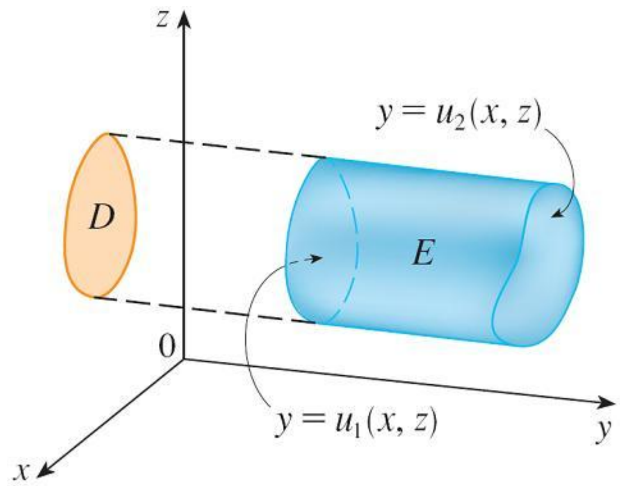
10
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)\, dx \right] dA$$

Triple Integrals

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

where D is the projection of E onto the xz-plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface (see Figure 8).



A type 3 region

Figure 8

Triple Integrals

For this type of region we have

$$\boxed{11} \qquad \iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)\, dy \right] dA$$

In each of Equations 10 and 11 there may be two possible expressions for the integral depending on whether *D* is a type I or type II plane region (and corresponding to Equations 7 and 8).

Applications of Triple Integrals

Applications of Triple Integrals

Recall that if $f(x) \geq 0$, then the single integral $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b, and if $f(x, y) \geq 0$, then the double integral $\iint_D f(x, y)\,dA$ represents the volume under the surface $z = f(x, y)$ and above D.

The corresponding interpretation of a triple integral $\iiint_E f(x, y, z)\,dV$, where $f(x, y, z) \geq 0$, is not very useful because it would be the "hypervolume" of a four-dimensional object and, of course, that is very difficult to visualize. (Remember that E is just the *domain* of the function f; the graph of f lies in four-dimensional space.)

Applications of Triple Integrals

Nonetheless, the triple integral $\iiint_E f(x, y, z)\, dV$ can be interpreted in different ways in different physical situations, depending on the physical interpretations of x, y, z and $f(x, y, z)$.

Let's begin with the special case where $f(x, y, z) = 1$ for all points in E. Then the triple integral does represent the volume of E:

$$V(E) = \iiint_E dV \tag{12}$$

Applications of Triple Integrals

For example, you can see this in the case of a type 1 region by putting $f(x, y, z) = 1$ in Formula 6:

$$\iiint_E 1\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} dz \right] dA = \iint_D [u_2(x, y) - u_1(x, y)]\, dA$$

and we know this represents the volume that lies between the surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$.

Example 5

Use a triple integral to find the volume of the tetrahedron *T* bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Solution:

The tetrahedron *T* and its projection *D* onto the *xy*-plane are shown in Figures 14 and 15.

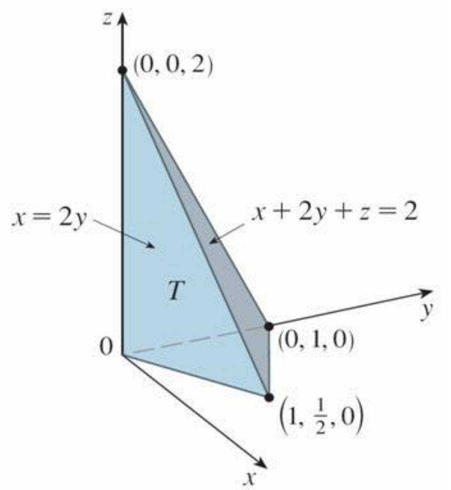


Figure 14

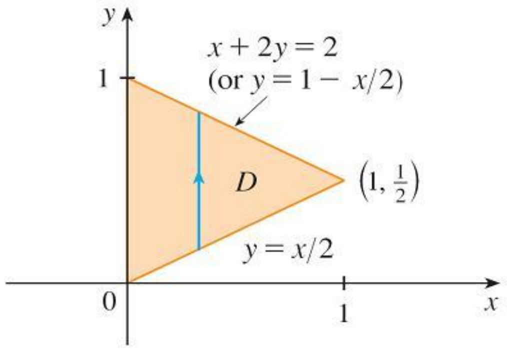


Figure 15

Example 5 – *Solution*

cont'd

The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$.

Therefore we have

$$V(T) = \iiint\limits_T dV$$

$$= \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\, dy\, dx$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx = \tfrac{1}{3}$$

(Notice that it is not necessary to use triple integrals to compute volumes. They simply give an alternative method for setting up the calculation.)

Applications of Triple Integrals

All the applications of double integrals can be immediately extended to triple integrals.

For example, if the density function of a solid object that occupies the region E is $\rho(x, y, z)$, in units of mass per unit volume, at any given point (x, y, z), then its **mass** is

13
$$m = \iiint_E \rho(x, y, z)\, dV$$

and its **moments** about the three coordinate planes are

14
$$M_{yz} = \iiint_E x\,\rho(x, y, z)\, dV \qquad M_{xz} = \iiint_E y\,\rho(x, y, z)\, dV$$

$$M_{xy} = \iiint_E z\,\rho(x, y, z)\, dV$$

Applications of Triple Integrals

The **center of mass** is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

$$\boxed{15} \qquad \bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

If the density is constant, the center of mass of the solid is called the **centroid** of *E*.

The **moments of inertia** about the three coordinate axes are

$$\boxed{16} \qquad I_x = \iiint_E (y^2 + z^2)\,\rho(x, y, z)\,dV \qquad I_y = \iiint_E (x^2 + z^2)\,\rho(x, y, z)\,dV$$

$$I_z = \iiint_E (x^2 + y^2)\,\rho(x, y, z)\,dV$$

Applications of Triple Integrals

The total **electric charge** on a solid object occupying a region E and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z)\, dV$$

If we have three continuous random variables X, Y, and Z, their **joint density function** is a function of three variables such that the probability that (X, Y, Z) lies in E is

$$P\big((X, Y, Z) \in E\big) = \iiint_E f(x, y, z)\, dV$$

Applications of Triple Integrals

In particular,

$$P(a \leqslant X \leqslant b,\ c \leqslant Y \leqslant d,\ r \leqslant Z \leqslant s) = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx$$

The joint density function satisfies

$$f(x, y, z) \geqslant 0 \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\, dz\, dy\, dx = 1$$

15 Multiple Integrals

15.8 Triple Integrals in Cylindrical Coordinates

Triple Integrals in Cylindrical Coordinates

In plane geometry the polar coordinate system is used to give a convenient description of certain curves and regions.

Figure 1 enables us to recall the connection between polar and Cartesian coordinates.

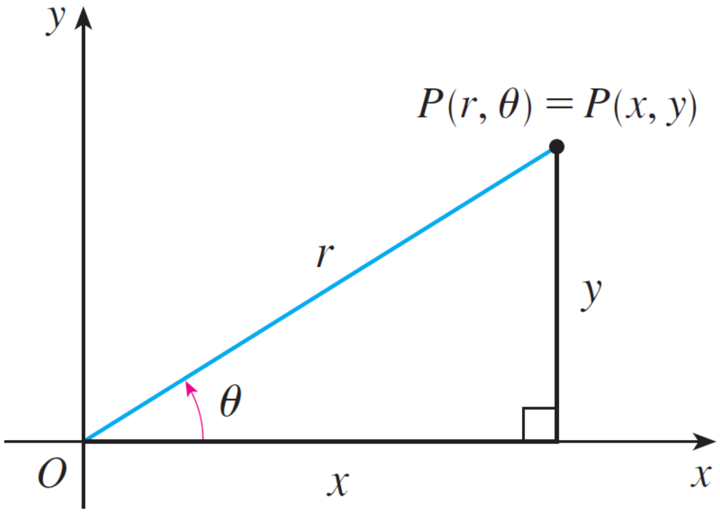


Figure 1

Triple Integrals in Cylindrical Coordinates

If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, from the figure,

$$x = r \cos \theta \qquad y = r \sin \theta$$

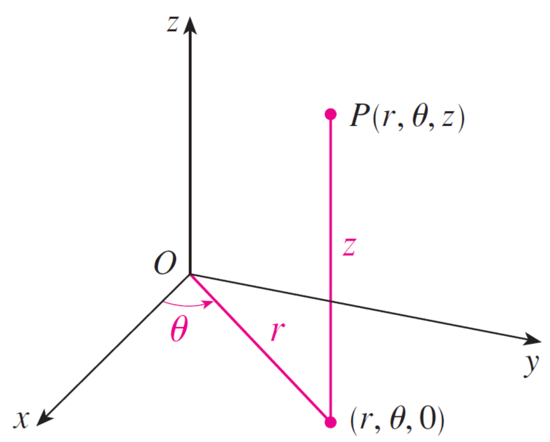
$$r^2 = x^2 + y^2 \qquad \tan\theta \; \frac{y}{x}$$

In three dimensions there is a coordinate system, called *cylindrical coordinates*, that is similar to polar coordinates and gives convenient descriptions of some commonly occurring surfaces and solids. As we will see, some triple integrals are much easier to evaluate in cylindrical coordinates.

Cylindrical Coordinates

Cylindrical Coordinates

In the **cylindrical coordinate system**, a point *P* in three-dimensional space is represented by the ordered triple (r, θ, z) where r and θ are polar coordinates of the projection of *P* onto the *xy*-plane and z is the directed distance from the *xy*-plane to *P*. (See Figure 2.)



The cylindrical coordinates of a point

Figure 2

Cylindrical Coordinates

To convert from cylindrical to rectangular coordinates, we use the equations

1

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

whereas to convert from rectangular to cylindrical coordinates, we use

2

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Example 1

(a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, –3, –7)$.

Solution:

(a) The point with cylindrical coordinates $(2, 2\pi/3, 1)$ is plotted in Figure 3.

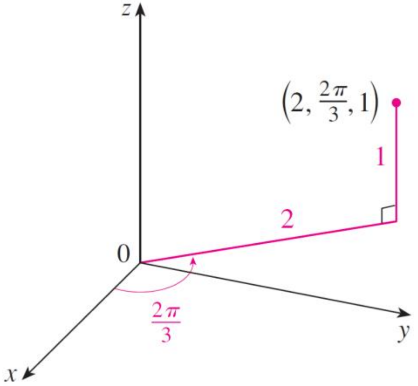


Figure 3

Example 1 – *Solution*

cont'd

From Equations 1, its rectangular coordinates are

$$x = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

$$y = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = 1$$

Thus the point is (–1, $\sqrt{3}$, 1) in rectangular coordinates.

Example 1 – *Solution*

cont'd

(b) From Equations 2 we have

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1 \qquad \text{so} \qquad \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

Therefore one set of cylindrical coordinates is $(3\sqrt{2}, 7\pi/4, -7)$. Another is $(3\sqrt{2}, -\pi/4, -7)$.

As with polar coordinates, there are infinitely many choices.

Evaluating Triple Integrals with Cylindrical Coordinates

Evaluating Triple Integrals with Cylindrical Coordinates

Suppose that E is a type 1 region whose projection D onto the xy-plane is conveniently described in polar coordinates (see Figure 6).

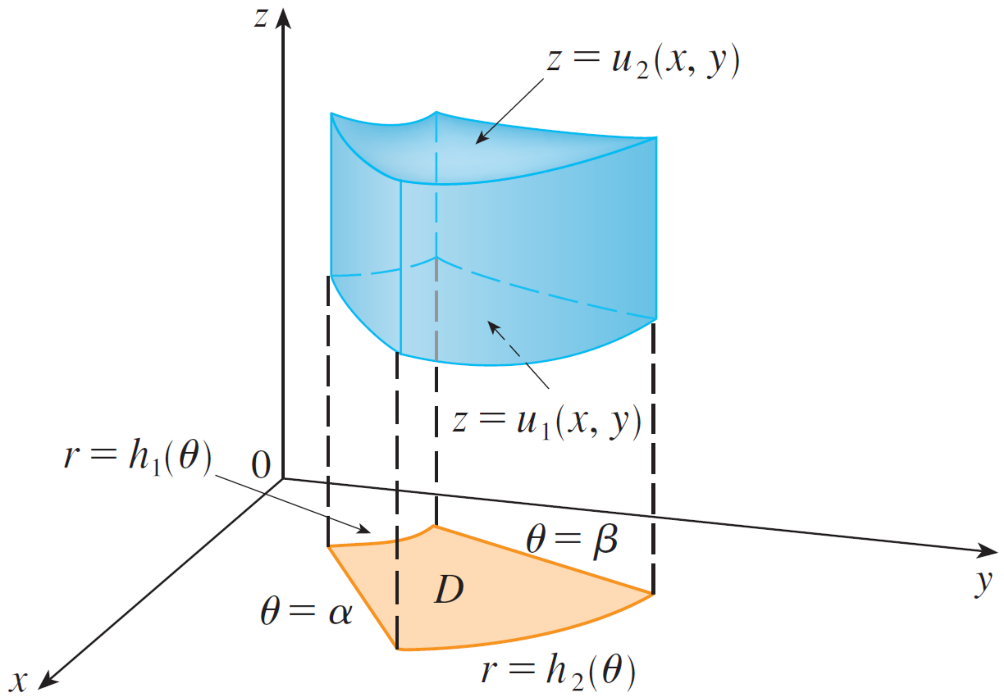


Figure 6

Evaluating Triple Integrals with Cylindrical Coordinates

In particular, suppose that *f* is continuous and

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$$

where *D* is given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

We know

3 $$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

Evaluating Triple Integrals with Cylindrical Coordinates

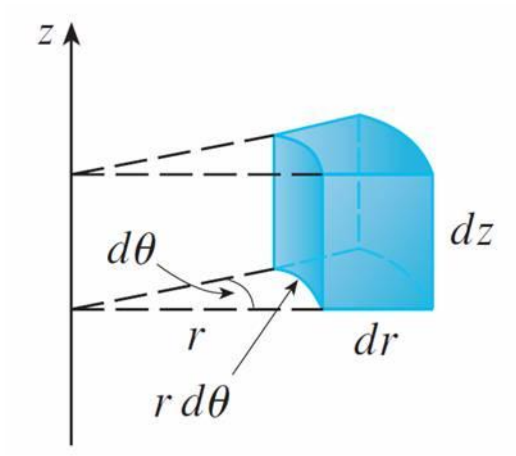
But to evaluate double integrals in polar coordinates, we have the formula

$$\boxed{4} \quad \iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Formula 4 is the **formula for triple integration in cylindrical coordinates.**

Evaluating Triple Integrals with Cylindrical Coordinates

It says that we convert a triple integral from rectangular to cylindrical coordinates by writing $x = r \cos \theta$, $y = r \sin \theta$, leaving z as it is, using the appropriate limits of integration for z, r, and θ, and replacing dV by $r\,dz\,dr\,d\theta$.
(Figure 7 shows how to remember this.)



Volume element in cylindrical coordinates: $dV = r\,dz\,dr\,d\theta$

Figure 7

Evaluating Triple Integrals with Cylindrical Coordinates

It is worthwhile to use this formula when E is a solid region easily described in cylindrical coordinates, and especially when the function $f(x, y, z)$ involves the expression $x^2 + y^2$.

Example 3

A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See Figure 8.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

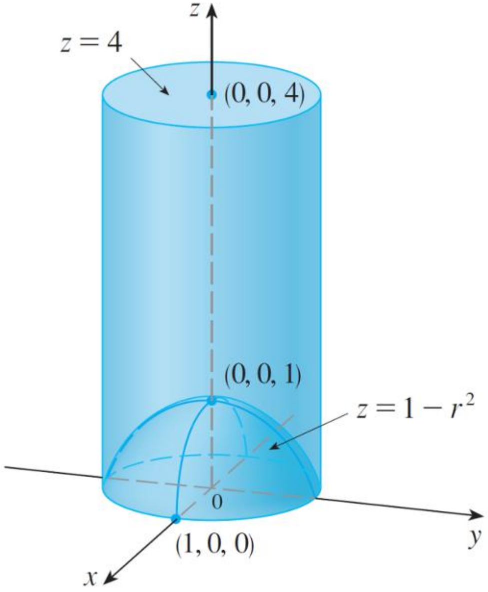


Figure 8

Example 3 – *Solution*

In cylindrical coordinates the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$, so we can write

$$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 1 - r^2 \leq z \leq 4\}$$

Since the density at (x, y, z) is proportional to the distance from the z-axis, the density function is

$$f(x, y, z) = K\sqrt{x^2 + y^2} = Kr$$

where K is the proportionality constant.

Example 3 – *Solution*

cont'd

Therefore, the mass of *E* is

$$m = \iiint_E K\sqrt{x^2 + y^2}\, dV = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (Kr)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 Kr^2[4 - (1 - r^2)]\, dr\, d\theta$$

$$= K \int_0^{2\pi} d\theta \int_0^1 (3r^2 + r^4)\, dr$$

$$= 2\pi K\left[r^3 + \frac{r^5}{5}\right]_0^1 = \frac{12\pi K}{5}$$

15 Multiple Integrals

15.9 Triple Integrals in Spherical Coordinates

Triple Integrals in Spherical Coordinates

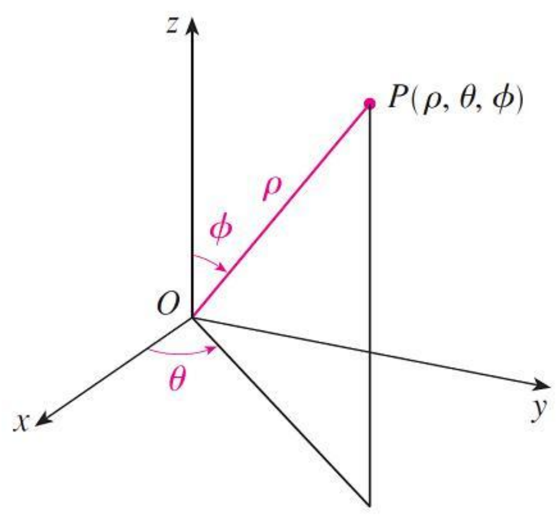
Another useful coordinate system in three dimensions is the *spherical coordinate system*.

It simplifies the evaluation of triple integrals over regions bounded by spheres or cones.

Spherical Coordinates

Spherical Coordinates

The **spherical coordinates** (ρ, θ, ϕ) of a point *P* in space are shown in Figure 1, where $\rho = |OP|$ is the distance from the origin to *P*, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive *z*-axis and the line segment *OP*.



The spherical coordinates of a point

Figure 1

Spherical Coordinates

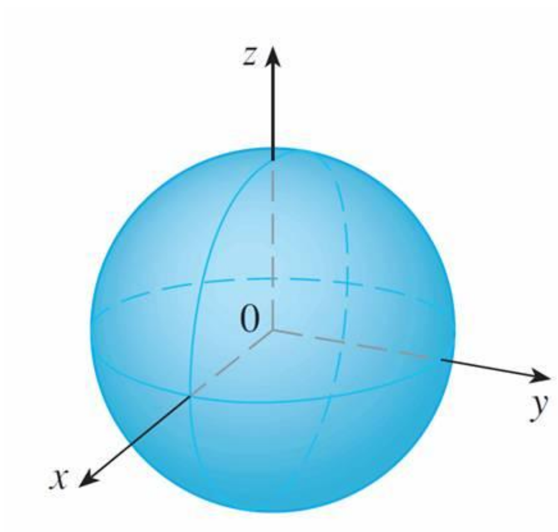
Note that

$$\rho \geq 0 \qquad\qquad 0 \leq \phi \leq \pi$$

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

Spherical Coordinates

For example, the sphere with center the origin and radius c has the simple equation $\rho = c$ (see Figure 2); this is the reason for the name “spherical” coordinates.

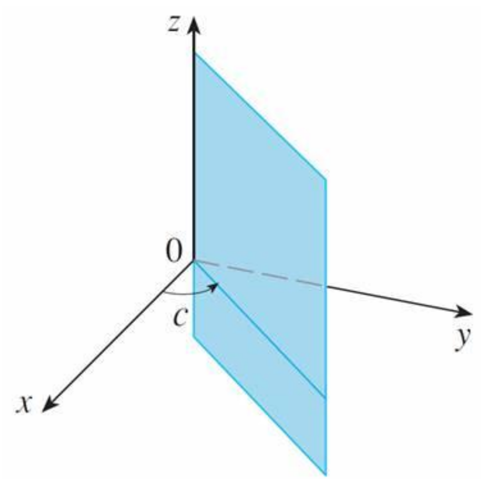


$\rho = c$, a sphere

Figure 2

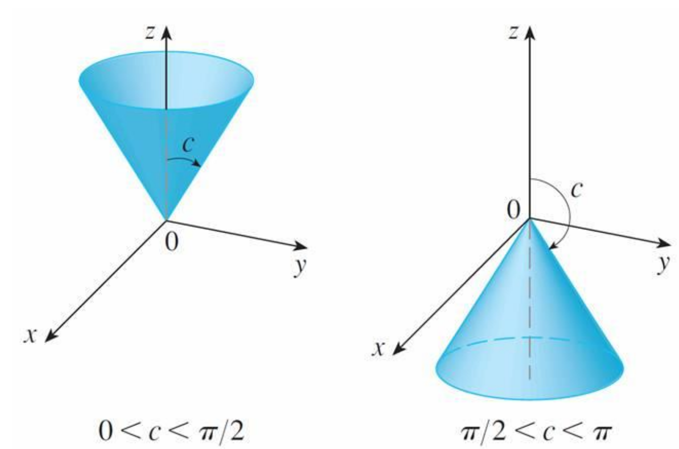
Spherical Coordinates

The graph of the equation $\theta = c$ is a vertical half-plane (see Figure 3), and the equation $\phi = c$ represents a half-cone with the z-axis as its axis (see Figure 4).



$\theta = c$, a half-plane

Figure 3



$\phi = c$, a half-plane

Figure 4

Spherical Coordinates

The relationship between rectangular and spherical coordinates can be seen from Figure 5.

From triangles *OPQ* and *OPP′* we have

$$z = \rho \cos \phi \qquad\qquad r = \rho \sin \phi$$

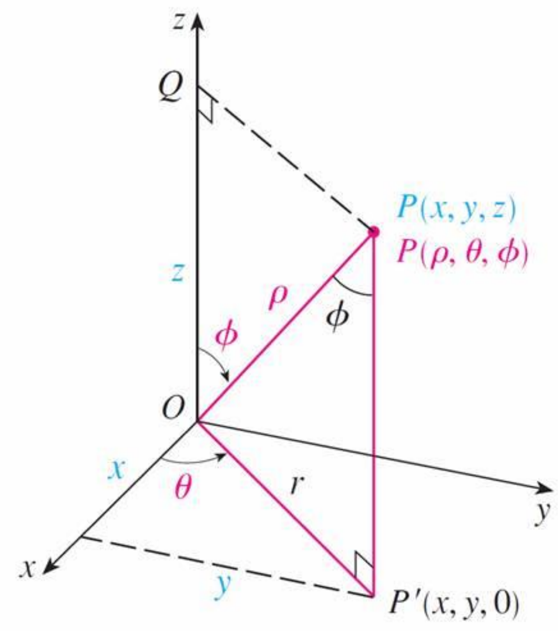


Figure 5

Spherical Coordinates

But $x = r \cos\theta$ and $y = r \sin\theta$, so to convert from spherical to rectangular coordinates, we use the equations

1

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

Also, the distance formula shows that

2

$$\rho^2 = x^2 + y^2 + z^2$$

We use this equation in converting from rectangular to spherical coordinates.

Example 1

The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

Solution:

We plot the point in Figure 6.

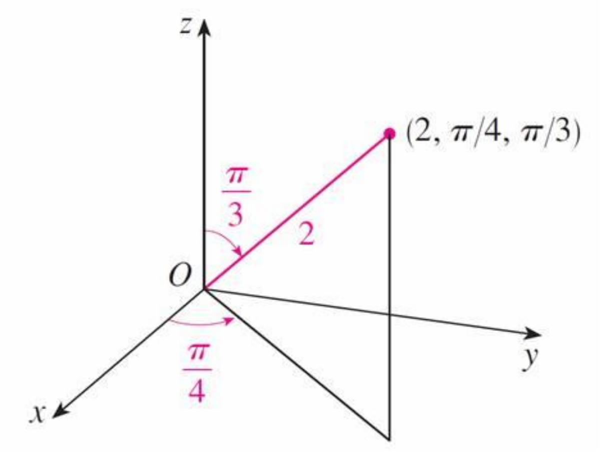


Figure 6

Example 1 – *Solution*

cont'd

From Equations 1 we have

$$x = \rho \sin \phi \cos \theta \; = 2 \sin \frac{\pi}{3} \cos \frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin \phi \sin \theta \; = 2 \sin \frac{\pi}{3} \sin \frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos \phi \; = 2 \cos \frac{\pi}{3} = 2\left(\tfrac{1}{2}\right) = 1$$

Thus the point (2, π/4, π/3) is $\left(\sqrt{3/2}, \sqrt{3/2}, 1\right)$ in rectangular coordinates.

Evaluating Triple Integrals with Spherical Coordinates

Evaluating Triple Integrals with Spherical Coordinates

In the spherical coordinate system the counterpart of a rectangular box is a **spherical wedge**

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

where $a \geq 0$ and $\beta - \alpha \leq 2\pi$, and $d - c \leq \pi$. Although we defined triple integrals by dividing solids into small boxes, it can be shown that dividing a solid into small spherical wedges always gives the same result.

So we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\phi = \phi_k$.

Evaluating Triple Integrals with Spherical Coordinates

Figure 7 shows that E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho$, $\rho_i \, \Delta \, \phi$ (arc of a circle with radius ρ_i, angle $\Delta\phi$), and $\rho_i \sin \phi_k \, \Delta \, \theta$ (arc of a circle with radius $\rho_i \sin \phi_k$, angle $\Delta\theta$).

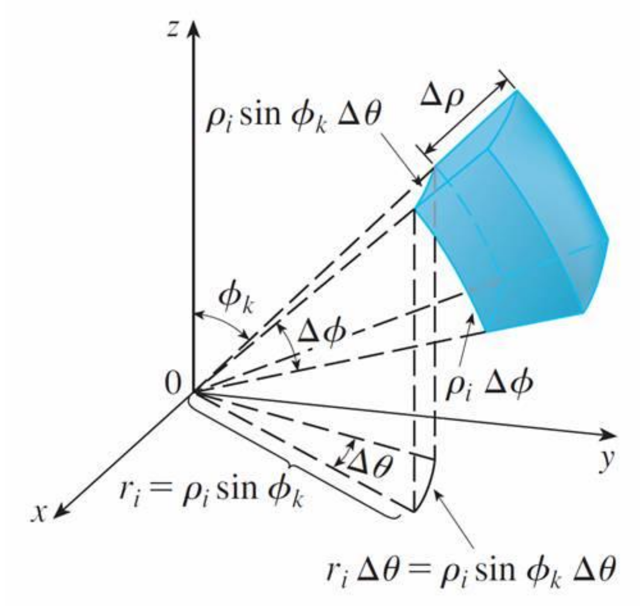


Figure 7

Evaluating Triple Integrals with Spherical Coordinates

So an approximation to the volume of E_{ijk} is given by

$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i \, \Delta\phi)(\rho_i \sin \phi_k \, \Delta\theta) = \rho_i^2 \sin \phi_k \, \Delta\rho \, \Delta\theta \, \Delta\phi$$

In fact, it can be shown, with the aid of the Mean Value Theorem that the volume of E_{ijk} is given exactly by

$$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin \tilde{\phi}_k \, \Delta\rho \, \Delta\theta \, \Delta\phi$$

where $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\phi}_k)$ is some point in E_{ijk}.

Evaluating Triple Integrals with Spherical Coordinates

Let $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ be the rectangular coordinates of this point. Then

$$\iiint_E f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V_{ijk}$$

$$= \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\tilde{\rho}_i \sin \tilde{\phi}_k \cos \tilde{\theta}_j,\ \tilde{\rho}_i \sin \tilde{\phi}_k \sin \tilde{\theta}_j,\ \tilde{\rho}_i \cos \tilde{\phi}_k)\ \tilde{\rho}_i^2 \sin \tilde{\phi}_k\, \Delta\rho\, \Delta\theta\, \Delta\phi$$

Evaluating Triple Integrals with Spherical Coordinates

But this sum is a Riemann sum for the function

$$F(\rho, \theta, \phi) = f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)\, \rho^2 \sin \phi$$

Consequently, we have arrived at the following **formula for triple integration in spherical coordinates**.

3 $$\iiint_E f(x, y, z)\, dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)\, \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$

Evaluating Triple Integrals with Spherical Coordinates

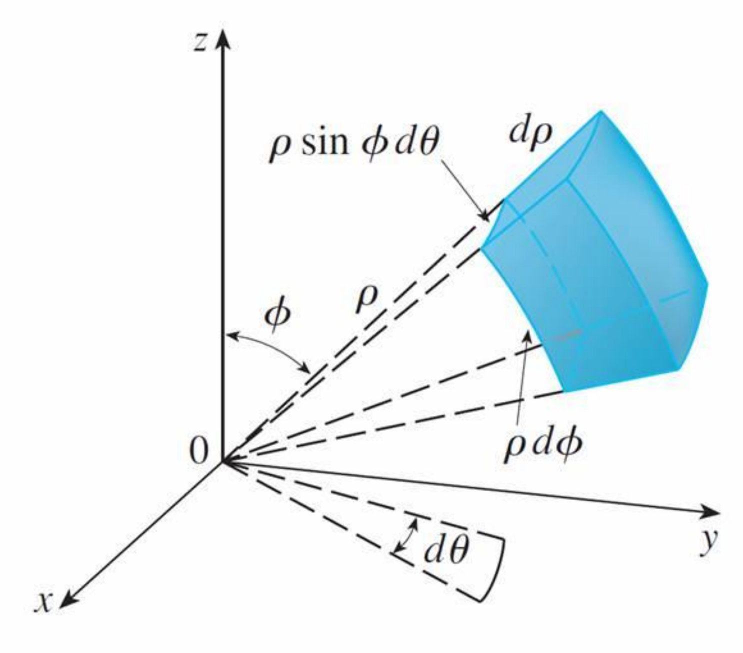
Formula 3 says that we convert a triple integral from rectangular coordinates to spherical coordinates by writing

$$x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi$$

using the appropriate limits of integration, and replacing dv by $\rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$.

Evaluating Triple Integrals with Spherical Coordinates

This is illustrated in Figure 8.



Volume element in spherical coordinates: $dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$

Figure 8

Evaluating Triple Integrals with Spherical Coordinates

This formula can be extended to include more general spherical regions such as

$$E = \{(\rho, \theta, \phi) \mid \alpha \le \theta \le \beta, c \le \phi \le d, g_1(\theta, \phi) \le \rho \le g_2(\theta, \phi)\}$$

In this case the formula is the same as in $\boxed{3}$ except that the limits of integration for ρ are $g_1(\theta, \phi)$ and $g_2(\theta, \phi)$.

Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

Example 4

Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. (See Figure 9.)

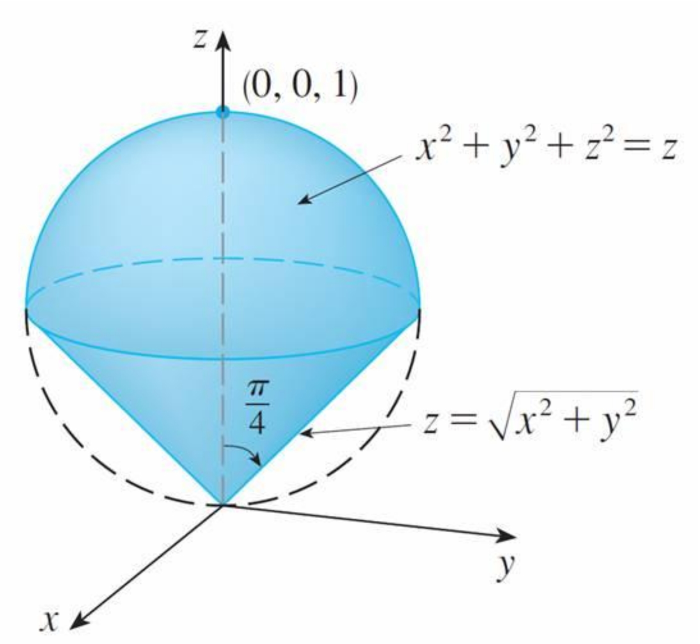


Figure 9

Example 4 – *Solution*

Notice that the sphere passes through the origin and has center $(0, 0, \frac{1}{2})$. We write the equation of the sphere in spherical coordinates as

$$\rho^2 = \rho \cos \phi \qquad \text{or} \qquad \rho = \rho \cos \phi$$

The equation of the cone can be written as

$$\rho \cos \phi = \sqrt{\rho^2 \sin^2\phi \, \cos^2\theta + \rho^2 \sin^2\phi \, \sin^2\theta}$$

$$= \rho \sin \phi$$

Example 4 – *Solution*

cont'd

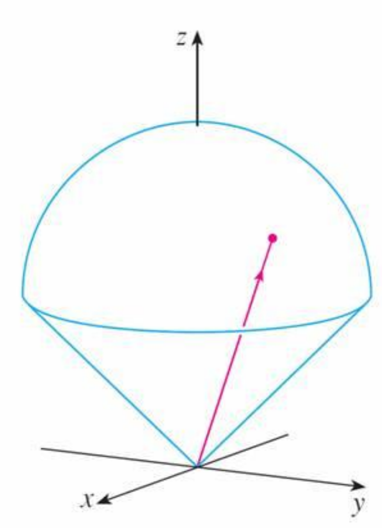
This gives *sin* $\phi = \cos \phi$, or $\phi = \pi/4$. Therefore the description of the solid *E* in spherical coordinates is

$$E = \{(\rho, \theta, \phi) \mid 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/4, 0 \leq \rho \leq \cos \phi\}$$

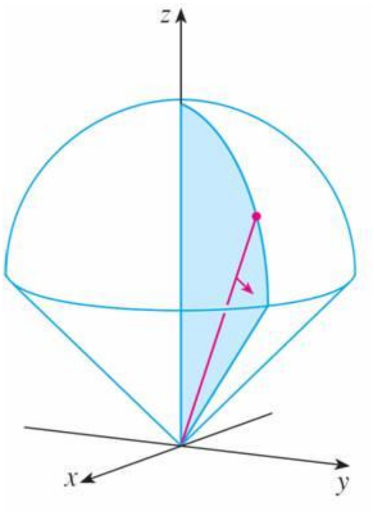
Example 4 – *Solution*

cont'd

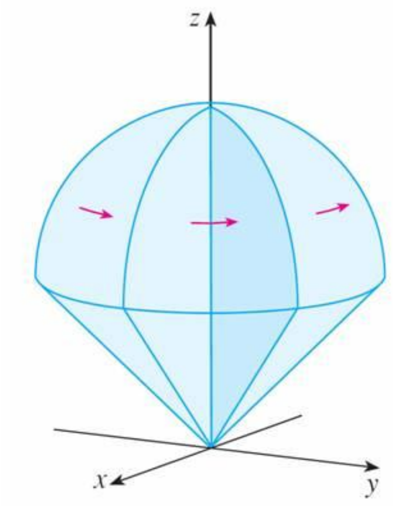
Figure 11 shows how E is swept out if we integrate first with respect to ρ, then ϕ, and then θ.



ρ varies from 0 to cos ϕ while ϕ and θ are constant.



ϕ varies from 0 to $\pi/4$ while θ is constant.



θ varies from 0 to 2π .

Figure 11

Example 4 – *Solution*

cont'd

The volume of *E* is

$$V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\rho=\cos\phi} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin\phi \cos^3\phi \, d\phi = \frac{2\pi}{3} \left[-\frac{\cos^4\phi}{4} \right]_0^{\pi/4} = \frac{\pi}{8}$$