9

Differential Equations

9.1

Modeling with Differential Equations

Modeling with Differential Equations

The mathematical model often takes the form of a *differential equation*, that is, an equation that contains an unknown function and some of its derivatives.

This is not surprising because in a real-world problem we often notice that changes occur and we want to predict future behavior on the basis of how current values change.

Let’s begin by examining several examples of how differential equations arise when we model physical phenomena.

Models of Population Growth

Models of Population Growth

One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population.

That is a reasonable assumption for a population of bacteria or animals under ideal conditions (unlimited environment, adequate nutrition, absence of predators, immunity from disease).

Let's identify and name the variables in this model:

t = time (the independent variable)

P = the number of individuals in the population (the dependent variable)

Models of Population Growth

The rate of growth of the population is the derivative *dP*/*dt*. So our assumption that the rate of growth of the population is proportional to the population size is written as the equation

$$\boxed{1} \qquad \frac{dP}{dt} = kP$$

where *k* is the proportionality constant. Equation 1 is our first model for population growth; it is a differential equation because it contains an unknown function *P* and its derivative *dP*/*dt*.

Models of Population Growth

Having formulated a model, let’s look at its consequences. If we rule out a population of 0, then $P(t) > 0$ for all t. So, if $k > 0$, then Equation 1 shows that $P'(t) > 0$ for all t.

This means that the population is always increasing. In fact, as $P(t)$ increases, Equation 1 shows that dP/dt becomes larger.

In other words, the growth rate increases as the population increases.

Let’s try to think of a solution of Equation 1. This equation asks us to find a function whose derivative is a constant multiple of itself.

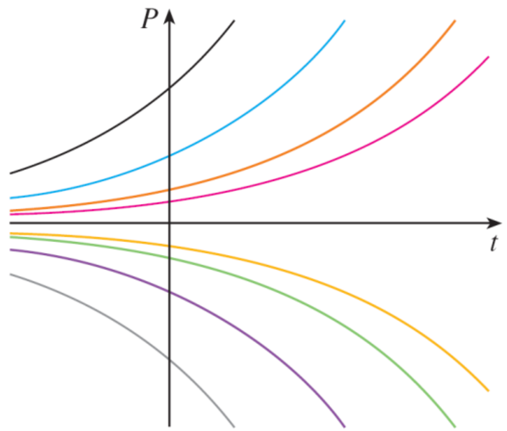
Models of Population Growth

We know that exponential functions have that property. In fact, if we let $P(t) = Ce^{kt}$, then

$$P'(t) = C(ke^{kt}) = k(Ce^{kt}) = kP(t)$$

Thus any exponential function of the form $P(t) = Ce^{kt}$ is a solution of Equation 1.

Allowing C to vary through all the real numbers, we get the *family* of solutions $P(t) = Ce^{kt}$ whose graphs are shown in Figure 1.

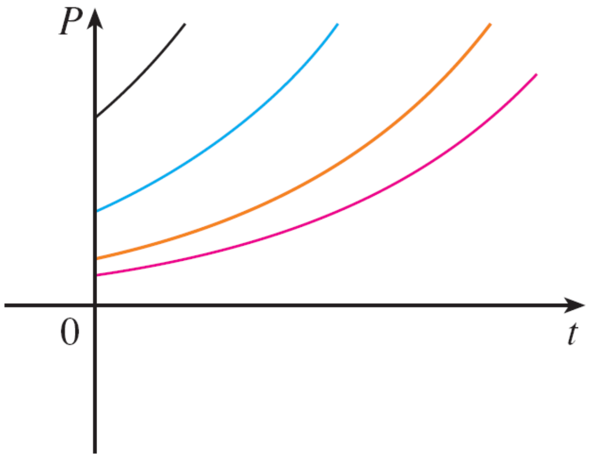


The family of solutions of $dP/dt = kP$

Figure 1

Models of Population Growth

But populations have only positive values and so we are interested only in the solutions with $C > 0$. And we are probably concerned only with values of t greater than the initial time $t = 0$. Figure 2 shows the physically meaningful solutions.



The family of solutions of $P(t) = Ce^{kt}$ with $C > 0$ and $t \geq 0$

Figure 2

Models of Population Growth

Putting $t = 0$, we get $P(0) = Ce^{k(0)} = C$, so the constant C turns out to be the initial population, $P(0)$.

Equation 1 is appropriate for modeling population growth under ideal conditions, but we have to recognize that a more realistic model must reflect the fact that a given environment has limited resources.

Many populations start by increasing in an exponential manner, but the population levels off when it approaches its *carrying capacity* M (or decreases toward M if it ever exceeds M).

Models of Population Growth

For a model to take into account both trends, we make two assumptions:

- $\dfrac{dP}{dt} \approx kP$ if *P* is small (Initially, the growth rate is proportional to *P*.)

- $\dfrac{dP}{dt} < 0$ if *P* > *M* (*P* decreases if it ever exceeds *M*.)

A simple expression that incorporates both assumptions is given by the equation

$$\boxed{2} \qquad \frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Models of Population Growth

Notice that if P is small compared with M, then P/M is close to 0 and so $dP/dt \approx kP$. If $P > M$, then $1 - P/M$ is negative and so $dP/dt < 0$.

Equation 2 is called the *logistic differential equation* and was proposed by the Dutch mathematical biologist Pierre-François Verhulst in the 1840s as a model for world population growth.

We first observe that the constant functions $P(t) = 0$ and $P(t) = M$ are solutions because, in either case, one of the factors on the right side of Equation 2 is zero. These two constant solutions are called *equilibrium solutions*.

Models of Population Growth

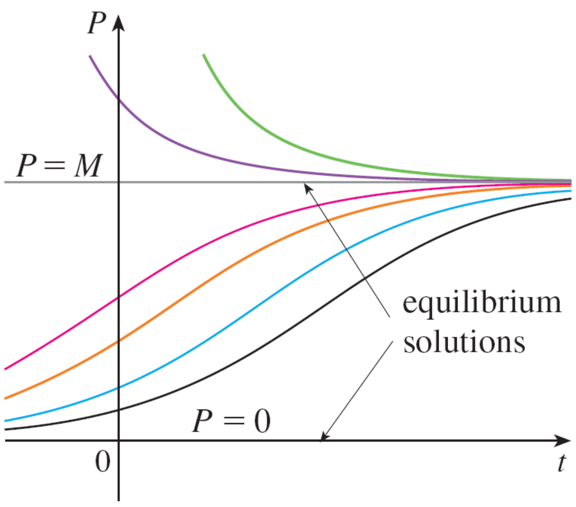
If the initial population $P(0)$ lies between 0 and M, then the right side of Equation 2 is positive, so $dP/dt > 0$ and the population increases. But if the population exceeds the carrying capacity ($P > M$), then $1 - P/M$ is negative, so $dP/dt < 0$ and the population decreases.

Notice that, in either case, if the population approaches the carrying capacity ($P \to M$), then $dP/dt \to 0$, which means the population levels off.

Models of Population Growth

So we expect that the solutions of the logistic differential equation have graphs that look something like the ones in Figure 3.

Notice that the graphs move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = M$.



Solutions of the logistic equation

Figure 3

A Model for the Motion of a Spring

A Model for the Motion of a Spring

Let’s now look at an example of a model from the physical sciences. We consider the motion of an object with mass *m* at the end of a vertical spring (as in Figure 4).

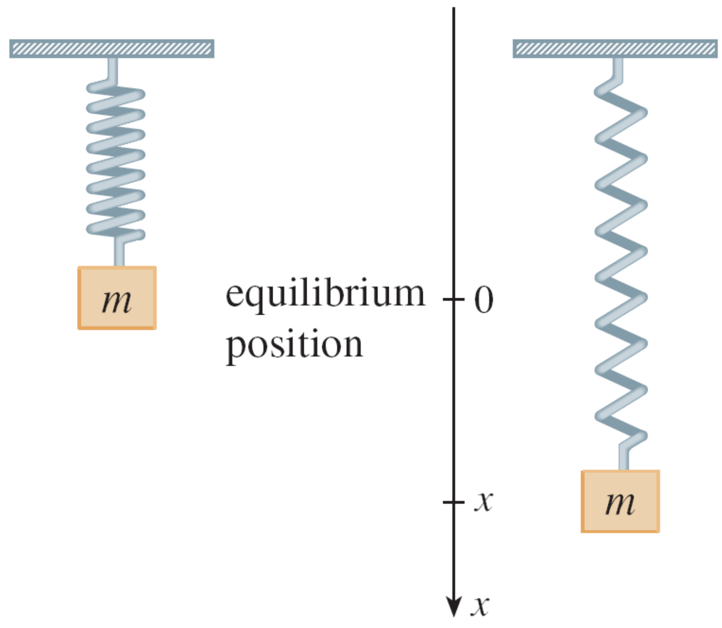


Figure 4

A Model for the Motion of a Spring

We have discussed Hooke's Law, which says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:

$$\text{restoring force} = -kx$$

where k is a positive constant (called the *spring constant*). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have

$$\boxed{3} \qquad m\frac{d^2x}{dt^2} = -kx$$

A Model for the Motion of a Spring

This is an example of what is called a *second-order differential equation* because it involves second derivatives.

Let's see what we can guess about the form of the solution directly from the equation. We can rewrite Equation 3 in the form

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

which says that the second derivative of x is proportional to x but has the opposite sign.

General Differential Equations

General Differential Equations

In general, a **differential equation** is an equation that contains an unknown function and one or more of its derivatives.

The **order** of a differential equation is the order of the highest derivative that occurs in the equation. Thus Equations 1 and 2 are first-order equations and Equation 3 is a second-order equation.

In all three of those equations the independent variable is called *t* and represents time, but in general the independent variable doesn't have to represent time.

General Differential Equations

For example, when we consider the differential equation

$$\boxed{4} \qquad y' = xy$$

it is understood that y is an unknown function of x.

A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation. Thus f is a solution of Equation 4 if

$$f'(x) = xf(x)$$

for all values of x in some interval.

General Differential Equations

When we are asked to *solve* a differential equation we are expected to find all possible solutions of the equation. We have already solved some particularly simple differential equations, namely, those of the form

$$y' = f(x)$$

For instance, we know that the general solution of the differential equation

$$y' = x^3$$

is given by

$$y = \frac{x^4}{4} + C$$

where C is an arbitrary constant.

Example 1

Show that every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$.

Example 1 – *Solution*

We use the Quotient Rule to differentiate the expression for *y*:

$$y' = \frac{(1 - ce^t)(ce^t) - (1 + ce^t)(-ce^t)}{(1 - ce^t)^2}$$

$$= \frac{ce^t - c^2e^{2t} + ce^t + c^2e^{2t}}{(1 - ce^t)^2}$$

$$= \frac{2ce^t}{(1 - ce^t)^2}$$

Example 1 – *Solution*

cont'd

The right side of the differential equation becomes

$$\begin{aligned}
\tfrac{1}{2}(y^2 - 1) &= \frac{1}{2}\left[\left(\frac{1 + ce^t}{1 - ce^t}\right)^2 - 1\right] \\
&= \frac{1}{2}\left[\frac{(1 + ce^t)^2 - (1 - ce^t)^2}{(1 - ce^t)^2}\right] \\
&= \frac{1}{2}\,\frac{4ce^t}{(1 - ce^t)^2} \\
&= \frac{2ce^t}{(1 - ce^t)^2}
\end{aligned}$$

Therefore, for every value of *c*, the given function is a solution of the differential equation.

General Differential Equations

When applying differential equations, we are usually not as interested in finding a family of solutions (the *general solution*) as we are in finding a solution that satisfies some additional requirement.

In many physical problems we need to find the particular solution that satisfies a condition of the form $y(t_0) = y_0$.

This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.

9

Differential Equations

9.2 Direction Fields and Euler's Method

Direction Fields and Euler's Method

Unfortunately, it's impossible to solve most differential equations in the sense of obtaining an explicit formula for the solution.

In this section we show that, despite the absence of an explicit solution, we can still learn a lot about the solution through a graphical approach (direction fields) or a numerical approach (Euler's method).

Direction Fields

Direction Fields

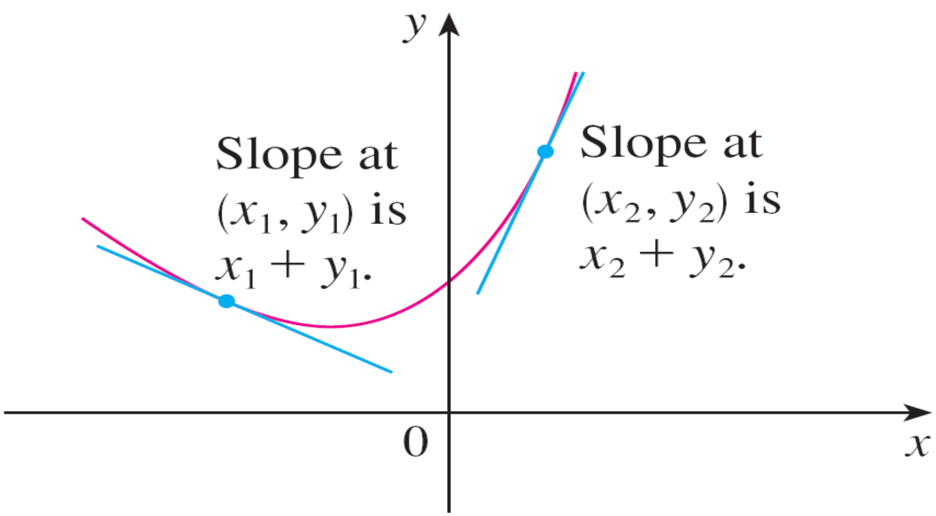
Suppose we are asked to sketch the graph of the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1$$

We don't know a formula for the solution, so how can we possibly sketch its graph? Let's think about what the differential equation means.

Direction Fields

The equation $y' = x + y$ tells us that the slope at any point (x, y) on the graph (called the *solution curve*) is equal to the sum of the *x*- and *y*-coordinates of the point (see Figure 1).

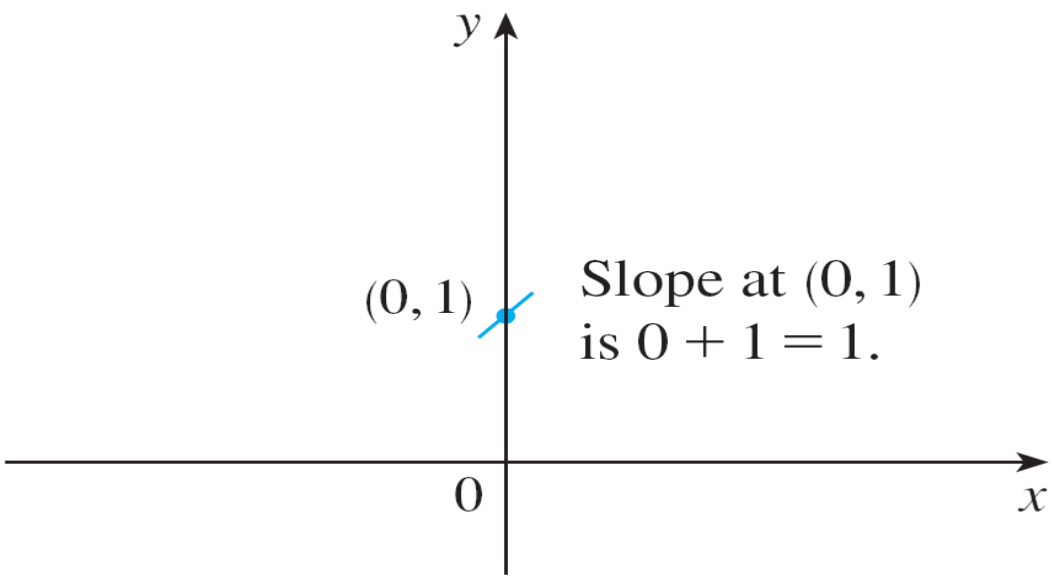


A solution of $y' = x + y$

Figure 1

Direction Fields

In particular, because the curve passes through the point (0, 1), its slope there must be 0 + 1 = 1. So a small portion of the solution curve near the point (0, 1) looks like a short line segment through (0, 1) with slope 1. (See Figure 2.)

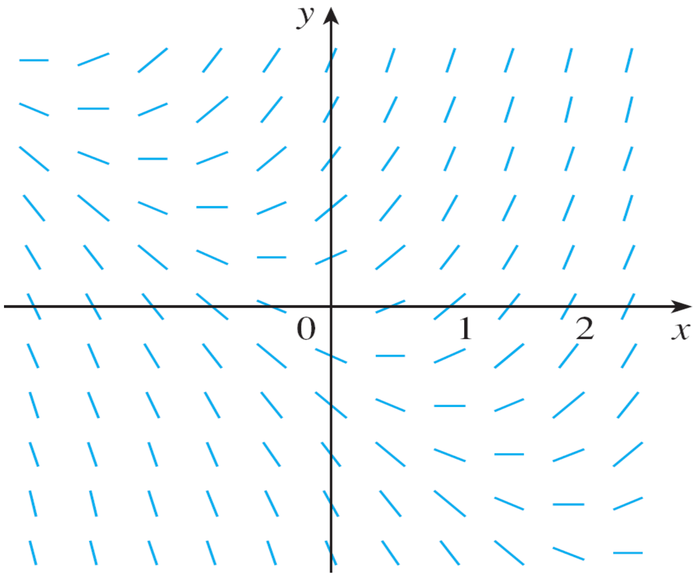


Beginning of the solution curve through (0, 1)

Figure 2

Direction Fields

As a guide to sketching the rest of the curve, let's draw short line segments at a number of points (x, y) with slope $x + y$. The result is called a *direction field* and is shown in Figure 3.



Direction field for $y' = x + y$

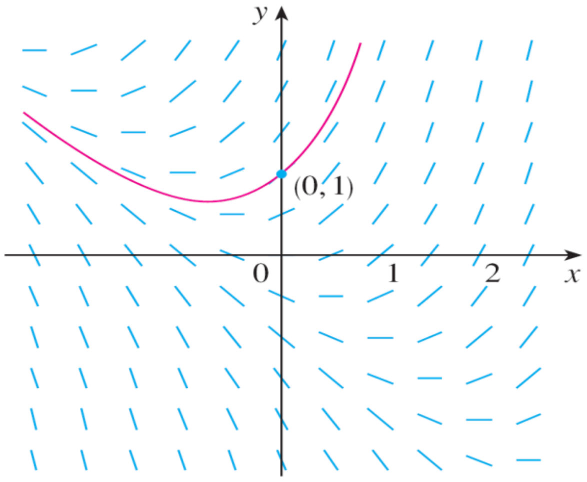
Figure 3

Direction Fields

For instance, the line segment at the point (1, 2) has slope $1 + 2 = 3$.

The direction field allows us to visualize the general shape of the solution curves by indicating the direction in which the curves proceed at each point.

Now we can sketch the solution curve through the point (0, 1) by following the direction field as in Figure 4.



The solution curve through (0, 1)

Figure 4

Direction Fields

Notice that we have drawn the curve so that it is parallel to nearby line segments.

In general, suppose we have a first-order differential equation of the form

$$y' = F(x, y)$$

where $F(x, y)$ is some expression in x and y. The differential equation says that the slope of a solution curve at a point (x, y) on the curve is $F(x, y)$.

Direction Fields

If we draw short line segments with slope $F(x, y)$ at several points (x, y), the result is called a **direction field** (or **slope field**).

These line segments indicate the direction in which a solution curve is heading, so the direction field helps us visualize the general shape of these curves.

Example 1

(a) Sketch the direction field for the differential equation

$y' = x^2 + y^2 - 1$.

(b) Use part (a) to sketch the solution curve that passes through the origin.

Solution:

(a) We start by computing the slope at several points in the following chart:

x	-2	-1	0	1	2	-2	-1	0	1	2	...
y	0	0	0	0	0	1	1	1	1	1	...
$y' = x^2 + y^2 - 1$	3	0	-1	0	3	4	1	0	1	4	...

Example 1 – *Solution*

cont'd

Now we draw short line segments with these slopes at these points. The result is the direction field shown in Figure 5.

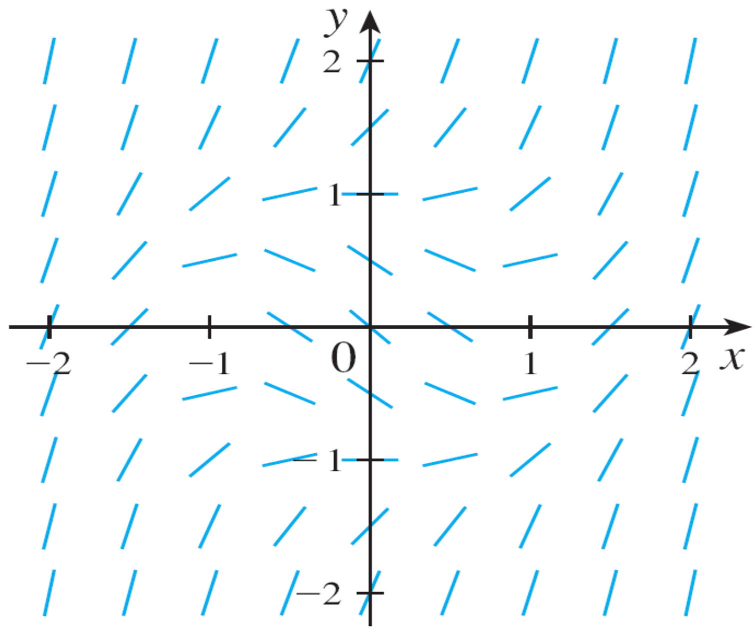


Figure 5

Example 1 – *Solution*

cont'd

(b) We start at the origin and move to the right in the direction of the line segment (which has slope –1).

We continue to draw the solution curve so that it moves parallel to the nearby line segments.

The resulting solution curve is shown in Figure 6.

Returning to the origin, we draw the solution curve to the left as well.

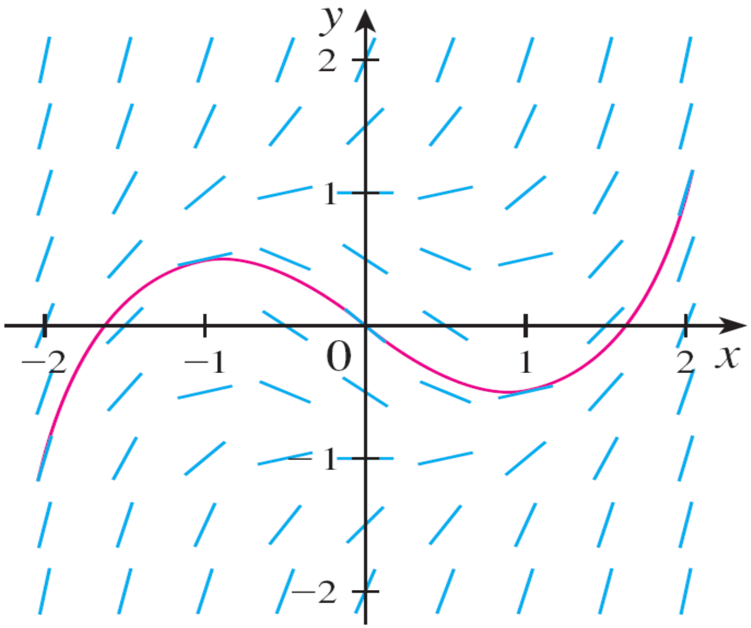


Figure 6

Direction Fields

Now let's see how direction fields give insight into physical situations.

The simple electric circuit shown in Figure 9 contains an electromotive force (usually a battery or generator) that produces a voltage of $E(t)$ volts (V) and a current of $I(t)$ amperes (A) at time t.

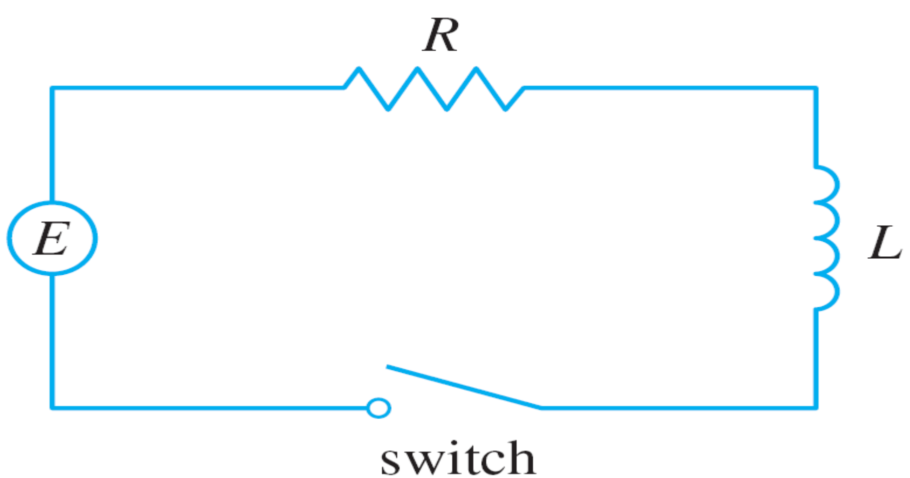


Figure 9

Direction Fields

The circuit also contains a resistor with a resistance of R ohms (Ω) and zan inductor with an inductance of L henries (H).

Ohm's Law gives the drop in voltage due to the resistor as RI. The voltage drop due to the inductor is $L(dI/dt)$. One of Kirchhoff's laws says that the sum of the voltage drops is equal to the supplied voltage $E(t)$. Thus we have

$$\boxed{1} \qquad L\frac{dI}{dt} + RI = E(t)$$

which is a first-order differential equation that models the current I at time t.

Direction Fields

A differential equation of the form

$$y' = f(y)$$

in which the independent variable is missing from the right side, is called **autonomous**.

For such an equation, the slopes corresponding to two different points with the same y-coordinate must be equal.

This means that if we know one solution to an autonomous differential equation, then we can obtain infinitely many others just by shifting the graph of the known solution to the right or left.

Euler's Method

Euler's Method

The basic idea behind direction fields can be used to find numerical approximations to solutions of differential equations.

We illustrate the method on the initial-value problem

that we used to introduce direction fields:

$$y' = x + y \qquad\qquad y(0) = 1$$

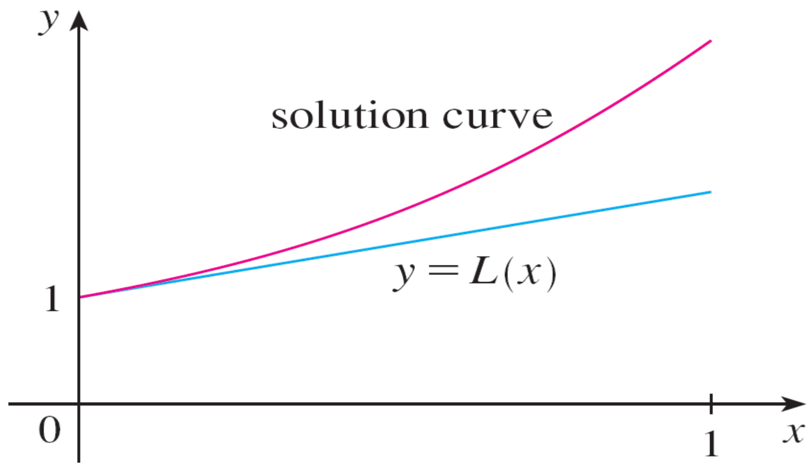
The differential equation tells us that $y'(0) = 0 + 1 = 1$, so the solution curve has slope 1 at the point (0, 1).

As a first approximation to the solution we could use the linear approximation $L(x) = x + 1$.

Euler’s Method

In other words, we could use the tangent line at (0, 1) as a rough approximation to the solution curve (see Figure 12).

Euler’s idea was to improve on this approximation by proceeding only a short distance along this tangent line and then making a midcourse correction by changing direction as indicated by the direction field.



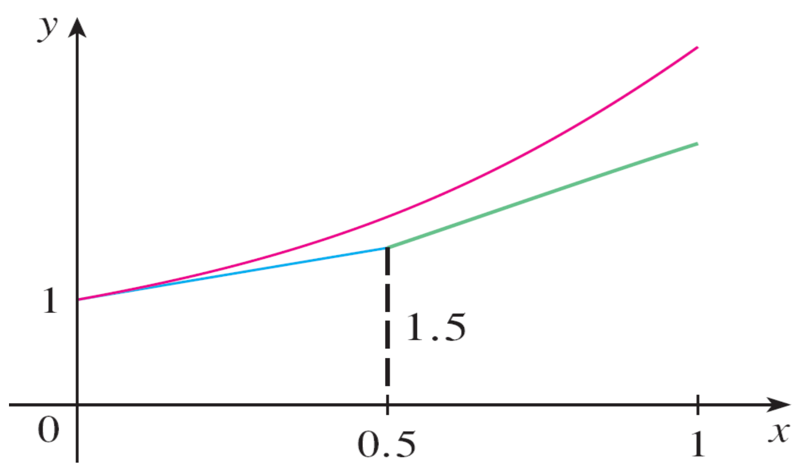
First Euler approximation

Figure 12

Euler’s Method

Figure 13 shows what happens if we start out along the tangent line but stop when $x = 0.5$. (This horizontal distance traveled is called the *step size.*)

Since $L(0.5) = 1.5$, we have $y(0.5) \approx 1.5$ and we take $(0.5, 1.5)$ as the starting point for a new line segment.



Euler approximation with step size 0.5

Figure 13

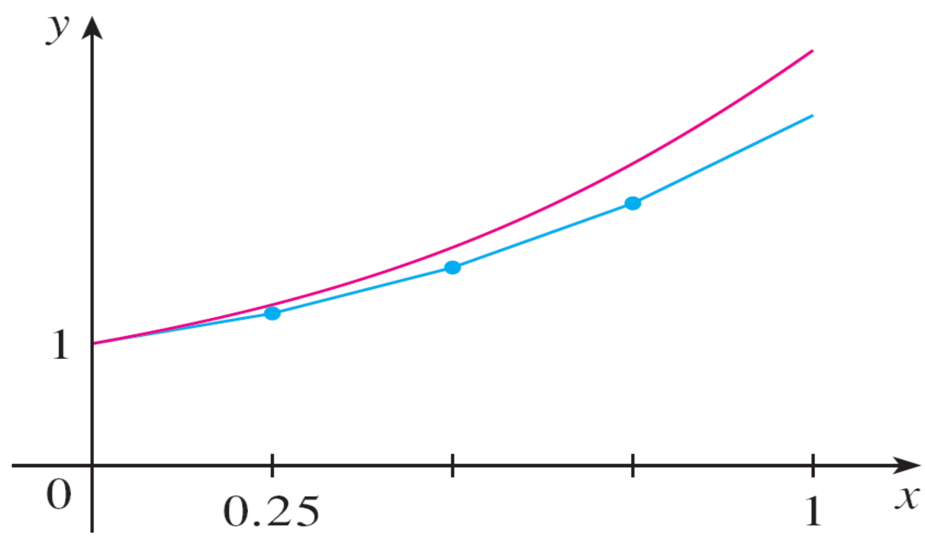
Euler's Method

The differential equation tells us that $y'(0.5) = 0.5 + 1.5 = 2$, so we use the linear function

$$y = 1.5 + 2(x - 0.5) = 2x + 0.5$$

as an approximation to the solution for $x > 0.5$ (the green segment in Figure 13).

If we decrease the step size from 0.5 to 0.25, we get the better Euler approximation shown in Figure 14.



Euler approximation with step size 0.25

Figure 14

Euler's Method

In general, Euler's method says to start at the point given by the initial value and proceed in the direction indicated by the direction field.

Stop after a short time, look at the slope at the new location, and proceed in that direction.

Keep stopping and changing direction according to the direction field.

Euler's method does not produce the exact solution to an initial-value problem—it gives approximations.

Euler's Method

But by decreasing the step size (and therefore increasing the number of midcourse corrections), we obtain successively better approximations to the exact solution. (Compare Figures 12, 13, and 14.)

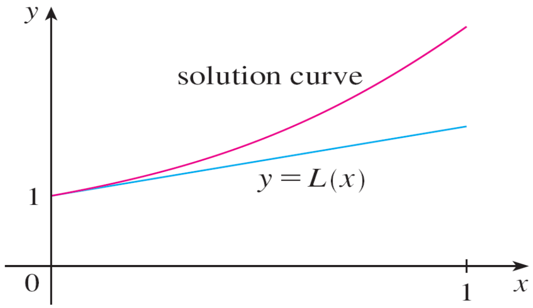


Figure 12

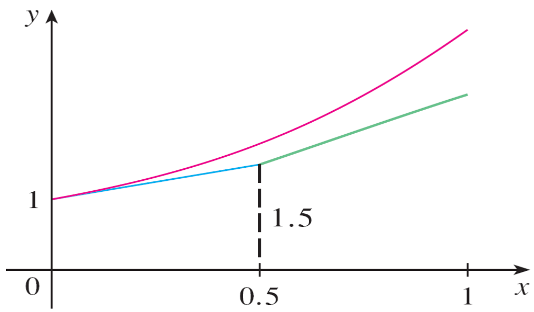


Figure 13

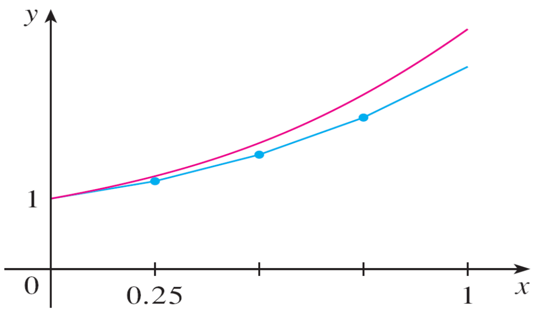


Figure 14

For the general first-order initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, our aim is to find approximate values for the solution at equally spaced numbers x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h, \ldots$, where h is the step size.

Euler's Method

The differential equation tells us that the slope at (x_0, y_0) is $y' = F(x_0, y_0)$, so Figure 15 shows that the approximate value of the solution when $x = x_1$ is

$$y_1 = y_0 + hF(x_0, y_0)$$

Similarly,

$$y_2 = y_1 + hF(x_1, y_1)$$

In general,

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

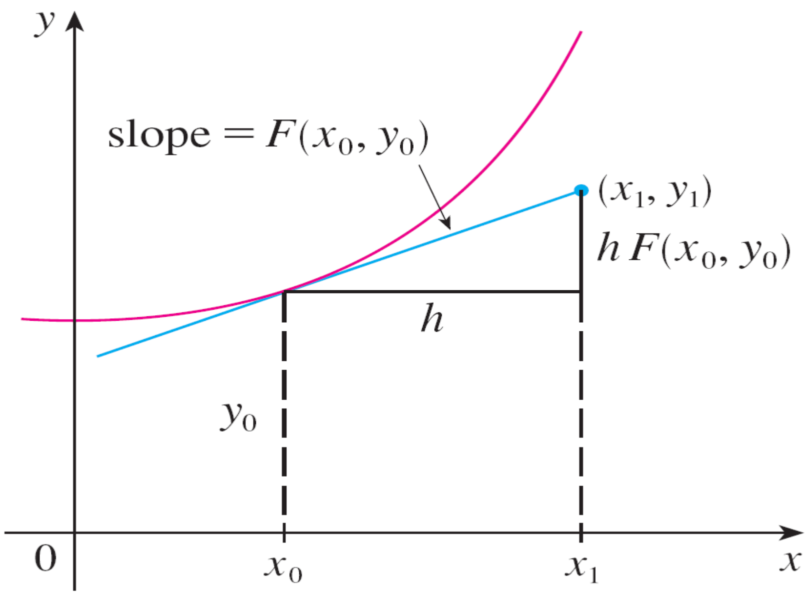


Figure 15

Euler's Method

Euler's Method Approximate values for the solution of the initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, with step size h, at $x_n = x_{n-1} + h$, are

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \qquad n = 1, 2, 3, \cdots$$

Example 3

Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the initial-value problem

$$y' = x + y \qquad\qquad y(0) = 1$$

Solution:

We are given that $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x + y$. So we have

$$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1$$

$$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22$$

$$y_3 = y_2 + hF(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362$$

This means that if $y(x)$ is the exact solution, then $y(0.3) \approx 1.362$.

Example 3 – *Solution*

cont'd

Proceeding with similar calculations, we get the values in the table:

n	x_n	y_n	n	x_n	y_n
1	0.1	1.100000	6	0.6	1.943122
2	0.2	1.220000	7	0.7	2.197434
3	0.3	1.362000	8	0.8	2.487178
4	0.4	1.528200	9	0.9	2.815895
5	0.5	1.721020	10	1.0	3.187485

Euler's Method

For a more accurate table of values in Example 3 we could decrease the step size.

But for a large number of small steps the amount of computation is considerable and so we need to program a calculator or computer to carry out these calculations.

Euler’s Method

The following table shows the results of applying Euler’s method with decreasing step size to the initial-value problem of Example 3.

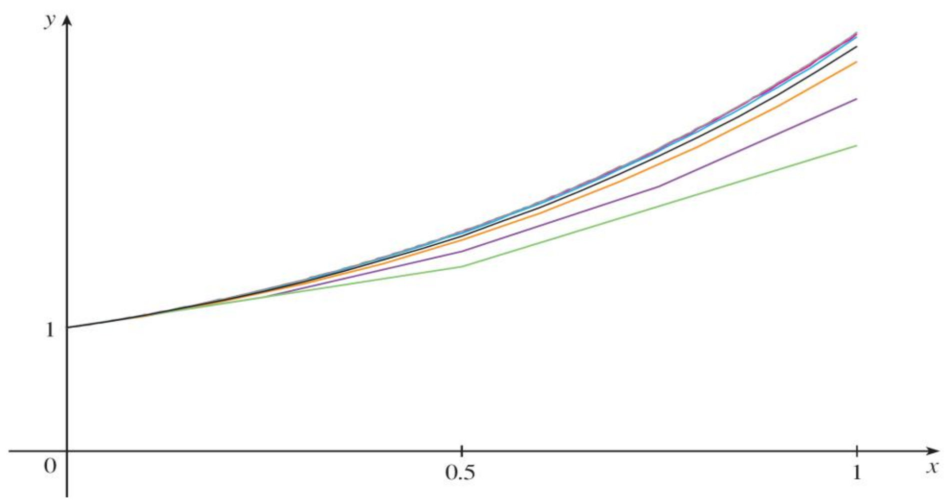
Step size	Euler estimate of $y(0.5)$	Euler estimate of $y(1)$
0.500	1.500000	2.500000
0.250	1.625000	2.882813
0.100	1.721020	3.187485
0.050	1.757789	3.306595
0.020	1.781212	3.383176
0.010	1.789264	3.409628
0.005	1.793337	3.423034
0.001	1.796619	3.433848

Notice that the Euler estimates in the table seem to be approaching limits, namely, the true values of $y(0.5)$ and $y(1)$.

Euler's Method

Figure 16 shows graphs of the Euler approximations with step sizes 0.5, 0.25, 0.1, 0.05, 0.02, 0.01, and 0.005.

They are approaching the exact solution curve as the step size *h* approaches 0.



Euler approximations approaching the exact solution

Figure 16

9

Differential Equations

9.3 Separable Equations

Separable Equations

A **separable equation** is a first-order differential equation in which the expression for *dy*/*dx* can be factored as a function of *x* times a function of *y*.

In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

The name *separable* comes from the fact that the expression on the right side can be "separated" into a function of *x* and a function of *y*.

Separable Equations

Equivalently, if $f(y) \neq 0$, we could write

$$\boxed{1} \qquad \frac{dy}{dx} = \frac{g(x)}{h(y)}$$

where $h(y) = 1/f(y)$.

To solve this equation we rewrite it in the differential form

$$h(y)\,dy = g(x)\,dx$$

so that all y's are on one side of the equation and all x's are on the other side.

Separable Equations

Then we integrate both sides of the equation:

$$\boxed{2} \qquad \int h(y)\,dy = \int g(x)\,dx$$

Equation 2 defines *y* implicitly as a function of *x*. In some cases we may be able to solve for *y* in terms of *x*.

Separable Equations

We use the Chain Rule to justify this procedure: If *h* and *g* satisfy (2), then

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dx}\left(\int g(x)\,dx\right)$$

so $\displaystyle \frac{d}{dy}\left(\int h(y)\,dy\right)\frac{dy}{dx} = g(x)$

and
$$h(y)\,\frac{dy}{dx} = g(x)$$

Thus Equation 1 is satisfied.

Example 1

(a) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

Solution:

(a) We write the equation in terms of differentials and integrate both sides:

$$y^2\,dy = x^2\,dx$$

$$\int y^2\,dy = \int x^2\,dx$$

Example 1 – *Solution*

cont'd

$$\tfrac{1}{3}y^3 = \tfrac{1}{3}x^3 + C$$

where C is an arbitrary constant. (We could have used a constant C_1 on the left side and another constant C_2 on the right side. But then we could combine these constants by writing $C = C_2 - C_1$.)

Solving for y, we get

$$y = \sqrt[3]{x^3 + 3C}$$

Example 1 – *Solution*

cont'd

We could leave the solution like this or we could write it in the form

$$y = \sqrt[3]{x^3 + K}$$

where $K = 3C$. (Since C is an arbitrary constant, so is K.)

(b) If we put $x = 0$ in the general solution in part (a), we get $y(0) = \sqrt[3]{K}$. To satisfy the initial condition $y(0) = 2$, we must have $\sqrt[3]{K} = 2$ and so $K = 8$.

Thus the solution of the initial-value problem is

$$y = \sqrt[3]{x^3 + 8}$$

Orthogonal Trajectories

Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles (see Figure 7).

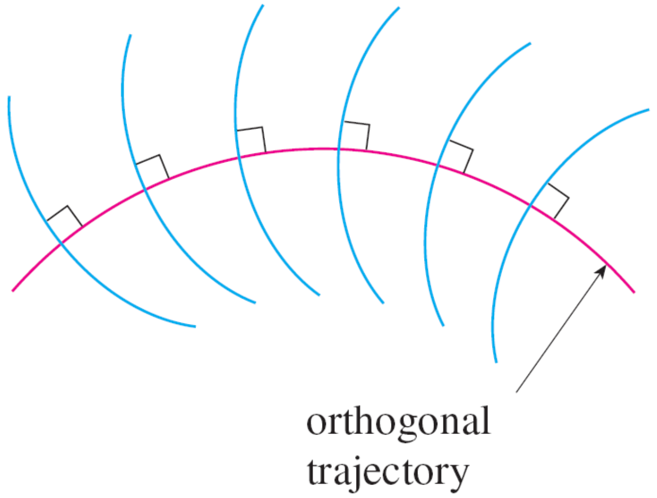


Figure 7

Orthogonal Trajectories

For instance, each member of the family $y = mx$ of straight lines through the origin is an orthogonal trajectory of the family $x^2 + y^2 = r^2$ of concentric circles with center the origin (see Figure 8). We say that the two families are orthogonal trajectories of each other.

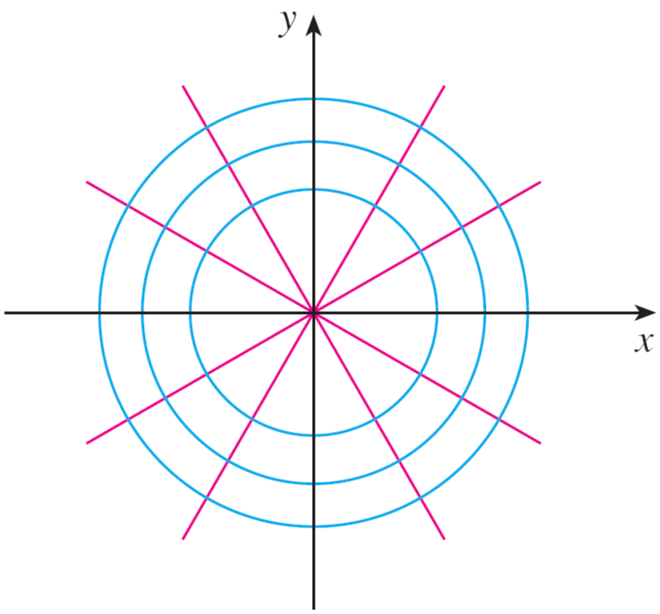


Figure 8

Example 5

Find the orthogonal trajectories of the family of curves $x = ky^2$, where is k an arbitrary constant.

Solution:

The curves $x = ky^2$ form a family of parabolas whose axis of symmetry is the x-axis.

The first step is to find a single differential equation that is satisfied by all members of the family.

If we differentiate $x = ky^2$, we get

$$1 = 2ky\frac{dy}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{2ky}$$

Example 5 – *Solution*

cont'd

This differential equation depends on k, but we need an equation that is valid for all values of k simultaneously.

To eliminate k we note that, from the equation of the given general parabola $x = ky^2$, we have $k = x/y^2$ and so the differential equation can be written as

$$\frac{dy}{dx} = \frac{1}{2ky} = \frac{1}{2\,\dfrac{x}{y^2}\,y}$$

or

$$\frac{dy}{dx} = \frac{y}{2x}$$

Example 5 – *Solution*

cont'd

This means that the slope of the tangent line at any point (x, y) on one of the parabolas is $y' = y/(2x)$.

On an orthogonal trajectory the slope of the tangent line must be the negative reciprocal of this slope.

Therefore the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = -\frac{2x}{y}$$

This differential equation is separable, and we solve it as follows:

$$\int y\,dy = -\int 2x\,dx$$

Example 5 – *Solution*

cont'd

$$\frac{y^2}{2} = -x^2 + C$$

$$\boxed{4} \qquad x^2 + \frac{y^2}{2} = C$$

where *C* is an arbitrary positive constant.

Thus the orthogonal trajectories are the family of ellipses given by Equation 4 and sketched in Figure 9.

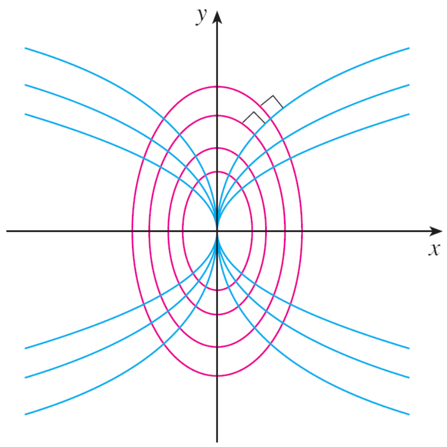


Figure 9

Mixing Problems

Mixing Problems

A typical mixing problem involves a tank of fixed capacity filled with a thoroughly mixed solution of some substance, such as salt.

A solution of a given concentration enters the tank at a fixed rate and the mixture, thoroughly stirred, leaves at a fixed rate, which may differ from the entering rate.

If $y(t)$ denotes the amount of substance in the tank at time t, then $y'(t)$ is the rate at which the substance is being added minus the rate at which it is being removed.

Mixing Problems

The mathematical description of this situation often leads to a first-order separable differential equation.

We can use the same type of reasoning to model a variety of phenomena: chemical reactions, discharge of pollutants into a lake, injection of a drug into the bloodstream.

Example 6

A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

Solution:

Let $y(t)$ be the amount of salt (in kilograms) after t minutes.

We are given that $y(0) = 20$ and we want to find $y(30)$. We do this by finding a differential equation satisfied by $y(t)$.

Example 6 – *Solution*

cont'd

Note that dy/dt is the rate of change of the amount of salt, so

$$\boxed{5} \qquad \frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

where (rate in) is the rate at which salt enters the tank and (rate out) is the rate at which salt leaves the tank.

We have

$$\text{rate in} = \left(0.03\,\frac{\text{kg}}{\text{L}}\right)\left(25\,\frac{\text{L}}{\text{min}}\right) = 0.75\,\frac{\text{kg}}{\text{min}}$$

The tank always contains 5000 L of liquid, so the concentration at time t is $y(t)/5000$ (measured in kilograms per liter).

Example 6 – *Solution*

cont'd

Since the brine flows out at a rate of 25 L/min, we have

$$\text{rate out} = \left(\frac{y(t)}{5000}\,\frac{\text{kg}}{\text{L}}\right)\left(25\,\frac{\text{L}}{\text{min}}\right) = \frac{y(t)}{200}\,\frac{\text{kg}}{\text{min}}$$

Thus, from Equation 5, we get

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

Solving this separable differential equation, we obtain

$$\int \frac{dy}{150 - y} = \int \frac{dt}{200}$$

$$-\ln|150 - y| = \frac{t}{200} + C$$

Example 6 – *Solution*

cont'd

Since $y(0) = 20$, we have $–\ln 130 = C$, so

$$-\ln|150 - y| = \frac{t}{200} - \ln 130$$

Therefore

$$|150 - y| = 130e^{-t/200}$$

Since $y(t)$ is continuous and $y(0) = 20$ and the right side is never 0, we deduce that $150 - y(t)$ is always positive.

Example 6 – *Solution*

cont'd

Thus $|150 - y| = 150 - y$ and so

$$y(t) = 150 - 130e^{-t/200}$$

The amount of salt after 30 min is

$$y(30) = 150 - 130e^{-30/200}$$

$$\approx 38.1\text{kg}$$

9

Differential Equations

9.4 Models for Population Growth

Models for Population Growth

In this section we investigate differential equations that are used to model population growth: the law of natural growth, the logistic equation, and several others.

The Law of Natural Growth

The Law of Natural Growth

In general, if $P(t)$ is the value of a quantity y at time t and if the rate of change of P with respect to t is proportional to its size $P(t)$ at any time, then

$$\boxed{1} \qquad \boxed{\frac{dP}{dt} = kP}$$

where k is a constant.

Equation 1 is sometimes called the **law of natural growth**. If k is positive, then the population increases; if k is negative, it decreases.

The Law of Natural Growth

Because Equation 1 is a separable differential equation, we can solve it by the methods given below:

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln|P| = kt + C$$

$$|P| = e^{kt + C} = e^{C}e^{kt}$$

$$P = Ae^{kt}$$

where A (= $\pm e^{C}$ or 0) is an arbitrary constant.

The Law of Natural Growth

To see the significance of the constant *A*, we observe that

$$P(0) = Ae^{k \cdot 0} = A$$

Therefore *A* is the initial value of the function.

2 The solution of the initial-value problem

$$\frac{dP}{dt} = kP \qquad P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}$$

The Law of Natural Growth

Another way of writing Equation 1 is

$$\frac{1}{P}\frac{dP}{dt} = k$$

which says that the **relative growth rate** (the growth rate divided by the population size) is constant.

Then $\boxed{2}$ says that a population with constant relative growth rate must grow exponentially.

The Law of Natural Growth

We can account for emigration (or "harvesting") from a population by modifying Equation 1: If the rate of emigration is a constant *m*, then the rate of change of the population is modeled by the differential equation

$$\boxed{3} \qquad \frac{dP}{dt} = kP - m$$

The Logistic Model

The Logistic Model

As we studied earlier, a population often increases exponentially in its early stages but levels off eventually and approaches its carrying capacity because of limited resources.

If $P(t)$ is the size of the population at time t, we assume that

$$\frac{dP}{dt} \approx kP \qquad \text{if } P \text{ is small}$$

This says that the growth rate is initially close to being proportional to size.

The Logistic Model

In other words, the relative growth rate is almost constant when the population is small. But we also want to reflect the fact that the relative growth rate decreases as the population *P* increases and becomes negative if *P* ever exceeds its **carrying capacity** *M*, the maximum population that the environment is capable of sustaining in the long run.

The simplest expression for the relative growth rate that incorporates these assumptions is

$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right)$$

The Logistic Model

Multiplying by *P*, we obtain the model for population growth known as the **logistic differential equation**:

$$\boxed{4} \qquad \boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)}$$

Example 1

Draw a direction field for the logistic equation with $k = 0.08$ and carrying capacity $M = 1000$. What can you deduce about the solutions?

Solution:

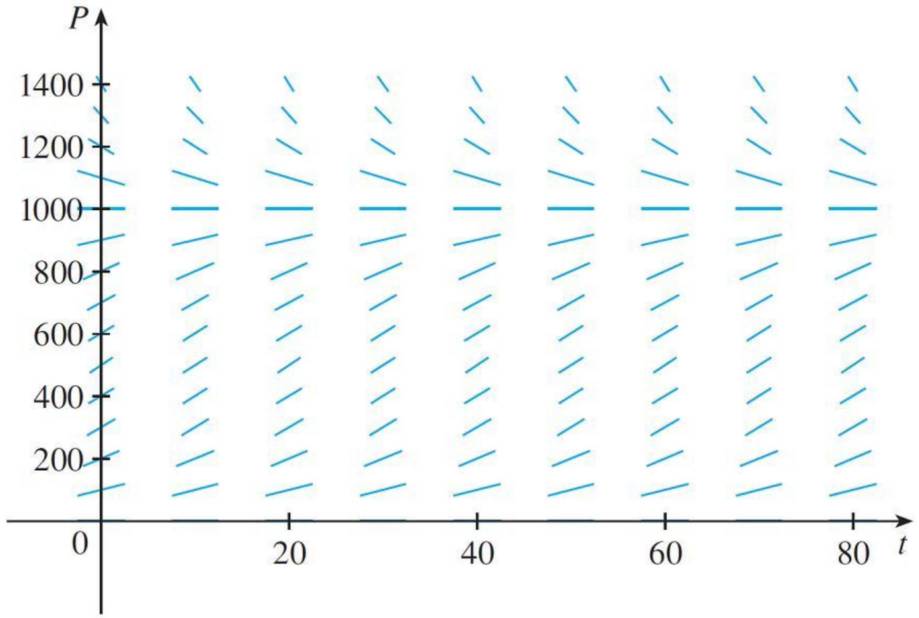
In this case the logistic differential equation is

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

Example 1 – *Solution*

cont'd

A direction field for this equation is shown in Figure 1.



Direction field for the logistic equation in Example 1

Figure 1

Example 1 – *Solution*

cont'd

We show only the first quadrant because negative populations aren't meaningful and we are interested only in what happens after $t = 0$.

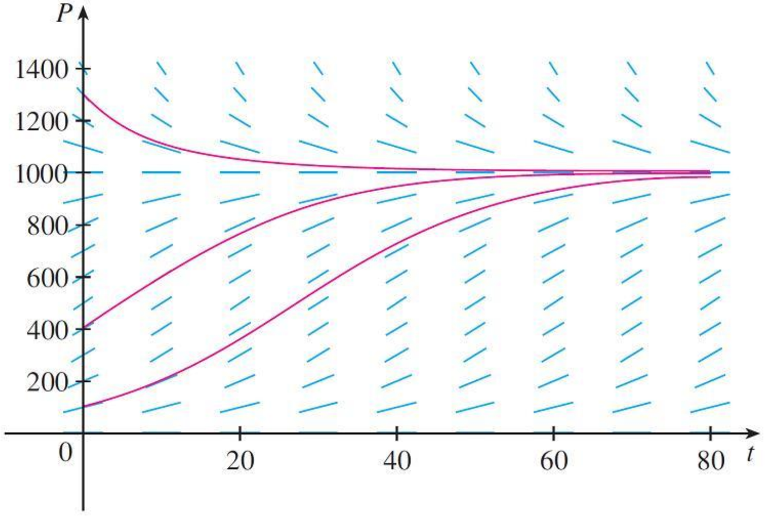
The logistic equation is autonomous (dP/dt depends only on P, not on t), so the slopes are the same along any horizontal line. As expected, the slopes are positive for $0 < P < 100$ and negative for $P > 1000$.

The slopes are small when P is close to 0 or 1000 (the carrying capacity). Notice that the solutions move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = 1000$.

Example 1 – *Solution*

cont'd

In Figure 2 we use the direction field to sketch solution curves with initial populations $P(0) = 100$, $P(0) = 400$, and $P(0) = 1300$.



Solution curves for the logistic equation in Example 1

Figure 2

Example 1 – *Solution*

cont'd

Notice that solution curves that start below $P = 1000$ are increasing and those that start above $P = 1000$ are decreasing.

The slopes are greatest when $P \approx 500$ and therefore the solution curves that start below $P = 1000$ have inflection points when $P \approx 500$.

In fact we can prove that all solution curves that start below $P = 500$ have an inflection point when P is exactly 500.

The Logistic Model

The logistic equation $\boxed{4}$ is separable and so we can solve it explicitly. Since

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

we have

$$\boxed{5} \qquad \int \frac{dP}{P(1 - P/M)} = \int k\, dt$$

The Logistic Model

To evaluate the integral on the left side, we write

$$\frac{1}{P(1 - P/M)} = \frac{M}{P(M - P)}$$

Using partial fractions, we get

$$\frac{M}{P(M - P)} = \frac{1}{P} + \frac{1}{M - P}$$

This enables us to rewrite Equation 5:

$$\int \left(\frac{1}{P} + \frac{1}{M - P} \right) dP = \int k\, dt$$

The Logistic Model

$$\ln|P| - \ln|M - P| = kt + C$$

$$\ln\left|\frac{M-P}{P}\right| = -kt - C$$

$$\left|\frac{M-P}{P}\right| = e^{-kt-C} = e^{-C}e^{-kt}$$

$$\boxed{6} \qquad \frac{M-P}{P} = Ae^{-kt}$$

where $A = \pm e^{-C}$.

The Logistic Model

Solving Equation 6 for *P*, we get

$$\frac{M}{P} - 1 = Ae^{-kt} \qquad \Rightarrow \qquad \frac{P}{M} = \frac{1}{1 + Ae^{-kt}}$$

so

$$P = \frac{M}{1 + Ae^{-kt}}$$

The Logistic Model

We find the value of A by putting $t = 0$ in Equation 6. If $t = 0$, then $P = P_0$ (the initial population), so

$$\frac{M - P_0}{P_0} = Ae^0 = A$$

Thus the solution to the logistic equation is

$$\boxed{7} \qquad \boxed{P(t) = \frac{M}{1 + Ae^{-kt}} \qquad \text{where } A = \frac{M - P_0}{P_0}}$$

The Logistic Model

Using the expression for $P(t)$ in Equation 7, we see that

$$\lim_{t \to \infty} P(t) = M$$

which is to be expected.

Example 2

Write the solution of the initial-value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) \qquad P(0) = 100$$

and use it to find the population sizes $P(40)$ and $P(80)$. At what time does the population reach 900?

Example 2 – *Solution*

The differential equation is a logistic equation with $k = 0.08$, carrying capacity $M = 1000$, and initial population $P_0 = 100$.

So Equation 7 gives the population at time t as

$$P(t) = \frac{1000}{1 + Ae^{-0.08t}}$$

where

$$A = \frac{1000 - 100}{100} = 9$$

Thus

$$P(t) = \frac{1000}{1 + 9e^{-0.08t}}$$

So the population sizes when $t = 40$ and 80 are

$$P(40) = \frac{1000}{1 + 9e^{-3.2}}$$

Example 2 – *Solution*

cont'd

$$\approx 731.6$$

$$P(80) = \frac{1000}{1 + 9e^{-6.4}}$$

$$\approx 985.3$$

The population reaches 900 when

$$\frac{1000}{1 + 9e^{-0.08t}} = 900$$

Example 2 – *Solution*

cont'd

Solving this equation for *t*, we get

$$1 + 9e^{-0.08t} = \tfrac{10}{9}$$

$$e^{-0.08t} = \tfrac{1}{81}$$

$$-0.08t = \ln \tfrac{1}{81}$$

$$= -\ln 81$$

$$t = \frac{\ln 81}{0.08}$$

$$\approx 54.9$$

So the population reaches 900 when *t* is approximately 55.

Example 2 – *Solution*

cont'd

As a check on our work, we graph the population curve in Figure 3 and observe where it intersects the line $P = 900$.

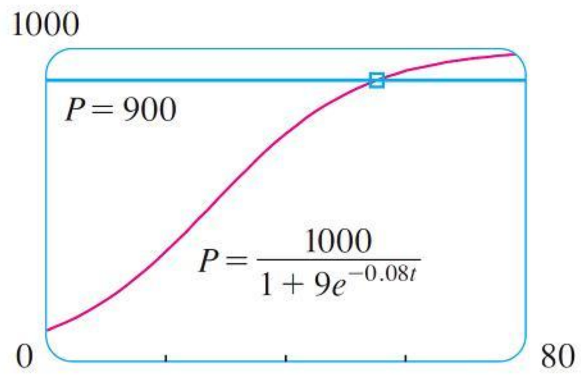


Figure 3

The cursor indicates that $t \approx 55$.

Comparison of the Natural Growth and Logistic Models

Comparison of the Natural Growth and Logistic Models

In the 1930s the biologist G. F. Gause conducted an experiment with the protozoan *Paramecium* and used a logistic equation to model his data. The table gives his daily count of the population of protozoa.

t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57

He estimated the initial relative growth rate to be 0.7944 and the carrying capacity to be 64.

Example 3

Find the exponential and logistic models for Gause’s data. Compare the predicted values with the observed values and comment on the fit.

Solution:

Given the relative growth rate $k = 0.7944$ and the initial population $P_0 = 2$, the exponential model is

$$P(t) = P_0 e^{kt}$$

$$= 2e^{0.7944t}$$

Example 3 – *Solution*

cont'd

Gause used the same value of k for his logistic model. [This is reasonable because $P_0 = 2$ is small compared with the carrying capacity ($M = 64$).

The equation

$$\frac{1}{P_0} \frac{dP}{dt}\bigg|_{t=0} = k\left(1 - \frac{2}{64}\right) \approx k$$

shows that the value of k for the logistic model is very close to the value for the exponential model.]

7

$$\boxed{P(t) = \frac{M}{1 + Ae^{-kt}} \qquad \text{where } A = \frac{M - P_0}{P_0}}$$

Example 3 – *Solution*

cont'd

Then the solution of the logistic equation in Equation 7 gives

$$P(t) = \frac{M}{1 + Ae^{-kt}}$$

$$= \frac{64}{1 + Ae^{-0.7944t}}$$

where

$$A = \frac{M - P_0}{P_0}$$

$$= \frac{64 - 2}{2} = 31$$

So

$$P(t) = \frac{64}{1 + 31e^{-0.7944t}}$$

Example 3 – *Solution*

cont'd

We use these equations to calculate the predicted values (rounded to the nearest integer) and compare them in the following table.

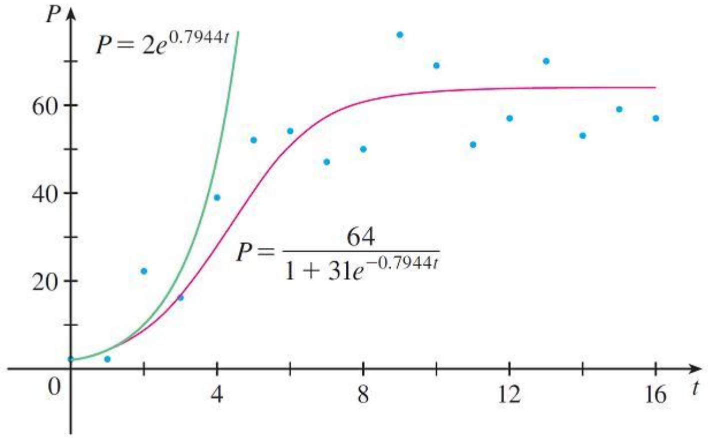
t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57
P (logistic model)	2	4	9	17	28	40	51	57	61	62	63	64	64	64	64	64	64
P (exponential model)	2	4	10	22	48	106	. . .										

Example 3 – *Solution*

cont'd

We notice from the table and from the graph in Figure 4 that for the first three or four days the exponential model gives results comparable to those of the more sophisticated logistic model.

For $t \geq 5$, however, the exponential model is hopelessly inaccurate, but the logistic model fits the observations reasonably well.



The exponential and logistic models for the *Paramecium* data

Figure 4

Other Models for Population Growth

Other Models for Population Growth

The Law of Natural Growth and the logistic differential equation are not the only equations that have been proposed to model population growth.

Two of the other models are modifications of the logistic model. The differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - c$$

has been used to model populations that are subject to harvesting of one sort or another. (Think of a population of fish being caught at a constant rate.)

Other Models for Population Growth

For some species there is a minimum population level *m* below which the species tends to become extinct. (Adults may not be able to find suitable mates.) Such populations have been modeled by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

where the extra factor, 1 – *m*/*p*, takes into account the consequences of a sparse population.

9

Differential Equations

9.5 Linear Equations

Linear Equations

A first-order **linear** differential equation is one that can be put into the form

$$\boxed{1} \qquad \frac{dy}{dx} + P(x)y = Q(x)$$

where *P* and *Q* are continuous functions on a given interval. This type of equation occurs frequently in various sciences, as we will see.

Linear Equations

An example of a linear equation is $xy' + y = 2x$ because, for $x \neq 0$, it can be written in the form

$$\boxed{2} \qquad y' + \frac{1}{x}y = 2$$

Notice that this differential equation is not separable because it's impossible to factor the expression for y' as a function of x times a function of y.

Linear Equations

But we can still solve the equation by noticing, by the Product Rule, that

$$xy' + y = (xy)'$$

and so we can rewrite the equation as

$$(xy)' = 2x$$

Linear Equations

If we now integrate both sides of this equation, we get

$$xy = x^2 + C \quad \text{or} \quad y = x + \frac{C}{x}$$

If we had been given the differential equation in the form of Equation 2, we would have had to take the preliminary step of multiplying each side of the equation by x.

It turns out that every first-order linear differential equation can be solved in a similar fashion by multiplying both sides of Equation 1 by a suitable function $I(x)$ called an *integrating factor.*

Linear Equations

We try to find I so that the left side of Equation 1, when multiplied by $I(x)$, becomes the derivative of the product $I(x)y$:

$$\boxed{3} \qquad I(x)(y' + P(x)y) = (I(x)y)'$$

If we can find such a function I, then Equation 1 becomes

$$(I(x)y)' = I(x)\,Q(x)$$

Linear Equations

Integrating both sides, we would have

$$I(x)y = \int I(x)\,Q(x)\,dx + C$$

so the solution would be

$$\boxed{4} \qquad y(x) = \frac{1}{I(x)}\left[\int I(x)\,Q(x)\,dx + C\right]$$

Linear Equations

To find such an I, we expand Equation 3 and cancel terms:

$$I(x)y' + I(x)P(x)y = (I(x)y)' = I'(x)y + I(x)y'$$

$$I(x)\,P(x) = I'(x)$$

This is a separable differential equation for I, which we solve as follows:

$$\int \frac{dI}{I} = \int P(x)\,dx$$

$$\ln |I| = \int P(x)\,dx$$

Linear Equations

$$I = Ae^{\int P(x)\,dx}$$

where $A = \pm e^C$. We are looking for a particular integrating factor, not the most general one, so we take $A = 1$ and use

$$\boxed{5} \qquad I(x) = e^{\int P(x)\,dx}$$

Thus a formula for the general solution to Equation 1 is provided by Equation 4, where I is given by Equation 5.

Linear Equations

Instead of memorizing this formula, however, we just remember the form of the integrating factor.

To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)\,dx}$ and integrate both sides.

Example 1

Solve the differential equation

$$\frac{dy}{dx} + 3x^2 y = 6x^2.$$

Solution:

The given equation is linear since it has the form of Equation 1 with $P(x) = 3x^2$ and $Q(x) = 6x^2$.

An integrating factor is

$$I(x) = e^{\int 3x^2\,dx} = e^{x^3}$$

Example 1 – *Solution*

cont'd

Multiplying both sides of the differential equation by e^{x^3}, we get

$$e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = 6x^2e^{x^3}$$

or

$$\frac{d}{dx}(e^{x^3}y) = 6x^2e^{x^3}$$

Integrating both sides, we have

$$e^{x^3}y = \int 6x^2e^{x^3}\,dx = 2e^{x^3} + C$$

$$y = 2 + Ce^{-x^3}$$

Example 2

Find the solution of the initial-value problem

$$x^2y' + xy = 1 \qquad x > 0 \qquad\qquad y(1) = 2$$

Solution:

We must first divide both sides by the coefficient of y' to put the differential equation into standard form:

$$\boxed{6} \qquad y' + \frac{1}{x}y = \frac{1}{x^2} \qquad x > 0$$

The integrating factor is

$$I(x) = e^{\int (1/x)\,dx} = e^{\ln x} = x$$

Example 2 – *Solution*

cont'd

Multiplication of Equation 6 by *x* gives

$$xy' + y = \frac{1}{x} \qquad \text{or} \qquad (xy)' = \frac{1}{x}$$

Then

$$xy = \int \frac{1}{x}\,dx = \ln x + C$$

and so

$$y = \frac{\ln x + C}{x}$$

Since $y(1) = 2$, we have

$$2 = \frac{\ln 1 + C}{1} = C$$

Example 2 – *Solution*

cont'd

Therefore the solution to the initial-value problem is

$$y = \frac{\ln x + 2}{x}$$

Application to Electric Circuits

Application to Electric Circuits

Here, we consider the simple electric circuit shown in Figure 4: An electro-motive force (usually a battery or generator) produces a voltage of $E(t)$ volts (V) and a current of $I(t)$ amperes (A) at time t.

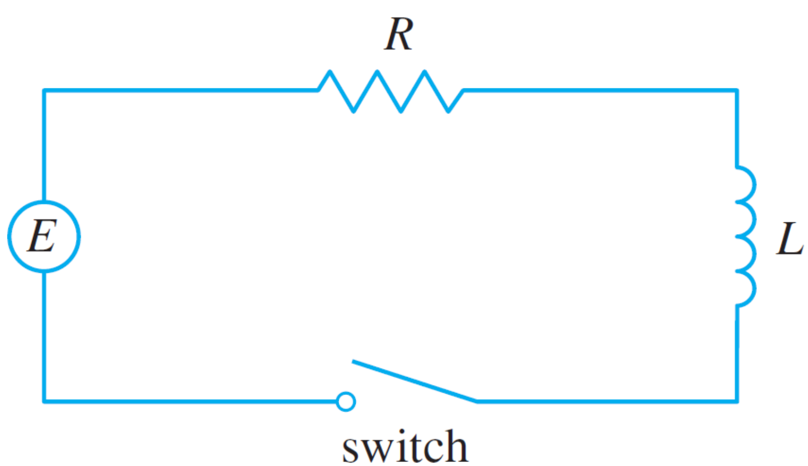


Figure 4

Application to Electric Circuits

The circuit also contains a resistor with a resistance of R ohms (Ω) and an inductor with an inductance of L henries (H).

Ohm's Law gives the drop in voltage due to the resistor as RI. The voltage drop due to the inductor is $L(dI/dt)$.

One of Kirchhoff's laws says that the sum of the voltage drops is equal to the supplied voltage $E(t)$.

Application to Electric Circuits

Thus we have

$$\boxed{7} \qquad L\frac{dI}{dt} + RI = E(t)$$

which is a first-order linear differential equation.

The solution gives the current *I* at time *t*.

Example 4

Suppose that in the simple circuit of Figure 4 the resistance is 12 Ω and the inductance is 4 H. If a battery gives a constant voltage of 60 V and the switch is closed when $t = 0$ so the current starts with $I(0) = 0$, find (a) $I(t)$, (b) the current after 1 s, and (c) the limiting value of the current.

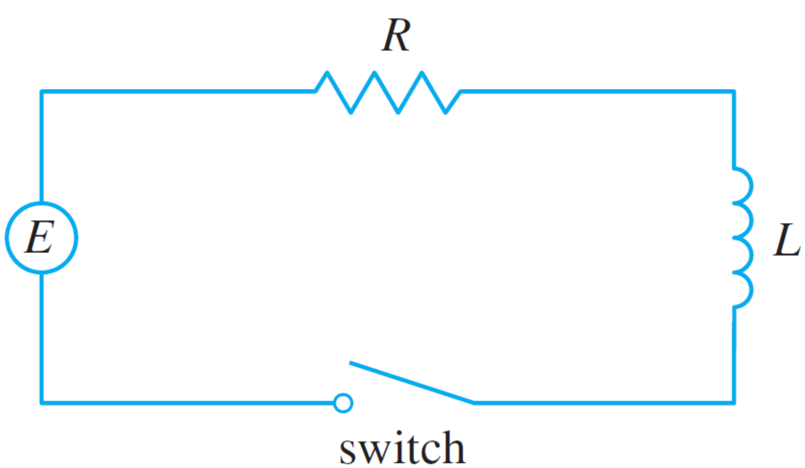


Figure 4

Example 4 – *Solution*

(a) If we put $L = 4$, $R = 12$, and $E(t) = 60$ in Equation 7, we obtain the initial-value problem

$$4\frac{dI}{dt} + 12I = 60 \qquad I(0) = 0$$

or

$$\frac{dI}{dt} + 3I = 15 \qquad I(0) = 0$$

Multiplying by the integrating factor $e^{\int 3\,dt} = e^{3t}$, we get

$$e^{3t}\frac{dI}{dt} + 3e^{3t}I = 15e^{3t}$$

Example 4 – *Solution*

cont'd

$$\frac{d}{dt}(e^{3t}I) = 15e^{3t}$$

$$e^{3t}I = \int 15e^{3t}\,dt = 5e^{3t} + C$$

$$I(t) = 5 + Ce^{-3t}$$

Since $I(0) = 0$, we have $5 + C = 0$, so $C = -5$ and

$$I(t) = 5(1 - e^{-3t})$$

Example 4 – *Solution*

cont'd

(b) After 1 second the current is

$$I(1) = 5(1 - e^{-3}) \approx 4.75 \text{ A}$$

(c) The limiting value of the current is given by

$$\lim_{t \to \infty} I(t) = \lim_{t \to \infty} 5(1 - e^{-3t}) = 5 - 5 \lim_{t \to \infty} e^{-3t} = 5 - 0 = 5$$

9

Differential Equations

9.6 Predator-Prey Systems

Predator-Prey Systems

We have looked at a variety of models for the growth of a single species that lives alone in an environment. In this section we consider more realistic models that take into account the interaction of two species in the same habitat.

We will see that these models take the form of a pair of linked differential equations. We first consider the situation in which one species, called the *prey*, has an ample food supply and the second species, called the *predators*, feeds on the prey.

Examples of prey and predators include rabbits and wolves in an isolated forest, food fish and sharks, aphids and ladybugs, and bacteria and amoebas.

Predator-Prey Systems

Our model will have two dependent variables and both are functions of time. We let $R(t)$ be the number of prey (using R for rabbits) and $W(t)$ be the number of predators (with W for wolves) at time t.

In the absence of predators, the ample food supply would support exponential growth of the prey, that is,

$$\frac{dR}{dt} = kR \qquad \text{where } k \text{ is a positive constant}$$

In the absence of prey, we assume that the predator population would decline at a rate proportional to itself, that is,

$$\frac{dW}{dt} = -rW \qquad \text{where } r \text{ is a positive constant}$$

Predator-Prey Systems

With both species present, however, we assume that the principal cause of death among the prey is being eaten by a predator, and the birth and survival rates of the predators depend on their available food supply, namely, the prey.

We also assume that the two species encounter each other at a rate that is proportional to both populations and is therefore proportional to the product RW. (The more there are of either population, the more encounters there are likely to be.)

Predator-Prey Systems

A system of two differential equations that incorporates these assumptions is as follows:

$$\boxed{1} \qquad \frac{dR}{dt} = kR - aRW \qquad \frac{dW}{dt} = -rW + bRW$$

W represents the predator.
R represents the prey.

where *k*, *r*, *a*, and *b* are positive constants.

Notice that the term –*aRW* decreases the natural growth rate of the prey and the term *bRW* increases the natural growth rate of the predators.

Predator-Prey Systems

The equations in $\boxed{1}$ are known as the **predator-prey equations**, or the **Lotka-Volterra equations**.

A **solution** of this system of equations is a pair of functions $R(t)$ and $W(t)$ that describe the populations of prey and predator as functions of time.

Because the system is coupled (R and W occur in both equations), we can't solve one equation and then the other; we have to solve them simultaneously.

Unfortunately, it is usually impossible to find explicit formulas for R and W as functions of t. We can, however, use graphical methods to analyze the equations.

Example 1

Suppose that populations of rabbits and wolves are described by the Lotka-Volterra equations $\boxed{1}$ with $k = 0.08$, $a = 0.001$, $r = 0.02$, and $b = 0.00002$. The time t is measured in months.

(a) Find the constant solutions (called the **equilibrium solutions**) and interpret the answer.

(b) Use the system of differential equations to find an expression for dW/dR.

(c) Draw a direction field for the resulting differential equation in the RW-plane. Then use that direction field to sketch some solution curves.

Example 1

cont'd

(d) Suppose that, at some point in time, there are 1000 rabbits and 40 wolves. Draw the corresponding solution curve and use it to describe the changes in both population levels.

(e) Use part (d) to make sketches of R and W as functions of t.

Solution:

(a) With the given values of k, a, r, and b, the Lotka-Volterra equations become

$$\frac{dR}{dt} = 0.08R - 0.001RW$$

Example 1 – *Solution*

cont'd

$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

Both R and W will be constant if both derivatives are 0, that is,

$$R' = R(0.08 - 0.001W) = 0$$

$$W' = W(-0.02 + 0.00002R) = 0$$

One solution is given by $R = 0$ and $W = 0$. (This makes sense: If there are no rabbits or wolves, the populations are certainly not going to increase.)

Example 1 – *Solution*

cont'd

The other constant solution is

$$W = \frac{0.08}{0.001} = 80 \qquad R = \frac{0.02}{0.00002} = 1000$$

So the equilibrium populations consist of 80 wolves and 1000 rabbits.

This means that 1000 rabbits are just enough to support a constant wolf population of 80.

There are neither too many wolves (which would result in fewer rabbits) nor too few wolves (which would result in more rabbits).

Example 1 – *Solution*

cont'd

(b) We use the Chain Rule to eliminate *t*:

$$\frac{dW}{dt} = \frac{dW}{dR}\frac{dR}{dt}$$

so
$$\frac{dW}{dR} = \frac{\frac{dW}{dt}}{\frac{dR}{dt}} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

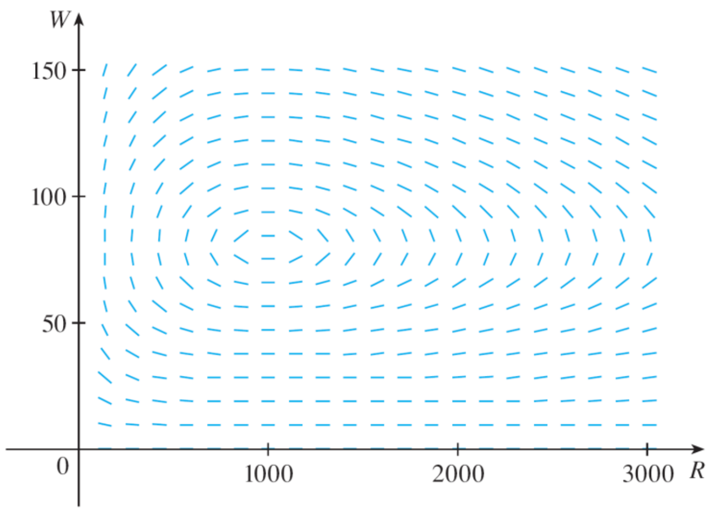
(c) If we think of *W* as a function of *R*, we have the differential equation

$$\frac{dW}{dR} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

Example 1 – *Solution*

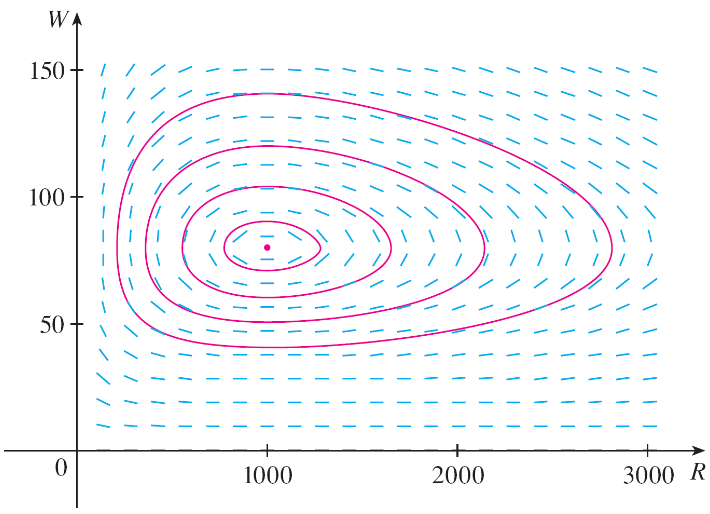
cont'd

We draw the direction field for this differential equation in Figure 1 and we use it to sketch several solution curves in Figure 2.



Direction field for the predator-prey system

Figure 1



Phase portrait of the system

Figure 2

Example 1 – *Solution*

cont'd

If we move along a solution curve, we observe how the relationship between R and W changes as time passes.

Notice that the curves appear to be closed in the sense that if we travel along a curve, we always return to the same point.

Notice also that the point (1000, 80) is inside all the solution curves.

That point is called an *equilibrium point* because it corresponds to the equilibrium solution $R = 1000$, $W = 80$.

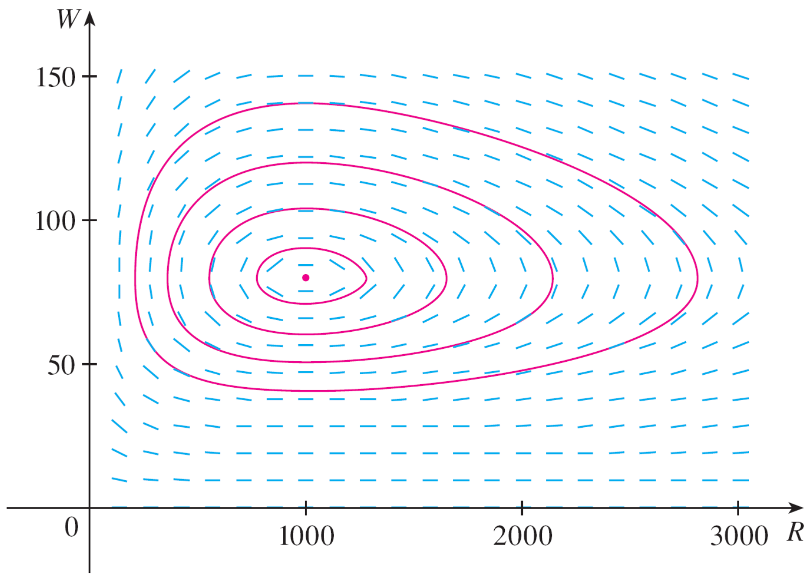
Example 1 – *Solution*

cont'd

When we represent solutions of a system of differential equations as in Figure 2, we refer to the *RW*-plane as the **phase plane**, and we call the solution curves **phase trajectories**.

So a phase trajectory is a path traced out by solutions (R, W) as time goes by.

A **phase portrait** consists of equilibrium points and typical phase trajectories, as shown in Figure 2.



Phase portrait of the system

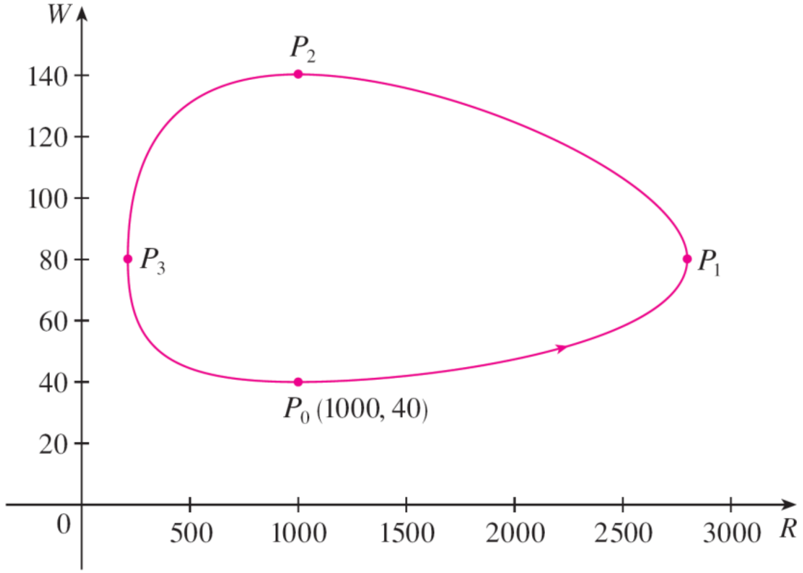
Figure 2

Example 1 – *Solution*

cont'd

(d) Starting with 1000 rabbits and 40 wolves corresponds to drawing the solution curve through the point $P_0(1000, 40)$.

Figure 3 shows this phase trajectory with the direction field removed.



Phase trajectory through (1000, 40)

Figure 3

Example 1 – *Solution*

cont'd

Starting at the point P_0 at time $t = 0$ and letting t increase, do we move clockwise or counterclockwise around the phase trajectory?

If we put $R = 1000$ and $W = 40$ in the first differential equation, we get

$$\frac{dR}{dt} = 0.08(1000) - 0.001(1000)(40)$$
$$= 80 - 40$$
$$= 40$$

Since $dR/dt > 0$, we conclude that R is increasing at P_0 and so we move counterclockwise around the phase trajectory.

Example 1 – *Solution*

cont'd

We see that at P_0 there aren't enough wolves to maintain a balance between the populations, so the rabbit population increases.

That results in more wolves and eventually there are so many wolves that the rabbits have a hard time avoiding them. So the number of rabbits begins to decline (at P_1, where we estimate that R reaches its maximum population of about 2800).

This means that at some later time the wolf population starts to fall (at P_2, where $R = 1000$ and $W \approx 140$).

Example 1 – *Solution*

cont'd

But this benefits the rabbits, so their population later starts to increase (at P_3, where $W = 80$ and $R \approx 210$). As a consequence, the wolf population eventually starts to increase as well.

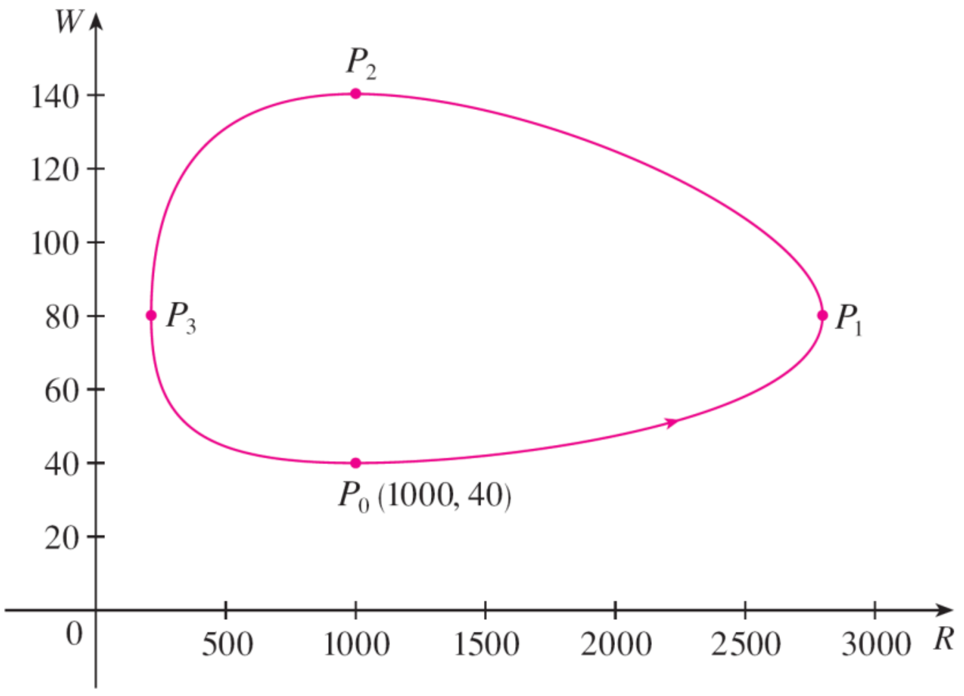
This happens when the populations return to their initial values of $R = 1000$ and $W = 40$, and the entire cycle begins again.

(e) From the description in part (d) of how the rabbit and wolf populations rise and fall, we can sketch the graphs of $R(t)$ and $W(t)$.

Example 1 – *Solution*

cont'd

Suppose the points P_1, P_2, and P_3 in Figure 3 are reached at times t_1, t_2, and t_3.



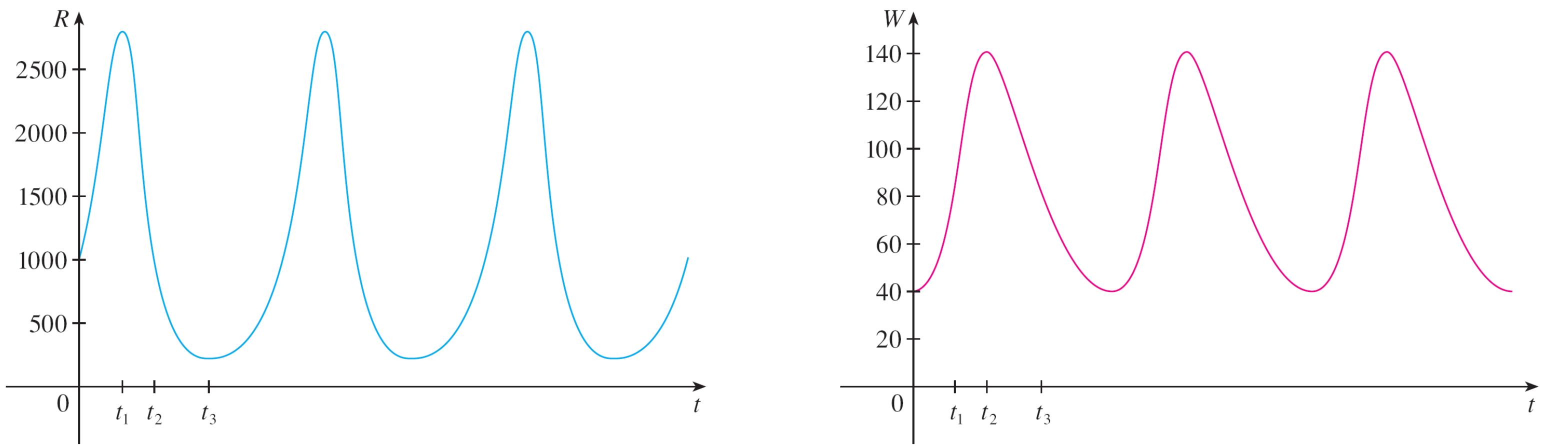
Phase trajectory through (1000, 40)

Figure 3

Example 1 – *Solution*

cont'd

Then we can sketch graphs of R and W as in Figure 4.



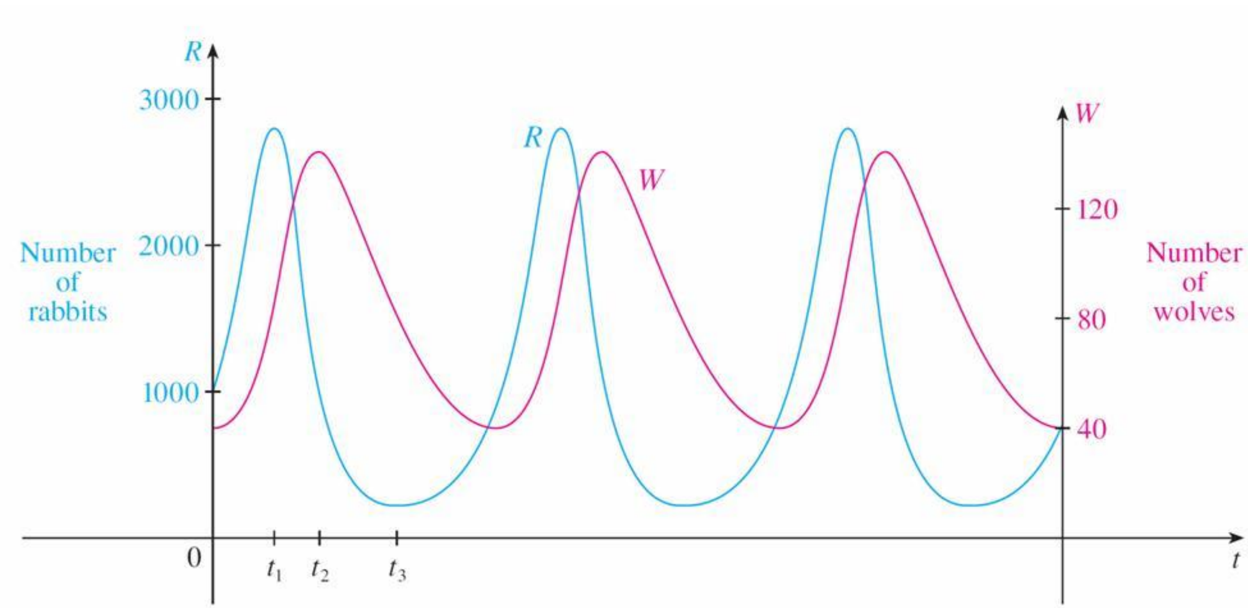
Graphs of the rabbit and wolf populations as functions of time

Figure 4

Example 1 – *Solution*

cont'd

To make the graphs easier to compare, we draw the graphs on the same axes but with different scales for *R* and *W*, as in Figure 5. Notice that the rabbits reach their maximum populations about a quarter of a cycle before the wolves.



Comparison of the rabbit and wolf populations

Figure 5