Chapter R Functions, Graphs and Models
MTH 201 - Business Calculus

Phan Xuân Thành

Faculty of Mathematics and Informatics
Hanoi University of Science and Technology
e-mail: thanh.phanxuan@hust.edu.vn

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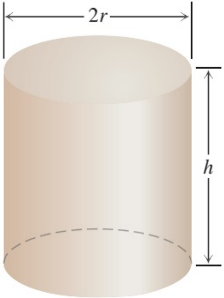

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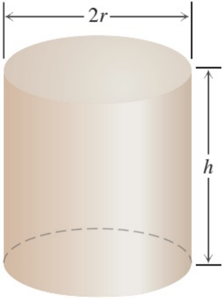

Answer: $V = \pi r^2 h$, we find that the soup can has a radius of approximately 3.4 cm and a height of 6.8 cm.

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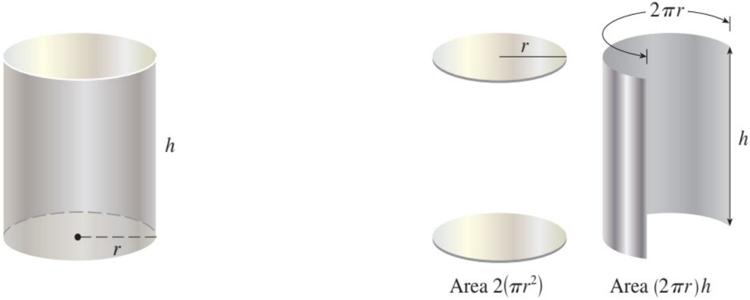

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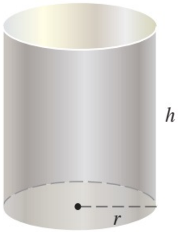

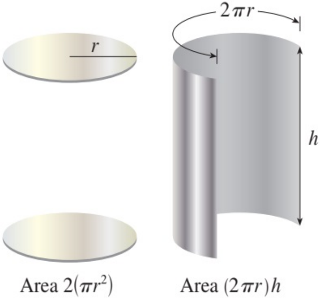

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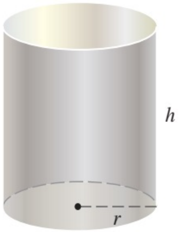

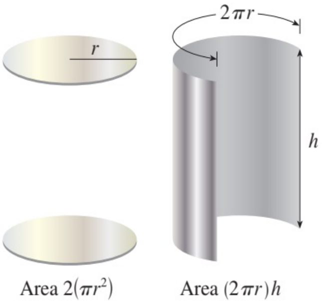

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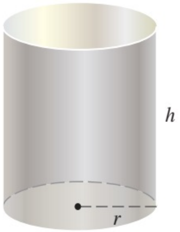

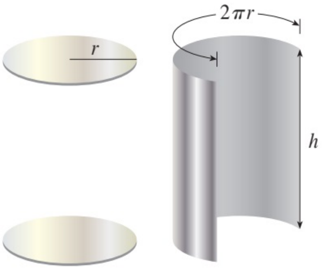

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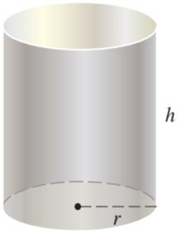

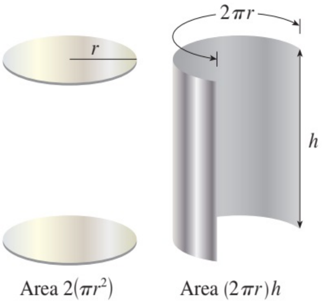

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Determine the value of r for which C has a minimum value. We need the tools of calculus to answer this.

Outline

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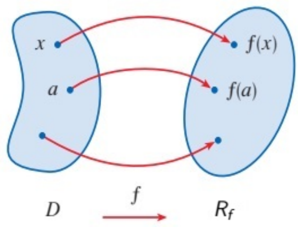

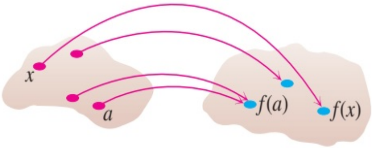

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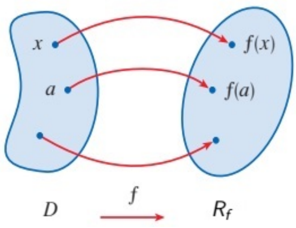

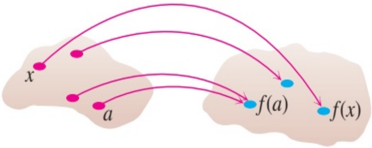

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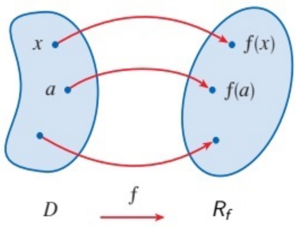

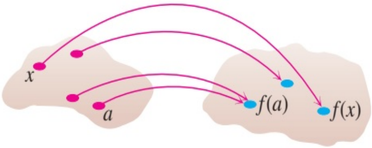

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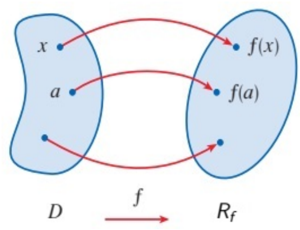

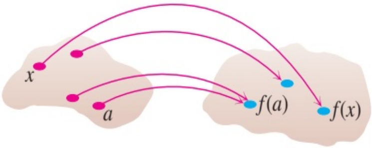

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- The set of all values of $f(x)$ as x varies throughout D is called the range of the function.

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a) $y = x^2$, b) $y = 1/x$, c) $y = \sqrt{x}$, d) $y = \sqrt{4 - x}$, e) $y = \sqrt{1 - x^2}$.

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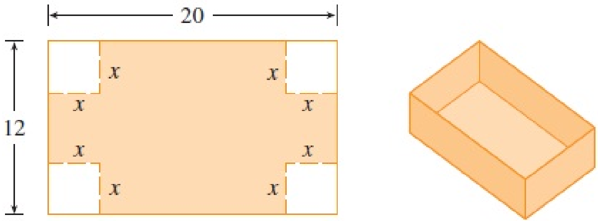

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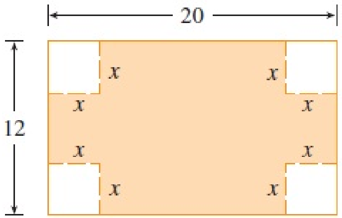

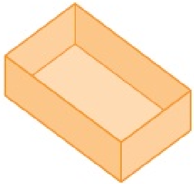

Express the volume V of the box as a function of x.

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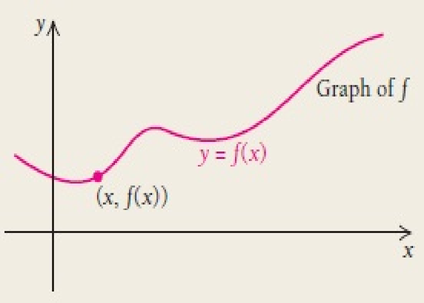

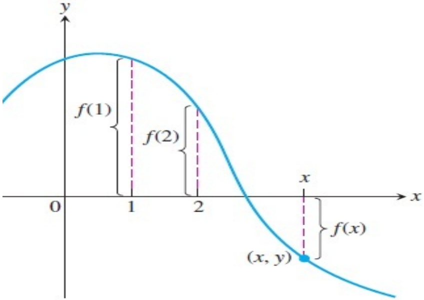

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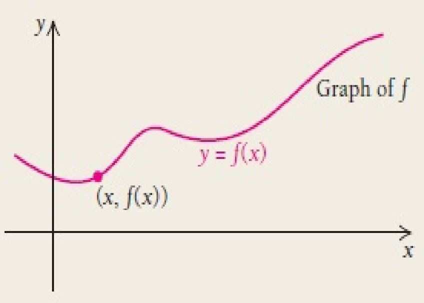

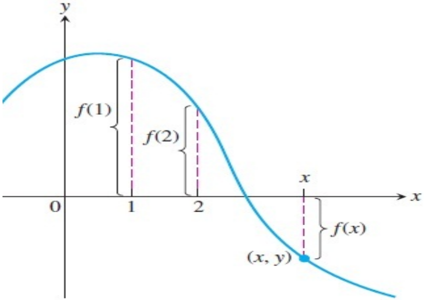

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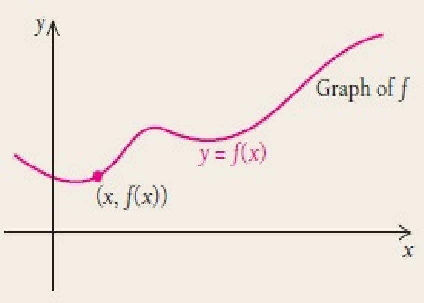

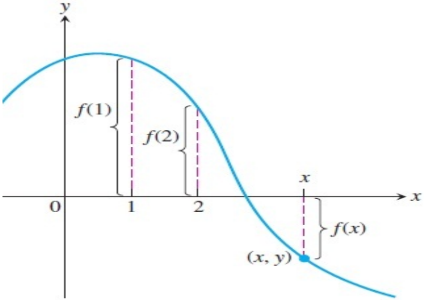

How to sketch the graph of a function? https://www.desmos.com

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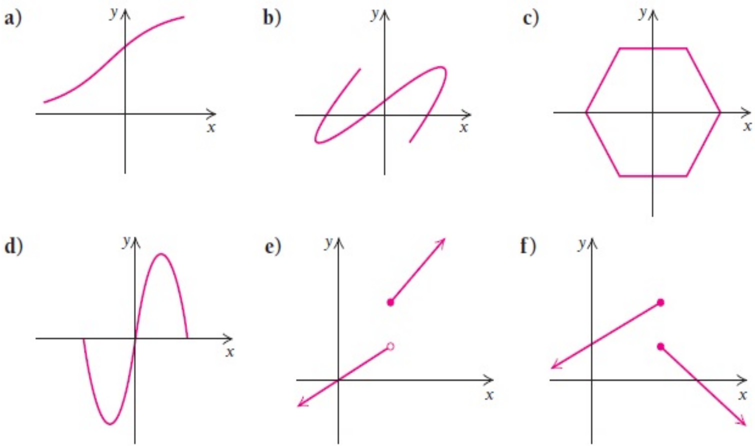

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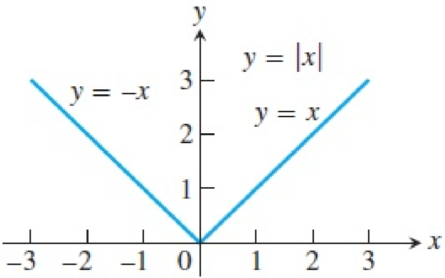

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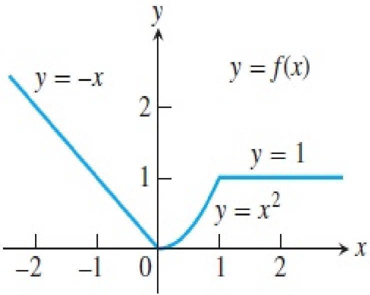

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Exercise 2. In a certain country, income tax is assessed as follows. There is no tax on income up to \$10000. Any income over \$10000 is taxed at a rate of 10%, up to an income of \$20000. Any income over \$20000 is taxed at 15%.

a) Sketch the graph of the tax rate R as a function of the income I.

b) How much tax is assessed on an income of \$14000? On \$26000?

c) Sketch the graph of the total assessed tax T as a function of the income I.

Slope and Linear Functions

Horizontal and Vertical Lines

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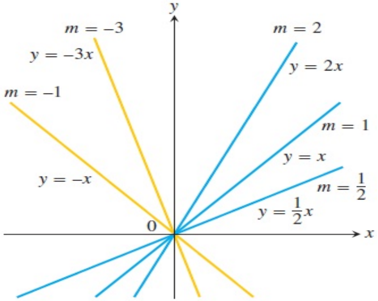

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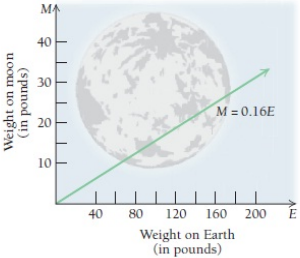

Linear Functions

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A *linear function* *is given by*

$$y = mx + b \quad \text{or} \quad f(x) = mx + b$$

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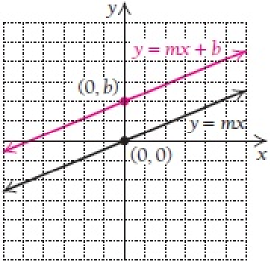

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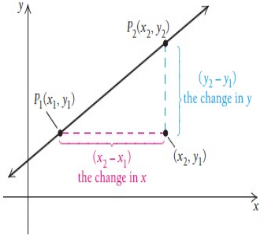

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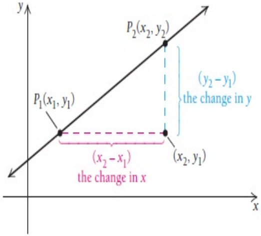

Theorem

The slope of a line containing points (x_1, y_1) *and* (x_2, y_2) *is* $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\textit{change in } y}{\textit{change in } x}$.

Applications of slope

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a) The **Total revenue** that a business receives is the product of the number of items sold and the price paid per item. Thus, if Raggs, Ltd., sells x suits at \$80 per suit, the total revenue $R(x)$, in dollars, is given by

$$R(x) = (\text{Unit price}) \times (\text{Quantity sold}) = 80x.$$

If $C(x) = 20x + 10000$, graph R and C using the same set of axes.

b) The **total profit** that business receives is the amount left after all costs have been subtracted from the total revenue. Thus, if $P(x)$ represents the total profit when x items are producted and sold, we have

$$P(x) = (\text{Total revenue}) - (\text{Total costs}) = R(x) - C(x).$$

Determine $P(x)$ and draw its graph using the same set of axes as was used for the graph in part a).

c) The company will *break even* at that value of x for which $P(x) = 0$ (that is, no profit and no loss). This is the point at which $R(x) = C(x)$. Find the break-even value of x.

Outline

Nonlinear Functions and models

Definition

A *quadratic function* f *is given by*

$$f(x) = ax^2 + bx + c, \quad \text{where } a \neq 0.$$

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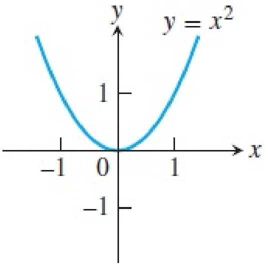

Nonlinear Functions and models

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A function p is a polynomial if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, ..., a_n$ are real constants (called the **coefficients** *of the polynomial).*

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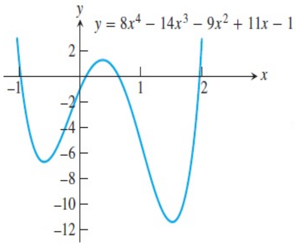

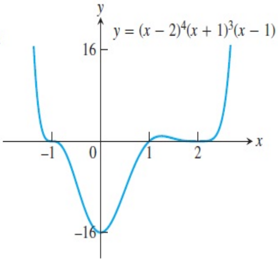

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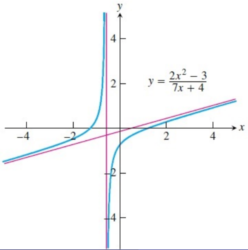

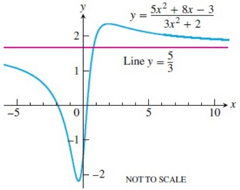

Nonlinear Functions and models

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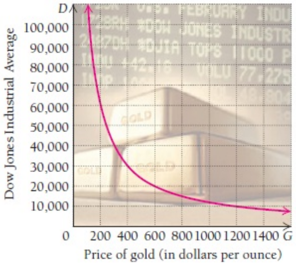

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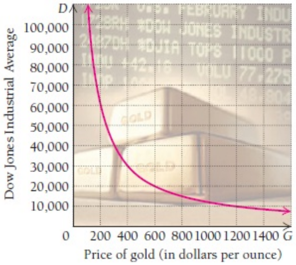

Absolute-Value Functions $y = |x|$, **Square-Root Functions** $y = \sqrt{x}$.

Supply and Demand Functions

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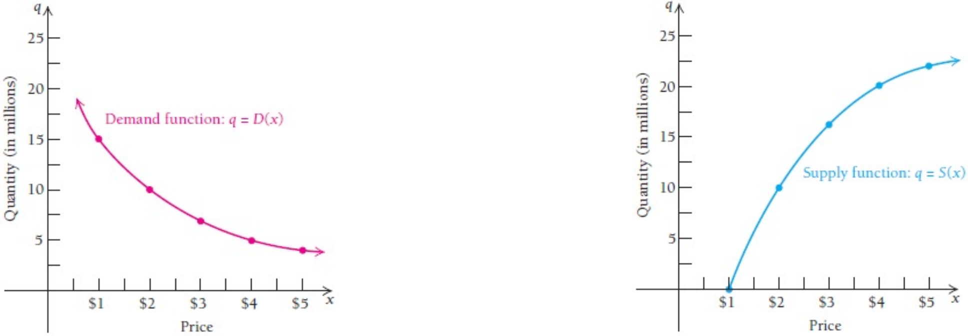

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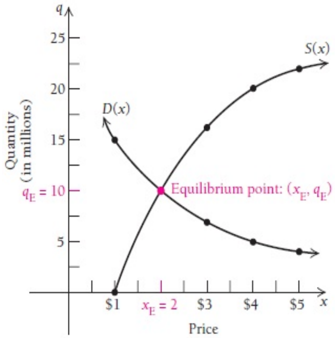

Outline

Fitting Functions to Data

Mathematical modeling and curve fitting

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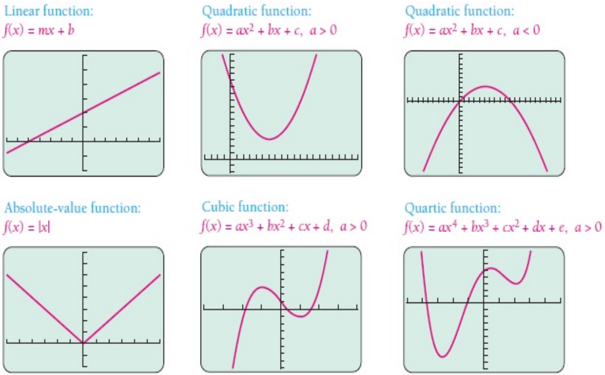

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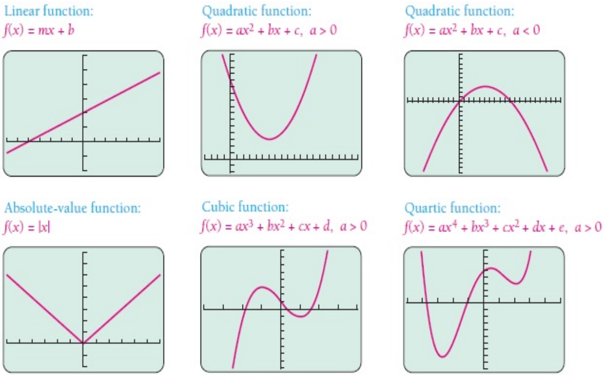

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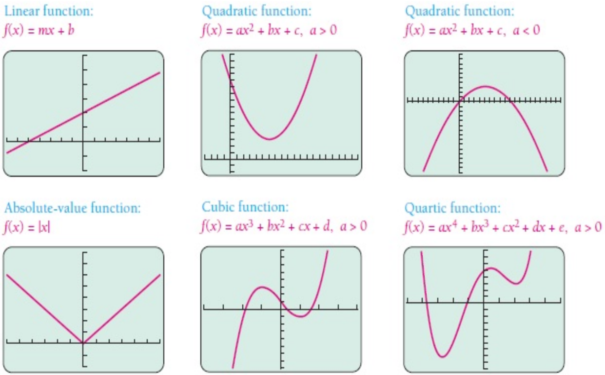

Let's consider some real-world data. How can we decide which, if any, type of function might fit the data?
One simple way is to examine a graph of the data called a **scatterplot**.

Example Business: Cable TV Subscribers. The following table shows the number of U.S. households with cable television and a scatterplot of the data. It appears that the data can be represented or modeled by a linear function.

Mathematical modeling and curve fitting

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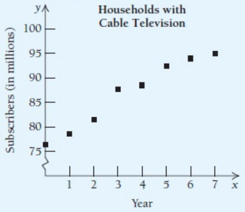
Year, x	Number of Cable Television Subscribers (in millions), y
1999, 0	76.4
2000, 1	78.6
2001, 2	81.5
2002, 3	87.8
2003, 4	88.4
2004, 5	92.4
2005, 6	94.0
2006, 7	95.0

Scatterplot



(*Source*: Nielsen Media Research.)

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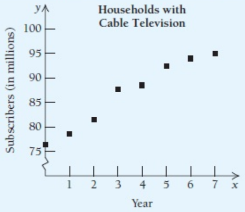

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a) Find a linear function that fits the data.

b) Use the model the predict the number of cable TV subscribers in 2014.

Slope of the line: we can choose any two of the data points...

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