Chapter 6: Functions of several variables
MTH 201 - Business Calculus

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Outline

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where h is the person's height in centimeters and w is the person's weight in kilograms. (*Source: www.halls.md.*) Use the Haycock approximation to estimate the surface area of a person whose height is 165 cm and whose weight is 70 kg. What is your own surface area?

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Example 1. Functions of two variables: $f(x,y) = x^2 + y^4 + 1$, $g(x,y) = \sqrt{4 - x^2 - y^2}$.

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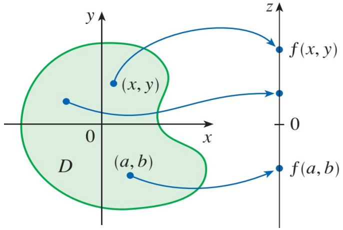
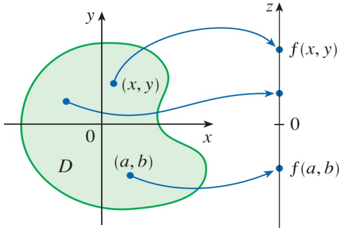
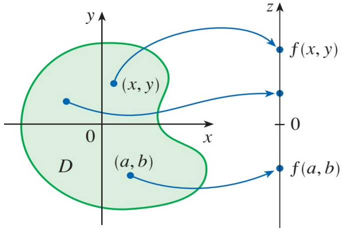




The domain D of $f(x, y)$ is the set of points in the xy-plane for which the function is defined. Its range is the set of values that f takes on, that is, $\{f(x, y) \,|\, (x, y) \in D\}$.



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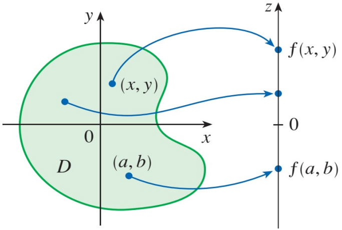

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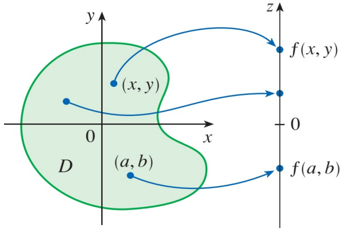

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Example 3. The total cost to Bradshaw Steel, in hundreds of dollars, for producing a spool of rolled steel is given by $C(x, y, z, w) = 4x^2 + 5y + z - \ln(w + 1)$, where x dollars are spent for labor, y dollars for raw materials, z dollars for advertising, and w dollars for machinery. This is a function of four variables.

Example 4. The *body mass index* (BMI) of a person is defined by

$$B(m,h) = \frac{m}{h^2}$$

where m is the person's mass (in kilograms) and h is the person's height (in meters). A rough guideline is that a person is underweight if the BMI is less than 18.5; optimal if the BMI lies between 18.5 and 25; overweight if the BMI lies between 25 and 30; and obese if the BMI exceeds 30. Does someone who weighs 62 kg and is 152 cm tall fall into the optimal category? What is your own BMI?

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The average adult female is 159 cm tall and weighs 68 kg (BMI $=$ 26.90).

Example 5. On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates. The National Weather Service has devised the *heat index* (also called the temperature-humidity index) to describe the combined effects of temperature and humidity. The heat index I is the perceived air temperature when the actual temperature is T and the relative humidity is H. So I is a function of T and H and we can write $I = f(T, H)$. The following table of values of I is an excerpt from a table compiled by the National Weather Service.

Relative humidity (%)

Actual temperature (°F) T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

The Fahrenheit to Celsius formula is expressed as $^oC = \frac{5}{9}(^oF - 32)$.

Geometric Interpretations

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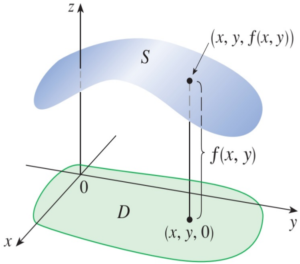

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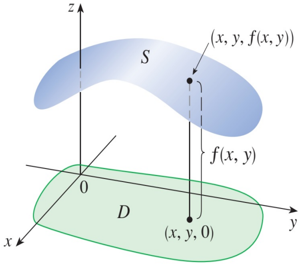

If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

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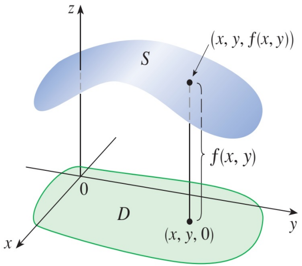

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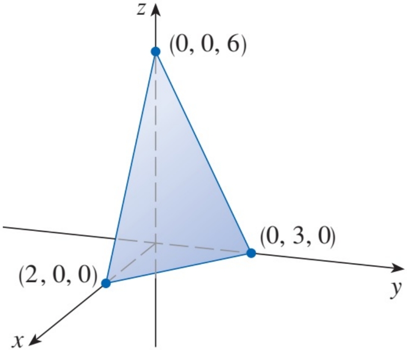

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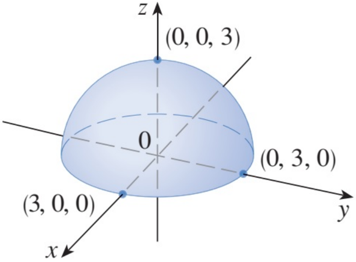

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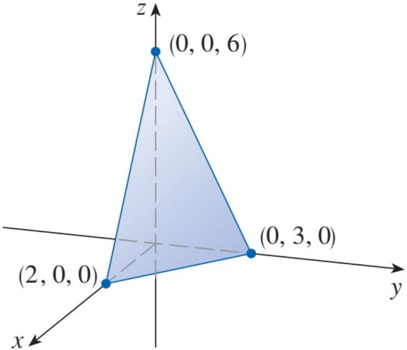

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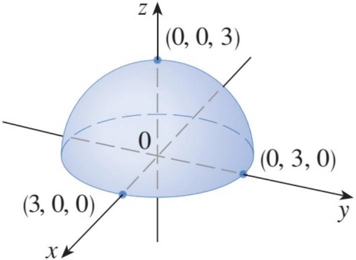

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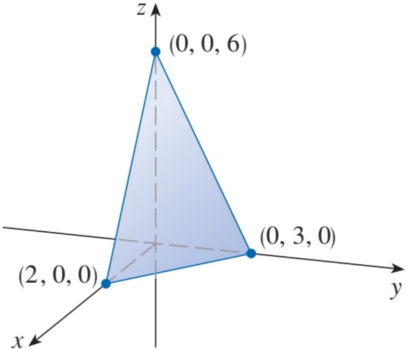

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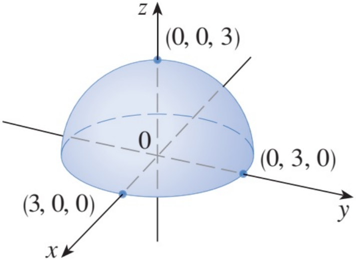

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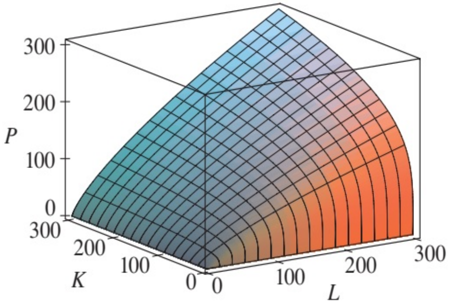
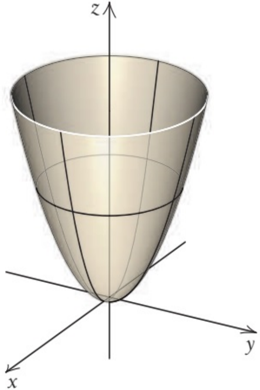
The graph of f is the plane $3x + 2y + z = 6$.
The graph of g is the upper hemisphere $x^2 + y^2 + z^2 = 9$.

Example 2. The graph of the Cobb-Douglas production function $P(L,K) = 1.01L^{0.75}K^{0.25}$ (left) and the elliptic paraboloid $z = x^2 + y^2$ (right).





The Cobb-Douglas production function

In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899–1922. They considered a simplified view of the economy in which production output is determined by the amount of labor involved and the amount of capital invested. While many other factors affect economic performance, this model proved to be remarkably accurate. The function Cobb and Douglas used to model production was of the form

$$P(L, K) = bL^{\alpha}K^{1-\alpha}$$

where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is the amount of capital invested (the monetary worth of all machinery, equipment, and buildings).

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Similarly, we find $\dfrac{\partial f}{\partial y}$ or $\dfrac{\partial z}{\partial y}$ by fixing x (treating it as a constant) and calculating the derivative with respect to y, we get $\dfrac{\partial f}{\partial y} = 0 - x^3(2y) + 3x(1) + 4y^3 = -2x^3y + 3x + 4y^3$.

Partial derivatives

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If f is a function of two variables, its partial derivatives with respect to x and y are defined by

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \quad \text{and} \quad \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}.$$

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There are many alternative notations for partial derivatives.

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Example. If $f(x,y) = x^5 - 2x^3y^2 + xy^3 + e^{2x-3y}$, find $f_x(x,y)$ and $f_y(x,y)$.

Finding Partial Derivatives

Rule for finding partial derivatives of $z = f(x, y)$.

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Answer. $f_x = 5x^4 - 6x^2y^2 + y^3 + 2e^{2x-3y}$ and $f_y = -4x^3y + 3xy^2 - 3e^{2x-3y}$.

Finding Partial Derivatives

Rule for finding partial derivatives of $z = f(x, y)$.

- To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
- To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example 1. If $f(x, y) = x^5 - 2x^3y^2 + xy^3 + e^{2x-3y}$, find $f_x(x, y)$ and $f_y(x, y)$.
Answer. $f_x = 5x^4 - 6x^2y^2 + y^3 + 2e^{2x-3y}$ and $f_y = -4x^3y + 3xy^2 - 3e^{2x-3y}$.

Example 2. For $f(x, y) = \dfrac{x}{x^2 + 2y^2} + x\ln(2x + y)$, find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$.

Example 3. For $z = f(x, y) = 5x^2y - x^4e^{2y} - \ln(x^2 + xy) + 3y^6$, find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

Example 4. For $u = x^3 + y^4 - z^2 + x^2yz^4 + e^{xy} - 2y\ln z$, find $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial z}$.

The Cobb–Douglas production function

The Cobb–Douglas production function: $p(x, y) = Ax^a y^{1-a}$, for $A > 0$ and $0 < a < 1$, where p is the number of units produced with x units of labor and y units of capital. (Capital is the cost of machinery, buildings, tools, and other supplies.) The partial derivatives $\dfrac{\partial p}{\partial x}$ and $\dfrac{\partial p}{\partial y}$ are called, respectively, the marginal productivity of labor and the marginal productivity of capital.

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Example. A publisher's production function for textbooks is given by $p(x, y) = 72x^{0.8}y^{0.2}$, where p is the number of books produced, x is units of labor, and y is units of capital. Determine the marginal productivities at $x = 90$ and $y = 50$.

Higher-order partial derivatives

Let $z = f(x, y)$ be a function of two variables. Then f_x and f_y are also functions of two variables.

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Note: f_{xy} means that we first differentiate with respect to x and then with respect to y.

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Note: f_{xy} means that we first differentiate with respect to x and then with respect to y.

Example 1. Find the second-order partial derivatives of $f(x, y) = x^4 - 3x^5y^2 + xe^{x+2y}$.

Example 2. Find the second-order partial derivatives of

$$f(x,y) = \ln(x^2 + 3y + 1), \quad g(x,y) = x^5 - 2x^3y^2 + xy^3 + e^x \ln y.$$

Higher-order partial derivatives

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$$f(x,y)=\ln(x^2+3y+1), \quad g(x,y)=x^5-2x^3y^2+xy^3+e^x\ln y.$$

Note that $f_{xy}=f_{yx}$ in these above examples.

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Theorem

Suppose that f is defined on a disk D that contains the point (x,y). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(x,y) = f_{yx}(x,y).$$

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$$f_{xy}(x,y) = f_{yx}(x,y).$$

Higher derivatives. Partial derivatives of order 3 or higher can also be defined. For instance,

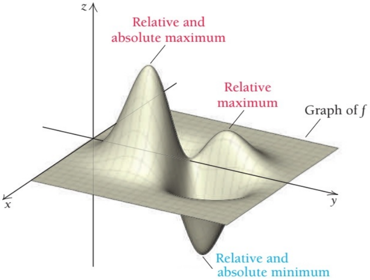
$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y^2 \partial x}.$$

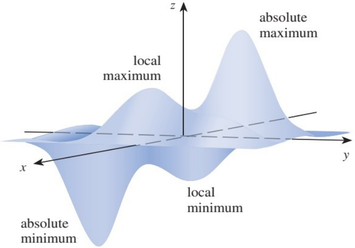
Outline

Definition

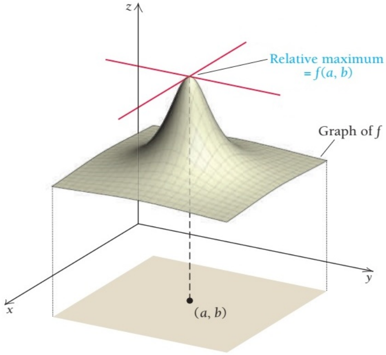
A function f of two variables:

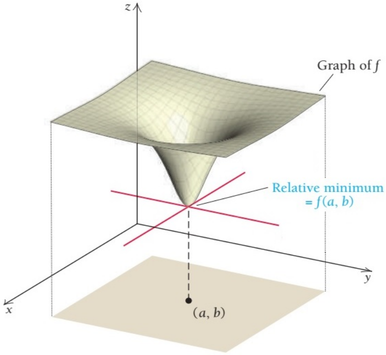
- *has a relative maximum at (a, b) if $f(x, y) \le f(a, b)$ for all points (x, y) in a region containing (a, b);*
- *has a relative minimum at (a, b) if $f(x, y) \ge f(a, b)$ for all points (x, y) in a region containing (a, b).*





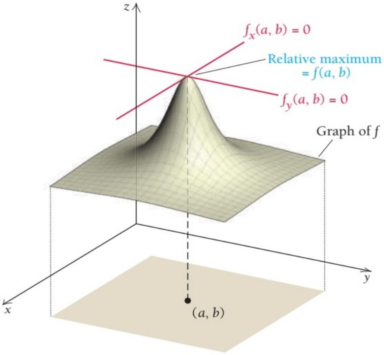
Properties at the extreme points.

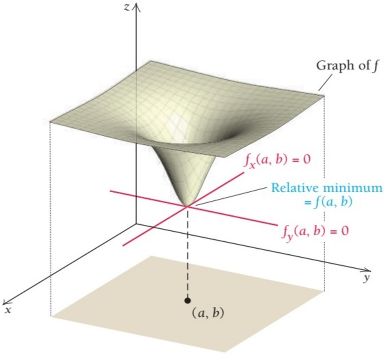




Determining maximum and minimum values

Properties at the extreme points.





Theorem

If f has a relative maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Determining maximum and minimum values

Theorem

If f has a relative maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Proof. Let $g(x) = f(x, b)$. If f has a relative maximum (or minimum) at (a, b), then g has a relative maximum (or minimum) at a, so $g'(a) = 0$ by Fermat's Theorem. But $g'(a) = f_x(a, b)$ and so $f_x(a, b) = 0$.

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Proof. Let $g(x) = f(x, b)$. If f has a relative maximum (or minimum) at (a, b), then g has a relative maximum (or minimum) at a, so $g'(a) = 0$ by Fermat's Theorem. But $g'(a) = f_x(a, b)$ and so $f_x(a, b) = 0$. Similarly, $f_y(a, b) = 0$.

A point (a, b) is called a critical point (or stationary point) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

Stationary point (a, b): $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Determining maximum and minimum values

Stationary points can be classified into one of three types

Minimum point, maximum point and saddle point.

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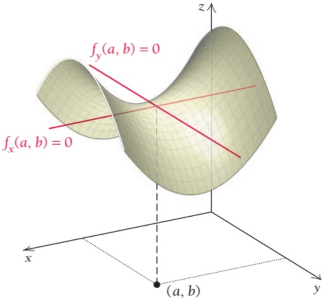

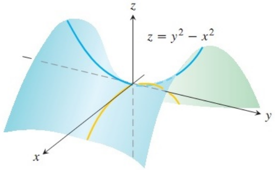

At a saddle point (a, b), $f_x(a, b) = 0$ and $f_y(a, b) = 0$ but f does not attain a relative maximum or minimum value at (a, b).

Maximum–minimum problems: The D-Test

Theorem (The D-Test)

If $f(x,y)$ is a differentiable function, to find the relative maximum and minimum values of f:

1. *Find f_x, f_y, f_{xx}, f_{yy} and f_{xy}.*
2. *Solve the system of equations $f_x = 0, f_y = 0$. Let (a,b) represent a solution.*
3. *Evaluate D, where $D = f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2$.*
4. *Then*

   a) f has a local maximum at (a,b) if $D > 0$ and $f_{xx}(a,b) < 0$.
   b) f has a local minimum at (a,b) if $D > 0$ and $f_{xx}(a,b) > 0$.
   c) f has neither a local maximum nor a local minimum at (a,b) if $D < 0$. In this case, f has a saddle point at (a,b).
   d) The test gives no information if $D = 0$ (not applicable).

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   d) The test gives no information if $D = 0$ (not applicable).

Example. Find the local maximum and minimum values and saddle points

$$f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2, \qquad g(x, y) = 1 + 3x^2 + 6xy - 12y + 2y^3.$$

See the textbook, page 563.

Exercise 1. (Maximizing profit) Safe Shades produces two kinds of sunglasses; one kind sells for \$17, and the other for \$21. The total revenue in thousands of dollars from the sale of x thousand sunglasses at \$17 each and y thousand at \$21 each is given by $R(x, y) = 17x + 21y$. The company determines that the total cost, in thousands of dollars, of producing x thousand of the \$17 sunglasses and y thousand of the \$21 sunglasses is given by

$$C(x, y) = 4x^2 - 4xy + 2y^2 - 11x + 25y - 3.$$

How many of each type of sunglasses must be produced and sold to maximize profit?

Exercise 2. (Maximizing profit) Humphrey's Medical Supply finds that its profit, P, in millions of dollars, is given by

$$P(a, n) = -5a^2 - 3n^2 + 48a - 4n + 2an + 290,$$

where a is the amount spent on advertising, in millions of dollars, and n is the number of items sold, in thousands. Find the maximum value of P and the values of a and n at which it occurs.

Maximum–minimum problems

Exercise 3. (Minimizing the cost of a container) ProHauling Services is designing an open-top, rectangular container that will have a volume of 320 ft^3. The cost of making the bottom of the container is \$5 per square foot, and the cost of the sides is \$4 per square foot. Find the dimensions of the container that will minimize total cost.

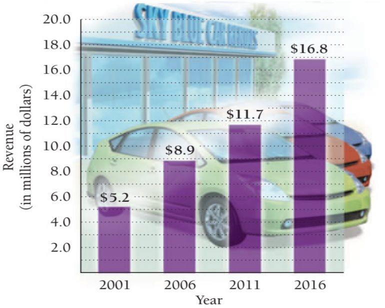
Outline

A regression line

Problems in business. Predict future sales on the basis of past data.

Problems in business. Predict future sales on the basis of past data.

Example. Sky Blue Car Rentals offers hybrid (gas–electric) vehicles and charts its revenue as shown in the following and the table below. We can use the data to predict the company's revenue for the year 2021.



Year, x	2001	2006	2011	2016	2021
Yearly Revenue, y (in millions of dollars)	5.2	8.9	11.7	16.8	?

A regression line

Example. Yearly revenue of Sky Blue Car Rentals

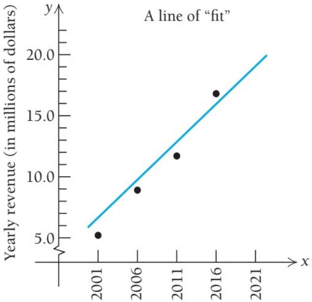
Year, x	2001	2006	2011	2016	2021
Yearly Revenue, y (in millions of dollars)	5.2	8.9	11.7	16.8	?

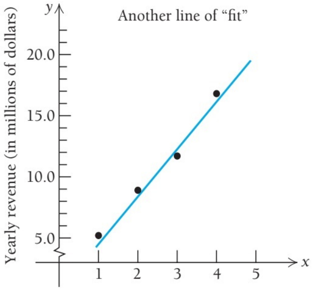
A regression line

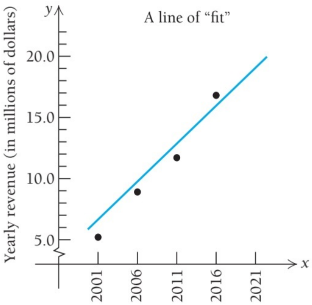
Example. Yearly revenue of Sky Blue Car Rentals

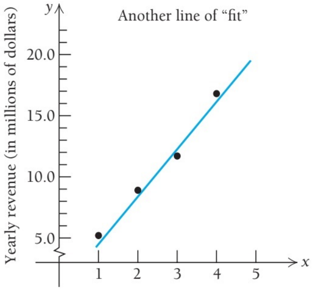
Year, x	2001	2006	2011	2016	2021
Yearly Revenue, y (in millions of dollars)	5.2	8.9	11.7	16.8	?

Solution. We plot these points and try to draw a line through them that fits. There are several ways in which this might be done. Each would give a different estimate of its revenue for 2021.





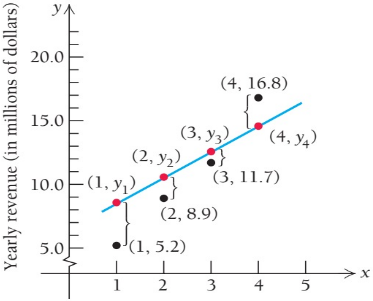




Note that the years for which revenue is given follow 5-yr increments. Thus, computations can be simplified if we use the data points $(1, 5.2)$, $(2, 8.9)$, $(3, 11.7)$ and $(4, 16.8)$, where each horizontal unit represents 5 years and $x = 1$ is 2001.

A regression line

To determine the equation of the line that "best" fits the data, we note that for each data point there is a *deviation*, or error, between its y-value and the y-value at the point on the line.



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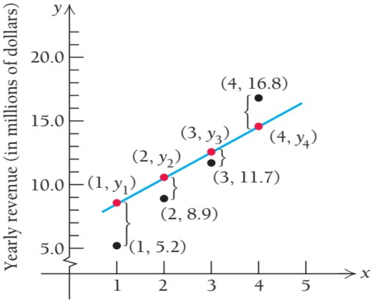
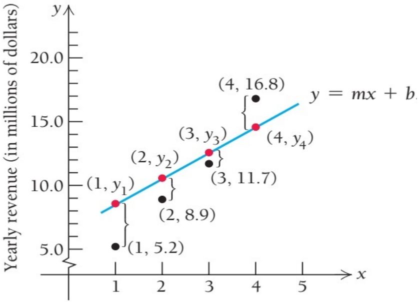
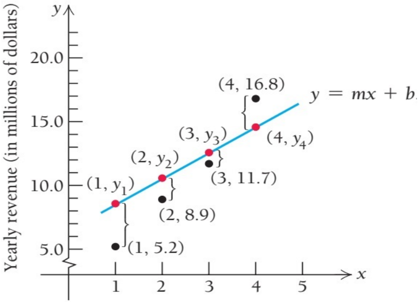


Definition

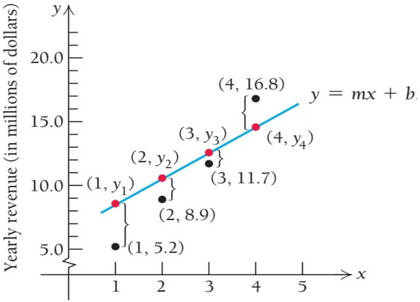
The line of best fit is the line for which the sum of the squares of the y-deviations is a minimum. This is called the regression line.



We want to minimize $(y_1 - 5.2)^2 + (y_2 - 8.9)^2 + (y_3 - 11.7)^2 + (y_4 - 16.8)^2$.



We want to minimize $(y_1 - 5.2)^2 + (y_2 - 8.9)^2 + (y_3 - 11.7)^2 + (y_4 - 16.8)^2$.
Since the points $(1, y_1)$, $(2, y_2)$, $(3, y_3)$ and $(4, y_4)$ must be solutions of $y = mx + b$, it follows that $y_i = mi + b$ for $i = 1, 2, 3, 4$.



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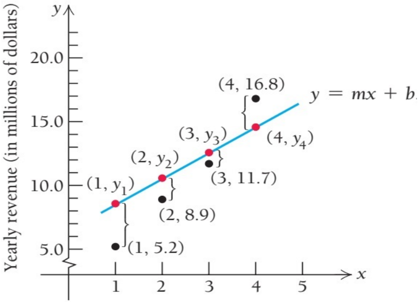
Since the points $(1, y_1)$, $(2, y_2)$, $(3, y_3)$ and $(4, y_4)$ must be solutions of $y = mx + b$, it follows that $y_i = mi + b$ for $i = 1, 2, 3, 4$. We form a function of two variables:

$$S(m, b) = (m + b - 5.2)^2 + (2m + b - 8.9)^2 + (3m + b - 11.7)^2 + (4m + b - 16.8)^2 \to \text{ min.}$$

The Least-Squares Technique



We want to minimize $(y_1 - 5.2)^2 + (y_2 - 8.9)^2 + (y_3 - 11.7)^2 + (y_4 - 16.8)^2$.
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$$S(m, b) = (m + b - 5.2)^2 + (2m + b - 8.9)^2 + (3m + b - 11.7)^2 + (4m + b - 16.8)^2 \to \text{ min.}$$

$\longrightarrow$ Apply the D-test to minimize $S(m, b)$.

The Least-Squares Technique

The function to be minimized

$$S(m,b) = (m+b-5.2)^2 + (2m+b-8.9)^2 + (3m+b-11.7)^2 + (4m+b-16.8)^2.$$

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To apply the D-test, we first find the partial derivatives:

$$\frac{\partial S}{\partial b} = \dots$$

The Least-Squares Technique

The function to be minimized

$$S(m,b) = (m+b-5.2)^2 + (2m+b-8.9)^2 + (3m+b-11.7)^2 + (4m+b-16.8)^2.$$

To apply the D-test, we first find the partial derivatives:

$$\frac{\partial S}{\partial b} = \dots = 20m + 8b - 85.2$$

The function to be minimized

$$S(m,b) = (m+b-5.2)^2 + (2m+b-8.9)^2 + (3m+b-11.7)^2 + (4m+b-16.8)^2.$$

To apply the D-test, we first find the partial derivatives:

$$\frac{\partial S}{\partial b} = ... = 20m + 8b - 85.2 \quad \text{and} \quad \frac{\partial S}{\partial m} = ...$$

The Least-Squares Technique

The function to be minimized

$$S(m, b) = (m + b - 5.2)^2 + (2m + b - 8.9)^2 + (3m + b - 11.7)^2 + (4m + b - 16.8)^2.$$

To apply the D-test, we first find the partial derivatives:

$$\frac{\partial S}{\partial b} = \ldots = 20m + 8b - 85.2 \quad \text{and} \quad \frac{\partial S}{\partial m} = \ldots = 60m + 20b - 250.6.$$

The Least-Squares Technique

The function to be minimized

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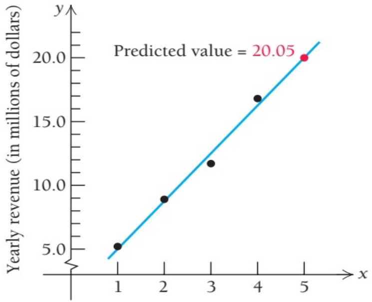
$$y = 3.76x + 1.25.$$

Using this regression line, we can now predict Sky Blue Car Rentals' yearly revenue in 2021

$$y = 3.76(5) + 1.25 = 20.05.$$

The yearly revenue in 2021 is predicted to be about $20.05 million.



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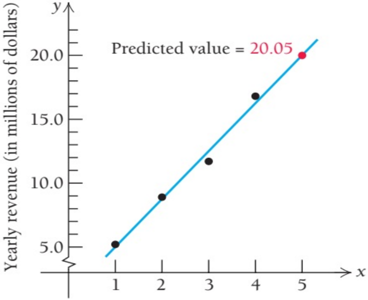

Exercise. Use the method of least squares to determine the regression line for the data points $(1, 25)$, $(2, 48)$, $(3, 76.7)$ and $(4, 104.8)$.

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To do exponential regression, we find $\tilde{y} = \ln y$ *and then find a regression line* $\tilde{y} = \tilde{b}x + \tilde{a}$. *The exponential regression curve is of the form* $y = a.b^x$, *where* $a = e^{\tilde{a}}$ *and* $b = e^{\tilde{b}}$.

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We find the natural logarithms of the given y-values:

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Outline

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The problem now is to maximize the utility function

$$U = x_1^{1/4} x_2^{3/4}$$

subject to the budgetary constraint

$$2x_1 + 3x_2 = 100.$$

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Step 3. Use the theory of stationary points of functions of one variable to optimize z.

Exercise 1. Find the maximum value of

$$z = 80x - 0.1x^2 + 100y - 0.2y^2$$

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Exercises

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Exercise 2. A firm's production function is given by

$$Q = 10K^{1/2}L^{1/4}.$$

Unit capital and labour costs are \$4 and \$5 respectively and the firm spends a total of \$60 on these inputs. Find the values of K and L which maximize output.

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Example 1. Use Lagrange multipliers to optimize $2x^2 - xy$ subject to $x + y = 12$.

Lagrange multipliers

Example 2. A consumer's utility function is given by

$$U(x_1, x_2) = 2x_1x_2 + 3x_1$$

where x_1 and x_2 denote the number of items of two goods G_1 and G_2 that are bought. Each item costs \$1 for G_1 and \$2 for G_2. Use Lagrange multipliers to find the maximum value of U if the consumer's income is \$83. Estimate the new optimal utility if the consumer's income rises by \$1.

The Lagrange multiplier λ gives the approximate change in the optimal value of f due to a 1 unit increase in M.