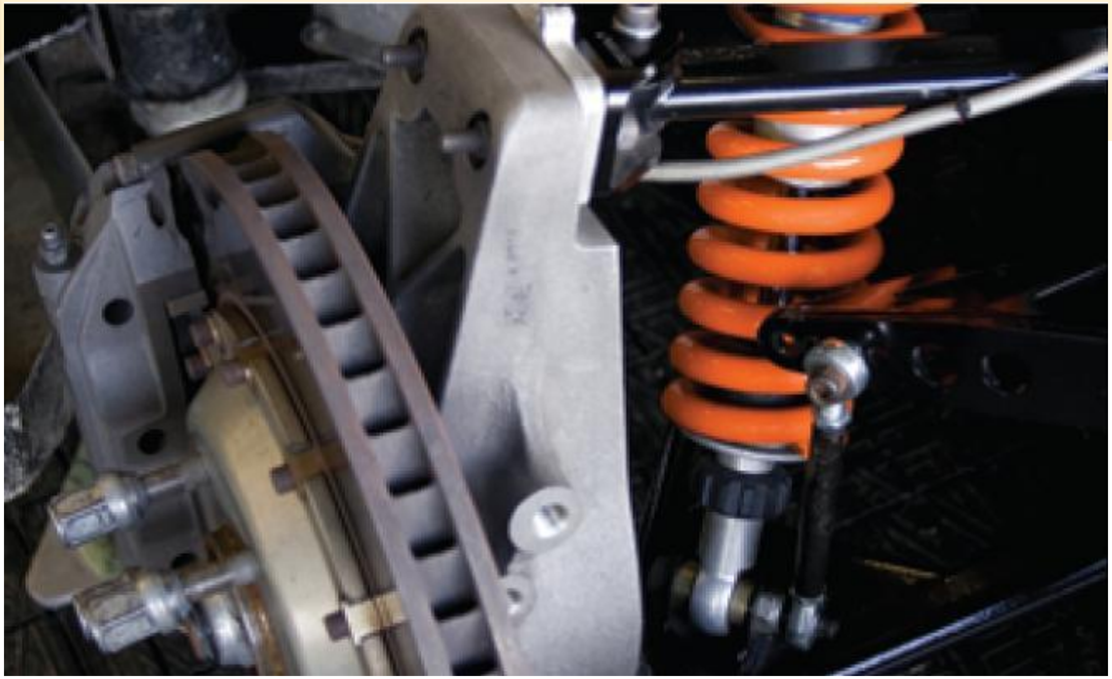
17

Second-Order Differential Equations



17.1 Second-Order Linear Equations

Second-Order Linear Equations

A **second-order linear differential equation** has the form

$$\boxed{1} \qquad P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

where *P*, *Q*, *R*, and *G* are continuous functions.

In this section we study the case where $G(x) = 0$, for all x, in Equation 1.

Such equations are called **homogeneous** linear equations.

Second-Order Linear Equations

Thus the form of a second-order linear homogeneous differential equation is

$$\boxed{2} \qquad P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

If $G(x) \neq 0$ for some x, Equation 1 is **nonhomogeneous**.

Second-Order Linear Equations

Two basic facts enable us to solve homogeneous linear equations.

The first of these says that if we know two solutions y_1 and y_2 of such an equation, then the **linear combination** $y = c_1y_1 + c_2y_2$ is also a solution.

3 Theorem If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation $\boxed{2}$ and c_1 and c_2 are any constants, then the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution of Equation 2.

Second-Order Linear Equations

The other fact we need is given by the following theorem, which is proved in more advanced courses.

It says that the general solution is a linear combination of two **linearly independent** solutions y_1 and y_2.

This means that neither y_1 nor y_2 is a constant multiple of the other.

For instance, the functions $f(x) = x^2$ and $g(x) = 5x^2$ are linearly dependent, but $f(x) = e^x$ and $g(x) = xe^x$ are linearly independent.

Second-Order Linear Equations

4 Theorem If y_1 and y_2 are linearly independent solutions of Equation 2 on an interval, and $P(x)$ is never 0, then the general solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are arbitrary constants.

Theorem 4 is very useful because it says that if we know *two* particular linearly independent solutions, then we know *every* solution.

In general, it's not easy to discover particular solutions to a second-order linear equation.

Second-Order Linear Equations

But it is always possible to do so if the coefficient functions P, Q, and R are constant functions, that is, if the differential equation has the form

5 $$ay'' + by' + cy = 0$$

where a, b, and c are constants and $a \neq 0$.

It's not hard to think of some likely candidates for particular solutions of Equation 5 if we state the equation verbally.

Second-Order Linear Equations

We are looking for a function y such that a constant times its second derivative y'' plus another constant times y' plus a third constant times y is equal to 0.

We know that the exponential function $y = e^{rx}$ (where r is a constant) has the property that its derivative is a constant multiple of itself: $y' = re^{rx}$. Furthermore, $y'' = r^2e^{rx}$. If we substitute these expressions into Equation 5, we see that $y = e^{rx}$ is a solution if

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

or

$$(ar^2 + br + c)e^{rx} = 0$$

Second-Order Linear Equations

But e^{rx} is never 0. Thus $y = e^{rx}$ is a solution of Equation 5 if r is a root of the equation

$$\boxed{6} \qquad ar^2 + br + c = 0$$

Equation 6 is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$.

Notice that it is an algebraic equation that is obtained from the differential equation by replacing y'' by r^2, y' by r, and y by 1.

Second-Order Linear Equations

Sometimes the roots r_1 and r_2 of the auxiliary equation can be found by factoring. In other cases they are found by using the quadratic formula:

$$\boxed{7} \qquad r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We distinguish three cases according to the sign of the discriminant $b^2 - 4ac$.

Second-Order Linear Equations

Case I: $b^2 - 4ac > 0$

In this case the roots r_1 and r_2 of the auxiliary equation are real and distinct, so $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are two linearly independent solutions of Equation 5. (Note that $e^{r_2 x}$ is not a constant multiple of $e^{r_1 x}$.)

Therefore, by Theorem 4, we have the following fact.

8 If the roots r_1 and r_2 of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Example 1

Solve the equation $y'' + y' - 6y = 0$.

Solution:

The auxiliary equation is

$$r^2 + r - 6 = (r - 2)(r + 3) = 0$$

whose roots are $r = 2, -3$.

Therefore, by $\boxed{8}$, the general solution of the given differential equation is

$$y = c_1e^{2x} + c_2e^{-3x}$$

Example 1 – *Solution*

cont'd

We could verify that this is indeed a solution by differentiating and substituting into the differential equation.

Second-Order Linear Equations

Case II: $b^2 - 4ac = 0$

In this case $r_1 = r_2$; that is, the roots of the auxiliary equation are real and equal. Let's denote by r the common value of r_1 and r_2. Then, from Equations 7, we have

$$\boxed{9} \qquad r = -\frac{b}{2a} \qquad \text{so} \qquad 2ar + b = 0$$

We know that $y_1 = e^{rx}$ is one solution of Equation 5. We now verify that $y_2 = xe^{rx}$ is also a solution:

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx} \end{aligned}$$

Second-Order Linear Equations

$$= 0(e^{rx}) + 0(xe^{rx}) = 0$$

The first term is 0 by Equations 9; the second term is 0 because r is a root of the auxiliary equation.

Since $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent solutions, Theorem 4 provides us with the general solution.

$\boxed{10}$ If the auxiliary equation $ar^2 + br + c = 0$ has only one real root r, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1e^{rx} + c_2xe^{rx}$$

Example 3

Solve the equation $4y'' + 12y' + 9y = 0$.

Solution:

The auxiliary equation is $4r^2 + 12r + 9 = 0$ can be factored as

$$(2r + 3)^2 = 0$$

so the only root is $r = -\frac{3}{2}$. By $\boxed{10}$, the general solution is

$$y = c_1 e^{-3x/2} + c_2 x e^{-3x/2}$$

Second-Order Linear Equations

Case III: $b^2 - 4ac < 0$

In this case the roots r_1 and r_2 of the auxiliary equation are complex numbers. We can write

$$r_1 = \alpha + i\beta \qquad\qquad r_2 = \alpha - i\beta$$

Where α and β are real numbers. [In fact, $\alpha = -b/(2a)$, $\beta = \sqrt{4ac - b^2}/(2a)$.] Then, using Euler's equation

$$e^{i\theta} = \cos\theta + i \sin\theta$$

Second-Order Linear Equations

We write the solution of the differential equation as

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

$$= C_1 e^{\alpha x}(\cos \beta x + i \sin \beta x) + C_2 e^{\alpha x}(\cos \beta x - i \sin \beta x)$$

$$= e^{\alpha x}[(C_1 + C_2) \cos \beta x + i(C_1 - C_2) \sin \beta x]$$

$$= e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

Where $c_1 = C_1 + C_2$, $c_2 = i(C_1 - C_2)$.

Second-Order Linear Equations

This gives all solutions (real or complex) of the differential equation. The solutions are real when the constants c_1 and c_2 are real.

We summarize the discussion as follows.

11 If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

Example 4

Solve the equation $y'' - 6y' + 13y = 0$.

Solution:

The auxiliary equation is $r^2 - 6r + 13 = 0$. By the quadratic formula, the roots are

$$r = \frac{6 \pm \sqrt{36 - 52}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2}$$

$$= 3 \pm 2i$$

Example 4 – *Solution*

cont'd

By $\boxed{11}$, the general solution of the differential equation is

$$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

Initial-Value and Boundary-Value Problems

Initial-Value and Boundary-Value Problems

An **initial-value problem** for the second-order Equation 1 or 2 consists of finding a solution y of the differential equation that also satisfies initial conditions of the form

$$y(x_0) = y_0 \qquad\qquad y'(x_0) = y_1$$

Where y_0 and y_1 are given constants.

If P, Q, R, and G are continuous on an interval and $P(x) \neq 0$ there, then a theorem found in more advanced books guarantees the existence and uniqueness of a solution to this initial-value problem. Examples 5 illustrate the technique for solving such a problem.

Example 5

Solve the initial-value problem

$$y'' + y' - 6y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

Solution:

From Example 1 we know that the general solution of the differential equation is

$$y(x) = c_1e^{2x} + c_2e^{-3x}$$

Differentiating this solution, we get

$$y'(x) = 2c_1e^{2x} - 3c_2e^{-3x}$$

Example 5 – *Solution*

cont'd

To satisfy the initial conditions we require that

$\boxed{12}$ $$y(0) = c_1 + c_2 = 1$$

$\boxed{13}$ $$y'(0) = 2c_1 - 3c_2 = 0$$

From $\boxed{13}$, we have $c_2 = \frac{2}{3}c_1$ and so $\boxed{12}$ gives

$$c_1 + \frac{2}{3}c_1 = 1 \qquad c_1 = \frac{3}{5} \qquad c_2 = \frac{2}{5}$$

Example 5 – *Solution*

cont'd

Thus the required solution of the initial-value problem is

$$y = \frac{3}{5} e^{2x} + \frac{2}{5} e^{-3x}$$

Initial-Value and Boundary-Value Problems

A **boundary-value problem** for Equation 1 or 2 consists of finding a solution *y* of the differential equation that also satisfies boundary conditions of the form

$$y(x_0) = y_0 \qquad\qquad y(x_1) = y_1$$

In contrast with the situation for initial-value problems, a boundary-value problem does not always have a solution. The method is illustrated in Example 7.

Example 7

Solve the boundary-value problem

$$y'' + 2y' + y = 0 \qquad y(0) = 1 \qquad y(1) = 3$$

Solution:

The auxiliary equation is

$$r^2 + 2r + 1 = 0 \quad \text{or} \quad (r + 1)^2 = 0$$

whose only root is $r = -1$.

Therefore the general solution is

$$y(x) = c_1e^{-x} + c_2xe^{-x}$$

Example 7 – *Solution*

cont'd

The boundary conditions are satisfied if

$$y(0) = c_1 = 1$$

$$y(1) = c_1e^{-1} + c_2e^{-1} = 3$$

The first condition gives $c_1 = 1$, so the second condition becomes

$$e^{-1} + c_2e^{-1} = 3$$

Example 7 – *Solution*

cont'd

Solving this equation for c_2 by first multiplying through by e, we get

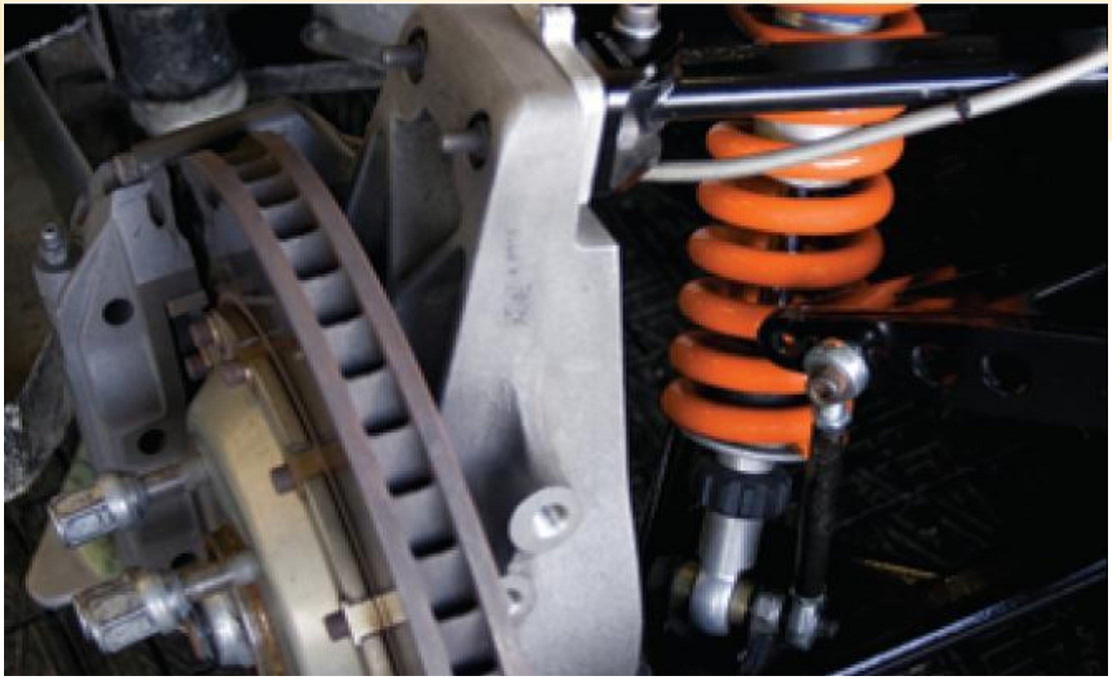
$$1 + c_2 = 3e \qquad \text{so} \qquad c_2 = 3e - 1$$

Thus the solution of the boundary-value problem is

$$y = e^{-x} + (3e - 1)xe^{-x}$$

17

Second-Order Differential Equations



17.2 Nonhomogeneous Linear Equations

Nonhomogeneous Linear Equations

In this section we learn how to solve second-order nonhomogeneous linear differential equations with constant coefficients, that is, equations of the form

$$\boxed{1} \qquad ay'' + by' + cy = G(x)$$

where *a*, *b*, and *c* are constants and *G* is a continuous function. The related homogeneous equation

$$\boxed{2} \qquad ay'' + by' + cy = 0$$

is called the **complementary equation** and plays an important role in the solution of the original nonhomogeneous equation $\boxed{1}$.

Nonhomogeneous Linear Equations

$\boxed{3}$ **Theorem** The general solution of the nonhomogeneous differential equation $\boxed{1}$ can be written as

$$y(x) = y_p(x) + y_c(x)$$

where y_p is a particular solution of Equation 1 and y_c is the general solution of the complementary Equation 2.

There are two methods for finding a particular solution:

The method of undetermined coefficients is straightforward but works only for a restricted class of functions *G*.

The method of variation of parameters works for every function *G* but is usually more difficult to apply in practice.

The Method of Undetermined Coefficients

The Method of Undetermined Coefficients

We first illustrate the method of undetermined coefficients for the equation

$$ay'' + by' + cy = G(x)$$

where $G(x)$ is a polynomial.

It is reasonable to guess that there is a particular solution y_p that is a polynomial of the same degree as G because if y is a polynomial, then $ay'' + by' + cy$ is also a polynomial.

We therefore substitute $y_p(x)$ = a polynomial (of the same degree as G) into the differential equation and determine the coefficients.

Example 1

Solve the equation $y'' + y' - 2y = x^2$.

Solution:

The auxiliary equation of $y'' + y' - 2y = 0$ is

$$r^2 + r - 2 = (r - 1)(r + 2) = 0$$

with roots $r = 1, -2$.

So the solution of the complementary equation is

$$y_c = c_1e^x + c_2e^{-2x}$$

Example 1 – *Solution*

cont'd

Since $G(x) = x^2$ is a polynomial of degree 2, we seek a particular solution of the form

$$y_p(x) = Ax^2 + Bx + C$$

Then $y_p' = 2Ax + B$ and $y_p'' = 2A$ so, substituting into the given differential equation, we have

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

or

$$-2Ax^2 + (2A - 2B)x + (2A + B - 2C) = x^2$$

Polynomials are equal when their coefficients are equal.

Example 1 – *Solution*

cont'd

Thus

$-2A = 1 \qquad 2A - 2B = 0 \qquad 2A + B - 2C = 0$

The solution of this system of equations is

$$A = -\tfrac{1}{2} \qquad B = -\tfrac{1}{2} \qquad C = -\tfrac{3}{4}$$

A particular solution is therefore

$$y_p(x) = -\tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{3}{4}$$

and, by Theorem 3, the general solution is

$$y = y_c + y_p = c_1e^x + c_2e^{-2x} - \tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{3}{4}$$

Example 3

Solve $y'' + y' - 2y = \sin x$.

Solution:

We try a particular solution

$$y_p(x) = A \cos x + B \sin x$$

Then

$$y_p' = -A \sin x + B \cos x$$

$$y_p'' = -A \cos x - B \sin x$$

Example 3 – *Solution*

cont'd

So substitution in the differential equation gives

$$(-A\cos x - B\sin x) + (-A\sin x + B\cos x) - 2(A\cos x + B\sin x) = \sin x$$

or
$$(-3A + B)\cos x + (-A - 3B)\sin x = \sin x$$

This is true if

$$-3A + B = 0 \qquad \text{and} \qquad -A - 3B = 1$$

The solution of this system is

$$A = -\tfrac{1}{10} \qquad B = -\tfrac{3}{10}$$

Example 3 – *Solution*

So a particular solution is

$$y_p(x) = -\tfrac{1}{10}\cos x - \tfrac{3}{10}\sin x$$

In Example 1 we determined that the solution of the complementary equation is

$$y_c = c_1e^x + c_2e^{-2x}.$$

Thus the general solution of the given equation is

$$y(x) = c_1e^x + c_2e^{-2x} - \tfrac{1}{10}(\cos x + 3\sin x)$$

The Method of Undetermined Coefficients

If $G(x)$ is a product of functions of the preceding types, then we take the trial solution to be a product of functions of the same type.

For instance, in solving the differential equation

$$y'' + 2y' + 4y = x \cos 3x$$

we would try

$$y_p(x) = (Ax + B) \cos 3x + (Cx + D) \sin 3x$$

The Method of Undetermined Coefficients

If $G(x)$ is a sum of functions of these types, we use the easily verified *principle of superposition*, which says that if y_{p_1} and y_{p_2} are solutions of

$$ay'' + by' + cy = G_1(x)$$

$$ay'' + by' + cy = G_2(x)$$

respectively, then $y_{p_1} + y_{p_2}$ is a solution of

$$ay'' + by' + cy = G_1(x) + G_2(x)$$

The Method of Undetermined Coefficients

Summary of the Method of Undetermined Coefficients

1. If $G(x) = e^{kx}P(x)$, where P is a polynomial of degree n, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an nth-degree polynomial (whose coefficients are determined by substituting in the differential equation).

2. If $G(x) = e^{kx}P(x)\cos mx$ or $G(x) = e^{kx}P(x)\sin mx$, where P is an nth-degree polynomial, then try

$$y_p(x) = e^{kx}Q(x)\cos mx + e^{kx}R(x)\sin mx$$

where Q and R are nth-degree polynomials.

Modification: If any term of y_p is a solution of the complementary equation, multiply y_p by x (or by x^2 if necessary).

Example 6

Determine the form of the trial solution for the differential equation

$$y'' - 4y + 13y = e^{2x} \cos 3x.$$

Solution:

Here $G(x)$ has the form of part 2 of the summary, where $k = 2$, $m = 3$, and $P(x) = 1$.

So, at first glance, the form of the trial solution would be

$$y_p(x) = e^{2x} (A \cos 3x + B \sin 3x)$$

Example 6 – *Solution*

cont'd

But the auxiliary equation is $r^2 - 4r + 13 = 0$, with roots $r = 2 \pm 3i$, so the solution of the complementary equation is

$$y_c(x) = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$$

This means that we have to multiply the suggested trial solution by x.

So, instead, we use

$$y_p(x) = xe^{2x}(A \cos 3x + B \sin 3x)$$

The Method of Variation of Parameters

The Method of Variation of Parameters

Suppose we have already solved the homogeneous equation $ay'' + by' + cy = 0$ and written the solution as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{4}$$

where y_1 and y_2 are linearly independent solutions.

Let's replace the constants (or parameters) c_1 and c_2 in Equation 4 by arbitrary functions and $u_1(x)$ and $u_2(x)$.

The Method of Variation of Parameters

We look for a particular solution of the nonhomogeneous equation $ay'' + by' + cy = G(x)$ of the form

$$\boxed{5} \qquad y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$$

(This method is called **variation of parameters** because we have varied the parameters c_1 and c_2 to make them functions.)

The Method of Variation of Parameters

Differentiating Equation 5, we get

$$\boxed{6} \qquad y_p' = (u_1' y_1 + u_2' y_2) + (u_1 y_1' + u_2 y_2')$$

Since u_1 and u_2 are arbitrary functions, we can impose two conditions on them.

One condition is that y_p is a solution of the differential equation; we can choose the other condition so as to simplify our calculations.

The Method of Variation of Parameters

In view of the expression in Equation 6, let's impose the condition that

7 $$u_1' y_1 + u_2' y_2 = 0$$

Then

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Substituting in the differential equation, we get

$$a(u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'') + b(u_1 y_1' + u_2 y_2') + c(u_1 y_1 + u_2 y_2) = G$$

or

8 $$u_1(a y_1'' + b y_1' + c y_1) + u_2(a y_2'' + b y_2' + c y_2) + a(u_1' y_1' + u_2' y_2') = G$$

The Method of Variation of Parameters

But y_1 and y_2 are solutions of the complementary equation, so

$$ay_1'' + by_1' + cy_1 = 0 \qquad \text{and} \qquad ay_2'' + by_2' + cy_2 = 0$$

and Equation 8 simplifies to

$$\boxed{9} \qquad a(u_1' y_1' + u_2' y_2') = G$$

Equations 7 and 9 form a system of two equations in the unknown functions u_1' and u_2'.

After solving this system we may be able to integrate to find u_1 and u_2 then the particular solution is given by Equation 5.

Example 7

Solve the equation

$$y'' + y = \tan x,\ 0 < x < \pi/2.$$

Solution:

The auxiliary equation is $r^2 + 1 = 0$ with roots $\pm i$, so the solution of $y'' + y = 0$ is $y(x) = c_1 \sin x + c_2 \cos x$.

Using variation of parameters, we seek a solution of the form

$$y_p(x) = u_1(x) \sin x + u_2(x) \cos x$$

Then

$$y_p' = (u_1' \sin x + u_2' \cos x) + (u_1 \cos x - u_2 \sin x)$$

Example 7 – *Solution*

cont'd

Set

$$\boxed{10} \qquad u_1' \sin x + u_2' \cos x = 0$$

Then

$$y_p'' = u_1' \cos x - u_2' \sin x - u_1 \sin x - u_2 \cos x$$

For y_p to be a solution we must have

$$\boxed{11} \qquad y_p'' + y_p = u_1' \cos x - u_2' \sin x = \tan x$$

Solving Equations 10 and 11, we get

$$u_1'(\sin^2 x + \cos^2 x) = \cos x \tan x$$

Example 7 – *Solution*

cont'd

$u_1' = \sin x \qquad u_1(x) = -\cos x$

(We seek a particular solution, so we don't need a constant of integration here.)

Then, from Equation 10, we obtain

$$u_2' = -\frac{\sin x}{\cos x}\,u_1'$$

$$= -\frac{\sin^2 x}{\cos x}$$

$$= \frac{\cos^2 x - 1}{\cos x}$$

Example 7 – *Solution*

cont'd

$$= \cos x - \sec x$$

So

$$u_2(x) = \sin x - \ln(\sec x + \tan x)$$

(Note that $\sec x + \tan x > 0$ for $0 < x < \pi/2$.)

Therefore

$$y_p(x) = -\cos x \sin x + [\sin x - \ln(\sec x + \tan x)] \cos x$$

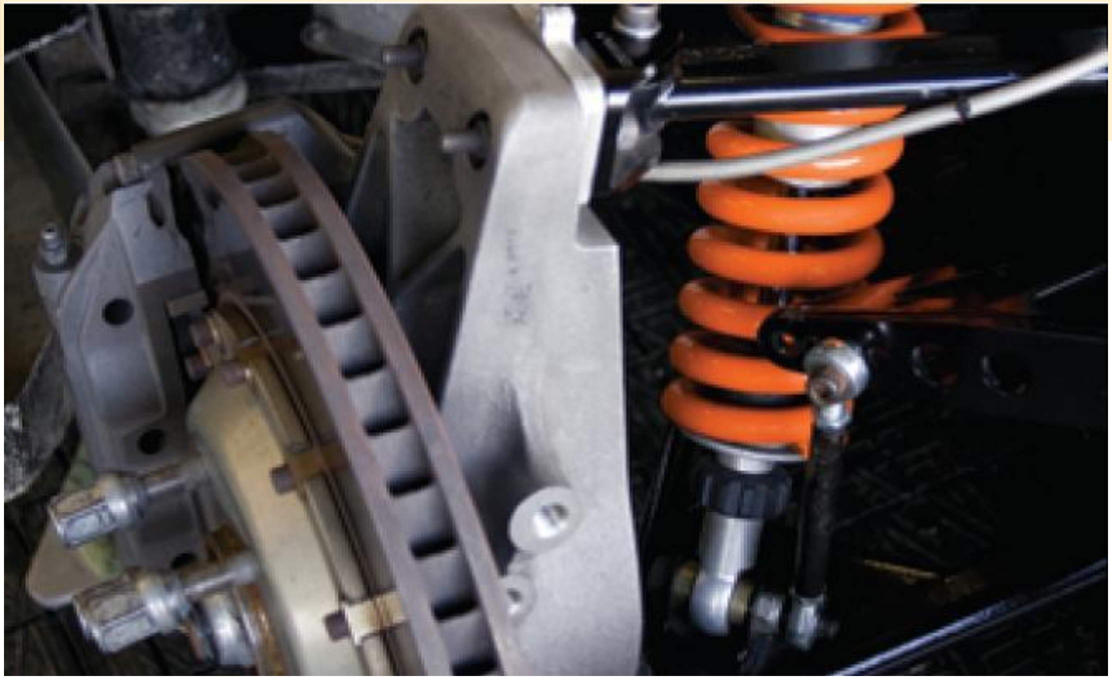
$$= -\cos x \ln(\sec x + \tan x)$$

and the general solution is

$$y(x) = c_1 \sin x + c_2 \cos x - \cos x \ln(\sec x + \tan x)$$

17

Second-Order Differential Equations



17.3 Applications of Second-Order Differential Equations

Applications of Second-Order Differential Equations

Second-order linear differential equations have a variety of applications in science and engineering. In this section we explore two of them: the vibration of springs and electric circuits.

Vibrating Springs

Vibrating Springs

We consider the motion of an object with mass *m* at the end of a spring that is either vertical (as in Figure 1) or horizontal on a level surface (as in Figure 2).

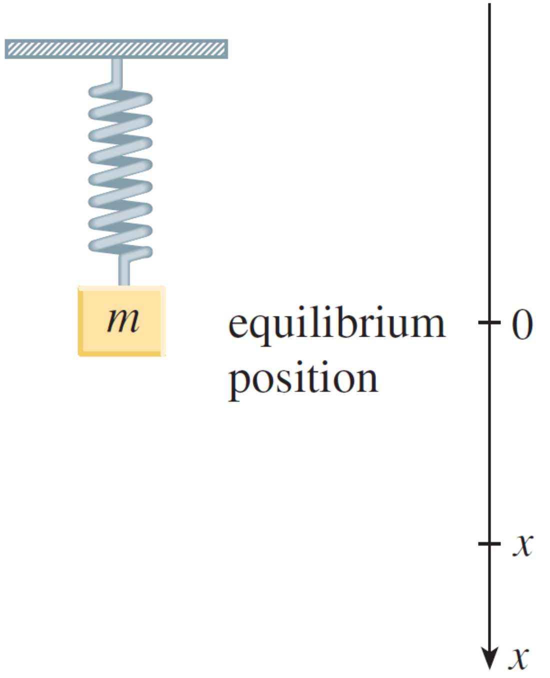


Figure 1

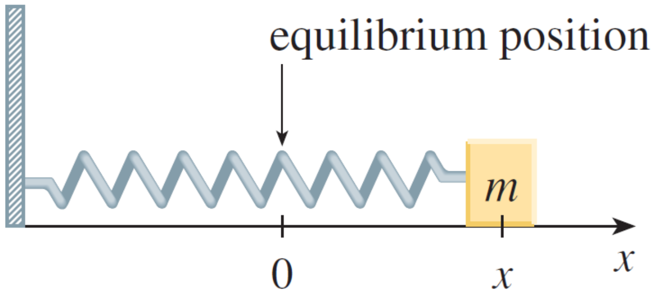


Figure 2

Vibrating Springs

The Hooke's Law, says that if the spring is stretched (or compressed) *x* units from its natural length, then it exerts a force that is proportional to *x*:

restoring force = –*kx*

where *k* is a positive constant (called the **spring constant**). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have

$$\boxed{1} \qquad m\frac{d^2x}{dt^2} = -kx \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + kx = 0$$

Vibrating Springs

This is a second-order linear differential equation. Its auxiliary equation is $mr^2 + k = 0$ with roots $r = \pm\omega i$, where $\omega = \sqrt{k/m}$. Thus the general solution is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

which can also be written as

$$x(t) = A \operatorname{cost}(\omega t + \delta)$$

where

$$\omega = \sqrt{k/m} \qquad \text{(frequency)}$$

$$A = \sqrt{c_1^2 + c_2^2} \qquad \text{(amplitude)}$$

$$\cos \delta = \frac{c_1}{A} \qquad \sin \delta = -\frac{c_2}{A} \qquad (\delta \text{ is the phase angle})$$

This type of motion is called **simple harmonic motion**.

Example 1

A spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. If the spring is stretched to a length of 0.7 m and then released with initial velocity 0, find the position of the mass at any time *t*.

Solution:

From Hooke's Law, the force required to stretch the spring is

$$k(0.2) = 25.6$$

so $k = 25.6/0.2 = 128$.

Example 1 – *Solution*

cont'd

Using this value of the spring constant k, together with $m = 2$ in Equation 1, we have

$$2\frac{d^2x}{dt^2} + 128x = 0$$

As in the earlier general discussion, the solution of this equation is

$$\boxed{2} \qquad x(t) = c_1 \cos 8t + c_2 \sin 8t$$

We are given the initial condition that $x(0) = 0.2$. But, from Equation 2, $x(0) = c_1$.

Example 1 – *Solution*

cont'd

Therefore $c_1 = 0.2$. Differentiating Equation 2, we get

$$x'(t) = -8c_1 \sin 8t + 8c_2 \cos 8t$$

Since the initial velocity is given as $x'(0) = 0$, we have $c_2 = 0$ and so the solution is

$$x(t) = \tfrac{1}{5} \cos 8t$$

Damped Vibrations

Damped Vibrations

We next consider the motion of a spring that is subject to a frictional force (in the case of the horizontal spring or a damping force (in the case where a vertical spring moves through a fluid.

An example is the damping force supplied by a shock absorber in a car or a bicycle.

We assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. (This has been confirmed, at least approximately, by some physical experiments.)

Damped Vibrations

Thus

$$\text{damping force} = -c\frac{dx}{dt}$$

where c is a positive constant, called the **damping constant**. Thus, in this case, Newton's Second Law gives

$$m\frac{d^2x}{dt^2} = \text{restoring force} + \text{damping force} = -kx - c\frac{dx}{dt}$$

or

3

$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0}$$

Damped Vibrations

Equation 3 is a second-order linear differential equation and its auxiliary equation is $mr^2 + cr + k = 0$. The roots are

$$\boxed{4} \qquad r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \qquad r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}$$

Case I: $c^2 - 4mk > 0$ (overdamping)

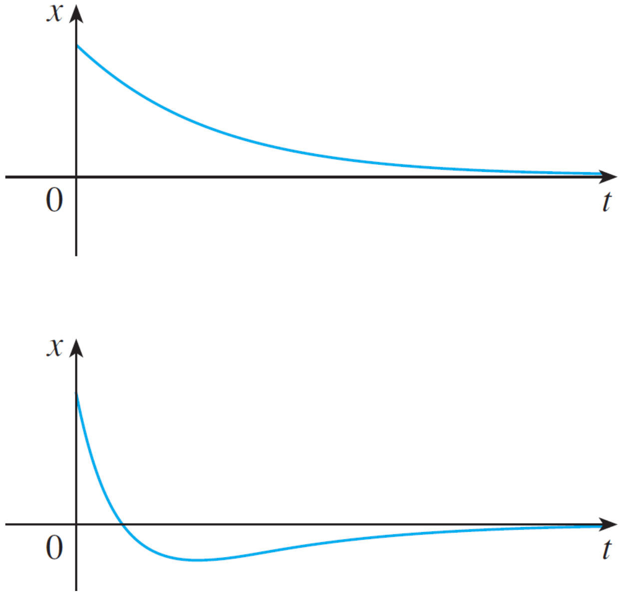
In this case r_1 and r_2 are distinct real roots and

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Since c, m, and k are all positive, we have $\sqrt{c^2 - 4mk} < c$, so the roots r_1 and r_2 given by Equations 4 must both be negative. This shows that $x \to 0$ as $t \to \infty$.

Damped Vibrations

Typical graphs of x as a function of t are shown in Figure 4. Notice that oscillations do not occur. (It's possible for the mass to pass through the equilibrium position once, but only once.)



Overdamping

Figure 4

Damped Vibrations

This is because $c^2 > 4mk$ means that there is a strong damping force (high-viscosity oil or grease) compared with a weak spring or small mass.

Case II: $\boldsymbol{c^2 - 4mk = 0}$ **(critical damping)**

This case corresponds to equal roots

$$r_1 = r_2 = -\frac{c}{2m}$$

and the solution is given by

$$x = (c_1 + c_2 t)e^{-(c/2m)t}$$

Damped Vibrations

It is similar to Case I, and typical graphs resemble those in Figure 4, but the damping is just sufficient to suppress vibrations. Any decrease in the viscosity of the fluid leads to the vibrations of the following case.

Case III: $c^2 - 4mk < 0$ (underdamping)

Here the roots are complex:

$$\left.\begin{array}{l} r_1 \\ r_2 \end{array}\right\} = -\frac{c}{2m} \pm \omega i$$

where

$$\omega = \frac{\sqrt{4mk - c^2}}{2m}$$

Damped Vibrations

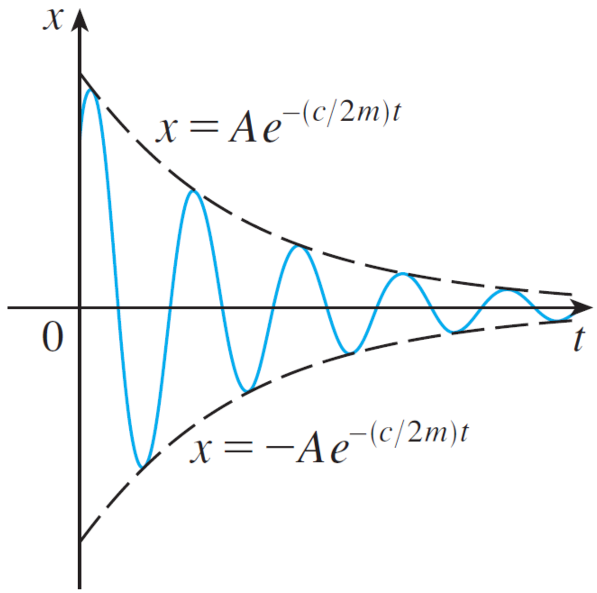
The solution is given by

$$x = e^{-(c/2m)t}(c_1 \cos \omega t + c_2 \sin \omega t)$$

We see that there are oscillations that are damped by the factor $e^{-(c/2m)t}$. Since $c > 0$ and $m > 0$, we have $-(c/2m) < 0$ so $e^{-(c/2m)t} \to 0$ as $t \to \infty$.

Damped Vibrations

This implies that $x \to 0$ as $t \to \infty$; that is, the motion decays to 0 as time increases. A typical graph is shown in Figure 5.



Underdamping

Figure 5

Forced Vibrations

Forced Vibrations

Suppose that, in addition to the restoring force and the damping force, the motion of the spring is affected by an external force $F(t)$. Then Newton's Second Law gives

$$m\frac{d^2x}{dt^2} = \text{restoring force} + \text{damping force} + \text{external force}$$

$$= -kx - c\frac{dx}{dt} + F(t)$$

Thus, instead of the homogeneous equation $\boxed{3}$, the motion of the spring is now governed by the following nonhomogeneous differential equation:

$$\boxed{5} \qquad \boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)}$$

Forced Vibrations

A commonly occurring type of external force is a periodic force function

$$F(t) = F_0 \cos \omega_0 t \quad \text{where} \quad \omega_0 \neq \omega = \sqrt{k/m}$$

In this case, and in the absence of a damping force ($c = 0$), you are asked to use the method of undetermined coefficients to show that

$$\boxed{6} \qquad x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t$$

If $\omega_0 = \omega$, then the applied frequency reinforces the natural frequency and the result is vibrations of large amplitude. This is the phenomenon of **resonance**.

Electric Circuits

Electric Circuits

Earlier we have seen how to use first-order separable and linear equations to analyze electric circuits that contain a resistor and inductor or a resistor and capacitor. Now that we know how to solve second-order linear equations, we are in a position to analyze the circuit shown in Figure 7.

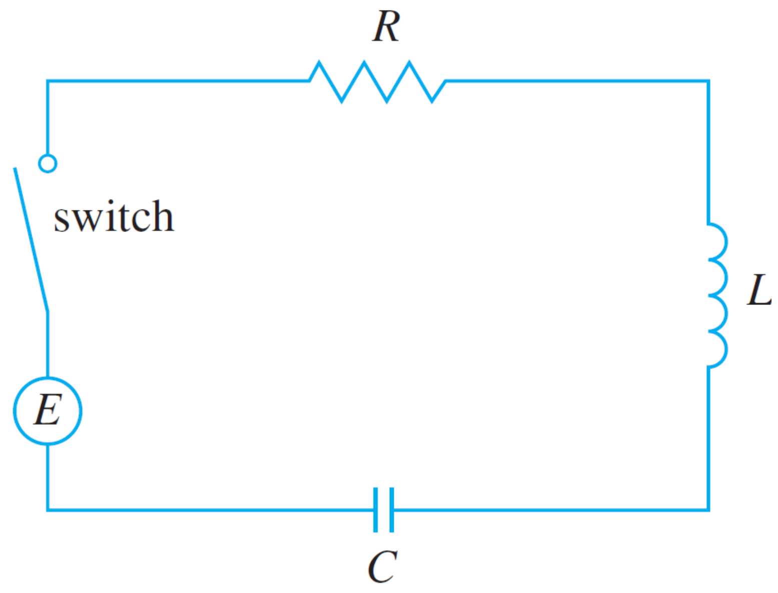


Figure 7

Electric Circuits

It contains an electromotive force E (supplied by a battery or generator), a resistor R, an inductor L, and a capacitor C, in series. If the charge on the capacitor at time t is $Q = Q(t)$, then the current is the rate of change of Q with respect to t: $I = dQ/dt$. It is known from physics that the voltage drops across the resistor, inductor, and capacitor are

$$RI \qquad L\frac{dI}{dt} \qquad \frac{Q}{C}$$

respectively. Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage:

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$$

Electric Circuits

Since $I = dQ/dt$, this equation becomes

$$\boxed{7} \qquad L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

which is a second-order linear differential equation with constant coefficients. If the charge Q_0 and the current I_0 are known at time 0, then we have the initial conditions

$$Q(0) = Q_0 \qquad Q'(0) = I(0) = I_0$$

Electric Circuits

A differential equation for the current can be obtained by differentiating Equation 7 with respect to *t* and remembering that $I = dQ/dt$:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t)$$

Example 3

Find the charge and current at time t in the circuit of Figure 7 if $R = 40\ \Omega$, $L = 1\text{H}$, $C = 16 \times 10^{-4}$ F, $E(t) = 100 \cos 10t$, and the initial charge and current are both 0.

Solution:

With the given values of L, R, C, and $E(t)$, Equation 7becomes

$$\boxed{8} \qquad \frac{d^2Q}{dt^2} + 40\frac{dQ}{dt} + 625Q = 100 \cos 10t$$

Example 3 – *Solution*

cont'd

The auxiliary equation is $r^2 + 40r + 625 = 0$ with roots

$$r = \frac{-40 \pm \sqrt{-900}}{2} = -20 \pm 15i$$

so the solution of the complementary equation is

$$Q_c(t) = e^{-20t}(c_1 \cos 15t + c_2 \sin 15t)$$

For the method of undetermined coefficients we try the particular solution

$$Q_p(t) = A \cos 10t + B \sin 10t$$

Example 3 – *Solution*

cont'd

Then

$$Q_p'(t) = -10A \sin 10t + 10B \cos 10t$$

$$Q_p''(t) = -100A \cos 10t - 100B \sin 10t$$

Substituting into Equation 8, we have

$$(-100A \cos 10t - 100B \sin 10t) + 40(-10A \sin 10t$$
$$+10B \cos 10t) + 625(A \cos 10t + B \sin 10t) = 100 \cos 10t$$

or

$$(525A + 400B) \cos 10t + (-400A + 525B) \sin 10t$$
$$= 100 \cos 10t$$

Example 3 – *Solution*

cont'd

Equating coefficients, we have

$$525A + 400B = 100 \qquad\qquad 21A + 16B = 4$$

or

$$-400A + 525B = 0 \qquad\qquad -16A + 21B = 0$$

Example 3 – *Solution*

cont'd

The solution of this system is $A = \frac{84}{697}$ and $B = \frac{64}{697}$, so a particular solution is

$$Q_p(t) = \frac{1}{697}(84 \cos 10t + 64 \sin 10t)$$

and the general solution is

$$Q(t) = Q_c(t) + Q_p(t)$$

$$= e^{-20t}(c_1 \cos 15t + c_2 \sin 15t) + \frac{4}{697}(21 \cos 10t + 16 \sin 10t)$$

Example 3 – *Solution*

cont'd

Imposing the initial condition $Q(0) = 0$, we get

$$Q(0) = c_1 + \frac{84}{697} = 0 \qquad c_1 = -\frac{84}{697}$$

To impose the other initial condition, we first differentiate to find the current:

$$I = \frac{dQ}{dt} = e^{-20t}[(-20c_1 + 15c_2)\cos 15t + (-15c_1 - 20c_2)\sin 15t]$$

$$+ \frac{40}{697}(-21 \sin 10t + 16 \cos 10t)$$

$$I(0) = -20c_1 + 15c_2 + \frac{640}{697} = 0 \qquad c_2 = -\frac{464}{2091}$$

Example 3 – *Solution*

cont'd

Thus the formula for the charge is

$$Q(t) = \frac{4}{697}\left[\frac{e^{-20t}}{3}(-63\cos 15t - 116\sin 15t) + (21\cos 10t + 16\sin 10t)\right]$$

and the expression for the current is

$$I(t) = \frac{1}{2091}[e^{-20t}(-1920\cos 15t + 13{,}060\sin 15t) + 120(-21\sin 10t + 16\cos 10t)]$$

Electric Circuits

Note 1:
In Example 3 the solution for $Q(t)$ consists of two parts. Since $e^{-20t} \rightarrow 0$ as $t \rightarrow \infty$ and both cos $15t$ and sin $15t$ are bounded functions,

$$Q_c(t) = \frac{4}{2091} e^{-20t} (-63 \cos 15t - 116 \sin 15t) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

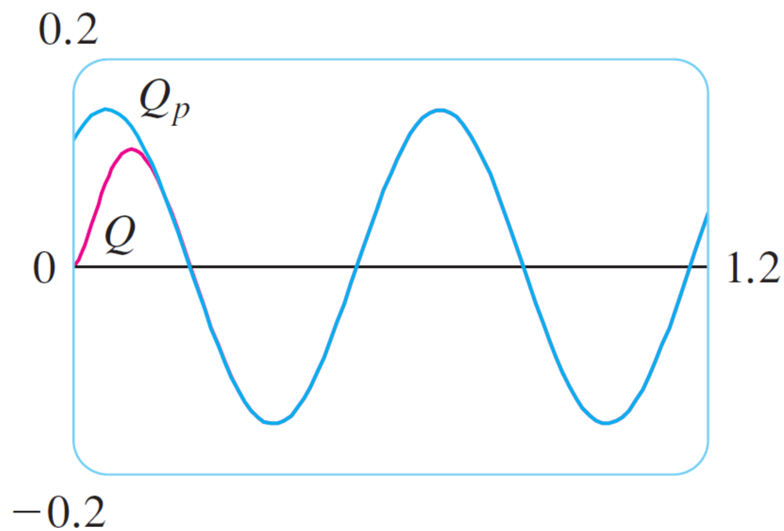
So, for large values of t,

$$Q(t) \approx Q_p(t) = \frac{4}{697} (21 \cos 10t + 16 \sin 10t)$$

and, for this reason, $Q_p(t)$, is called the **steady state solution**.

Electric Circuits

Figure 8 shows how the graph of the steady state solution compares with the graph of *Q* in this case.



$$\boxed{5} \quad m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

$$\boxed{7} \quad L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Figure 8

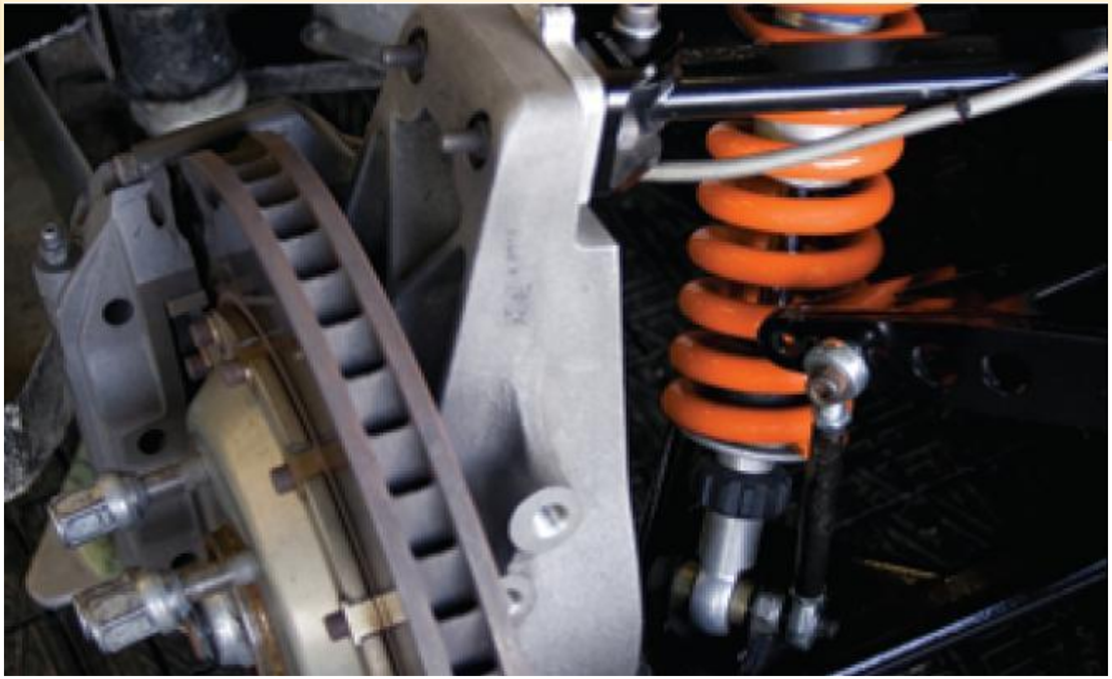
Electric Circuits

Note 2:
Comparing Equations 5 and 7, we see that mathematically they are identical. This suggests the analogies given in the following chart between physical situations that, at first glance, are very different.

Spring system		Electric circuit	
x	displacement	Q	charge
dx/dt	velocity	$I = dQ/dt$	current
m	mass	L	inductance
c	damping constant	R	resistance
k	spring constant	$1/C$	elastance
$F(t)$	external force	$E(t)$	electromotive force

17

Second-Order Differential Equations



17.4 Series Solutions

Series Solutions

Many differential equations can't be solved explicitly in terms of finite combinations of simple familiar functions.

This is true even for a simple-looking equation like

$$y'' - 2xy' + y = 0 \tag{1}$$

But it is important to be able to solve equations such as Equation 1 because they arise from physical problems and, in particular, in connection with the Schrödinger equation in quantum mechanics.

Series Solutions

In such a case we use the method of power series; that is, we look for a solution of the form

$$y = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

The method is to substitute this expression into the differential equation and determine the values of the coefficients $c_0, c_1, c_2, \ldots$. This technique resembles the method of undetermined coefficients.

Series Solutions

Before using power series to solve Equation 1, we illustrate the method on the simpler equation $y'' + y = 0$ in Example1.

But it's easier to understand the power series method when it is applied to this simpler equation.

Example 1

Use power series to solve the equation $y'' + y = 0$.

Solution:

We assume there is a solution of the form

$$\boxed{2} \qquad y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

We can differentiate power series term by term, so

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

Example 1 – *Solution*

cont'd

3 $$y'' = 2c_2 + 2 \cdot 3c_3 x + \cdots = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

In order to compare the expressions for y and y'' more easily, we rewrite y'' as follows:

4 $$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

Substituting the expressions in Equations 2 and 4 into the differential equation, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

Example 1 – *Solution*

cont'd

or

$$\boxed{5} \qquad \sum_{n=0}^{\infty} [(n + 2)(n + 1)c_{n+2} + c_n]x^n = 0$$

If two power series are equal, then the corresponding coefficients must be equal. Therefore the coefficients of x^n in Equation 5 must be 0:

$$(n + 2)(n + 1)c_{n+2} + c_n = 0$$

$$\boxed{6} \qquad c_{n+2} = -\frac{c_n}{(n+1)(n+2)} \qquad n = 0, 1, 2, 3, \ldots$$

Example 1 – *Solution*

cont'd

Equation 6 is called a *recursion relation.* If c_0 and c_1 are known, this equation allows us to determine the remaining coefficients recursively by putting n = 0, 1, 2, 3, . . . in succession.

Put n = 0: $$c_2 = -\frac{c_0}{1 \cdot 2}$$

Put n = 1: $$c_3 = -\frac{c_1}{2 \cdot 3}$$

Put n = 2: $$c_4 = -\frac{c_2}{3 \cdot 4} = \frac{c_0}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{c_0}{4!}$$

Example 1 – *Solution*

cont'd

Put n = 3: $$c_5 = -\frac{c_3}{4 \cdot 5} = \frac{c_1}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{c_1}{5!}$$

Put n = 4: $$c_6 = -\frac{c_4}{5 \cdot 6} = -\frac{c_0}{4!\,5 \cdot 6} = -\frac{c_0}{6!}$$

Put n = 5: $$c_7 = -\frac{c_5}{6 \cdot 7} = -\frac{c_1}{5!\,6 \cdot 7} = -\frac{c_1}{7!}$$

Example 1 – *Solution*

cont'd

By now we see the pattern:

For the even coefficients, $c_{2n} = (-1)^n \dfrac{c_0}{(2n)!}$

For the odd coefficients, $c_{2n+1} = (-1)^n \dfrac{c_1}{(2n+1)!}$

Putting these values back into Equation 2, we write the solution as

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

Example 1 – *Solution*

cont'd

$$= c_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots\right)$$

$$+ c_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right)$$

$$= c_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Notice that there are two arbitrary constants, c_0 and c_1.

Series Solutions

Note 1:
We recognize the series obtained in Example 1 as being the Maclaurin series for cos x and sin x. Therefore we could write the solution as

$$y(x) = c_0 \cos x + c_1 \sin x$$

But we are not usually able to express power series solutions of differential equations in terms of known functions.

Example 2

Solve $y'' - 2xy' + y = 0$.

Solution:

We assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$$

And

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

as in Example 1.

Example 2 – *Solution*

cont'd

Substituting in the differential equation, we get

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - 2x \sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - (2n-1)c_n]x^n = 0$$

Example 2 – *Solution*

cont'd

This equation is true if the coefficient of x^n is 0:

$$(n + 2)(n + 1)c_{n+2} - (2n - 1)c_n = 0$$

$$\boxed{7} \qquad c_{n+2} = \frac{2n - 1}{(n + 1)(n + 2)}\, c_n \qquad n = 0, 1, 2, 3, \ldots$$

We solve this recursion relation by putting $n = 0, 1, 2, 3, \ldots$ successively in Equation 7:

Put $n = 0$: $\quad c_2 = \dfrac{-1}{1 \cdot 2}\, c_0$

Example 2 – *Solution*

cont'd

Put n = 1: $$c_3 = \frac{1}{2 \cdot 3} c_1$$

Put n = 2: $$c_4 = \frac{3}{3 \cdot 4} c_2 = -\frac{3}{1 \cdot 2 \cdot 3 \cdot 4} c_0 = -\frac{3}{4!} c_0$$

Put n = 3: $$c_5 = \frac{5}{4 \cdot 5} c_3 = \frac{1 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 5} c_1 = \frac{1 \cdot 5}{5!} c_1$$

Put n = 4: $$c_6 = \frac{7}{5 \cdot 6} c_4 = -\frac{3 \cdot 7}{4!\, 5 \cdot 6} c_0 = -\frac{3 \cdot 7}{6!} c_0$$

Example 2 – *Solution*

cont'd

Put n = 5: $$c_7 = \frac{9}{6 \cdot 7} c_5 = \frac{1 \cdot 5 \cdot 9}{5!\, 6 \cdot 7} c_1 = \frac{1 \cdot 5 \cdot 9}{7!} c_1$$

Put n = 6: $$c_8 = \frac{11}{7 \cdot 8} c_6 = -\frac{3 \cdot 7 \cdot 11}{8!} c_0$$

Put n = 7: $$c_9 = \frac{13}{8 \cdot 9} c_7 = \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!} c_1$$

Example 2 – *Solution*

cont'd

In general, the even coefficients are given by

$$c_{2n} = -\frac{3 \cdot 7 \cdot 11 \cdot \cdots \cdot (4n-5)}{(2n)!} c_0$$

and the odd coefficients are given by

$$c_{2n+1} = \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} c_1$$

The solution is

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

Example 2 – *Solution*

cont'd

$$= c_0\left(1 - \frac{1}{2!}x^2 - \frac{3}{4!}x^4 - \frac{3 \cdot 7}{6!}x^6 - \frac{3 \cdot 7 \cdot 11}{8!}x^8 - \cdots\right)$$

$$+ c_1\left(x + \frac{1}{3!}x^3 + \frac{1 \cdot 5}{5!}x^5 + \frac{1 \cdot 5 \cdot 9}{7!}x^7 + \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!}x^9 + \cdots\right)$$

or

$$\boxed{8} \qquad y = c_0\left(1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n-5)}{(2n)!}x^{2n}\right)$$

$$+ c_1\left(x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!}x^{2n+1}\right)$$

Series Solutions

Note 2:
In Example 2 we had to *assume* that the differential equation had a series solution. But now we could verify directly that the function given by Equation 8 is indeed a solution.

Note 3:
Unlike the situation of Example 1, the power series that arise in the solution of Example 2 do not define elementary functions.

Series Solutions

The functions

$$y_1(x) = 1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n - 5)}{(2n)!} x^{2n}$$

and

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n - 3)}{(2n + 1)!} x^{2n+1}$$

are perfectly good functions but they can't be expressed in terms of familiar functions.

We can use these power series expressions for y_1 and y_2 to compute approximate values of the functions and even to graph them.

Series Solutions

Figure 1 shows the first few partial sums T_0, T_2, T_4, . . . (Taylor polynomials) for $y_1(x)$, and we see how they converge to y_1.

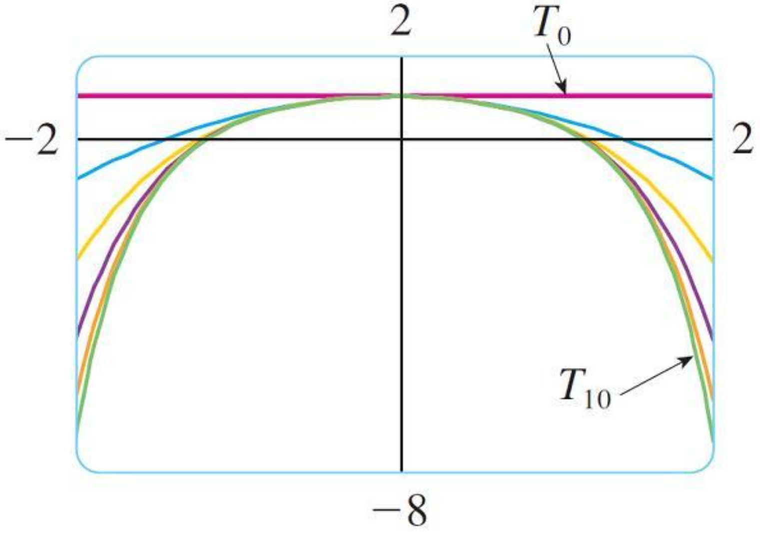


Figure 1

Series Solutions

In this way we can graph both y_1 and y_2 in Figure 2.

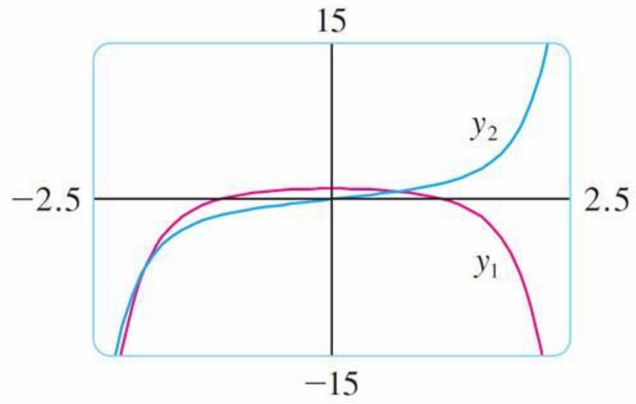


Figure 2

Note 4:

If we were asked to solve the initial-value problem

$$y'' - 2xy' + y = 0 \qquad y(0) = 1 \qquad y'(0) = 1$$

Series Solutions

we would observe from this Theorem that

$$c_0 = y(0) = 0 \qquad\qquad c_1 = y'(0) = 1$$

This would simplify the calculations in Example 2, since all of the even coefficients would be 0.

Series Solutions

The solution to the initial-value problem is

$$y(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} x^{2n+1}$$