Limits and continuity
MTH 201 - Business Calculus

Phan Xuân Thành

Faculty of Mathematics and Informatics
Hanoi University of Science and Technology
e-mail: thanh.phanxuan@hust.edu.vn

March 2026

Outline

1. **Limit of functions**

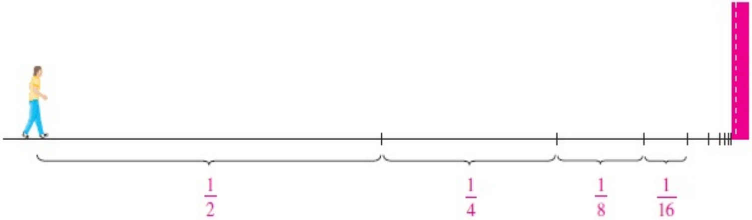
2. Continuity of functions

Zeno's paradoxes

A man standing in a room cannot walk to the wall. In order to do so, he would first have to go half the distance, then half the remaining distance, and then again half of what still remains.

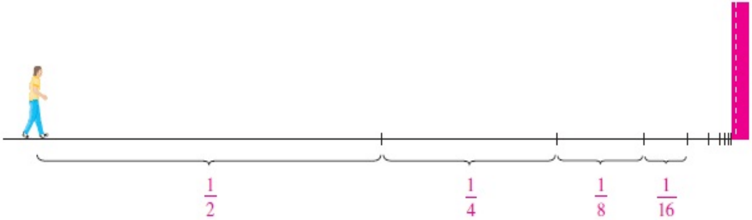
Zeno's paradoxes

A man standing in a room cannot walk to the wall. In order to do so, he would first have to go half the distance, then half the remaining distance, and then again half of what still remains.



Zeno's paradoxes

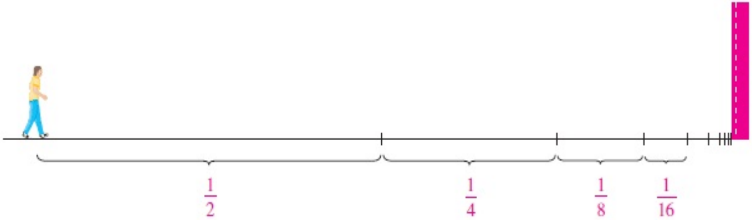
A man standing in a room cannot walk to the wall. In order to do so, he would first have to go half the distance, then half the remaining distance, and then again half of what still remains.



This process can always be continued and can never be ended.

Zeno's paradoxes

A man standing in a room cannot walk to the wall. In order to do so, he would first have to go half the distance, then half the remaining distance, and then again half of what still remains.



This process can always be continued and can never be ended.

We would say that the limit of the distance between the man and the wall is zero.

Definition

As x approaches a, the limit of $f(x)$ is L, written $\lim_{x \to a} f(x) = L$, if all values of $f(x)$ are close to L for all values of x that are sufficiently close, but not equal, to a. The limit L must be a unique real number.

Limit of functions

Definition

As x approaches a, the limit of $f(x)$ is L, written $\lim_{x\to a} f(x) = L$, if all values of $f(x)$ are close to L for all values of x that are sufficiently close, but not equal, to a. The limit L must be a unique real number.

Definition

The limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x\to a} f(x) = L,$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

Limit of functions

Definition

As x approaches a, the limit of $f(x)$ is L, written $\lim\limits_{x\to a} f(x) = L$, if all values of $f(x)$ are close to L for all values of x that are sufficiently close, but not equal, to a. The limit L must be a unique real number.

Definition

The limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x\to a} f(x) = L,$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |x-a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

Example 1. Find $\lim\limits_{x\to 1}(3x+2)$, $\lim\limits_{x\to 2} \dfrac{x^3-8}{x^2-4}$, $\lim\limits_{x\to 0} x\sin\dfrac{1}{x}$.

Limit of functions

One-side limits: *left-hand* and *right-hand limits*.

Limit of functions

One-side limits: *left-hand* and *right-hand limits*.

- $\lim\limits_{x \to a^-} f(x)$ to indicate the limit from the left (that is, where $x < a$),

$$\lim_{x \to a^-} f(x) = L \;\Leftrightarrow\; \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |f(x) - L| < \varepsilon \text{ if } a - \delta < x < a.$$

Limit of functions

One-side limits: *left-hand* and *right-hand limits*.

- $\lim\limits_{x\to a^-} f(x)$ to indicate the limit from the left (that is, where $x < a$),

$$\lim_{x\to a^-} f(x) = L \Leftrightarrow \forall\varepsilon > 0, \exists\delta > 0 \text{ such that } |f(x) - L| < \varepsilon \text{ if } a - \delta < x < a.$$

- $\lim\limits_{x\to a^+} f(x)$ to indicate the limit from the right (that is, where $x > a$),

$$\lim_{x\to a^+} f(x) = L \Leftrightarrow \forall\varepsilon > 0, \exists\delta > 0 \text{ such that } |f(x) - L| < \varepsilon \text{ if } a < x < a + \delta.$$

Limit of functions

One-side limits: *left-hand* and *right-hand limits*.

- $\lim\limits_{x\to a^-} f(x)$ to indicate the limit from the left (that is, where $x < a$),

$$\lim_{x\to a^-} f(x) = L \iff \forall\varepsilon > 0, \exists\delta > 0 \text{ such that } |f(x) - L| < \varepsilon \text{ if } a - \delta < x < a.$$

- $\lim\limits_{x\to a^+} f(x)$ to indicate the limit from the right (that is, where $x > a$),

$$\lim_{x\to a^+} f(x) = L \iff \forall\varepsilon > 0, \exists\delta > 0 \text{ such that } |f(x) - L| < \varepsilon \text{ if } a < x < a + \delta.$$

Theorem

As x approaches a, the limit of $f(x)$ is L if the limit from the left exists and the limit from the right exists and both limits are L. That is,

$$\textit{if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L, \textit{ then } \lim_{x\to a} f(x) = L.$$

One-side limits

Theorem

As x approaches a, the limit of $f(x)$ is L if the limit from the left exists and the limit from the right exists and both limits are L. That is,

$$\text{if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L, \text{ then } \lim_{x\to a} f(x) = L.$$

Limit of functions

One-side limits

Theorem

As x approaches a, the limit of $f(x)$ is L if the limit from the left exists and the limit from the right exists and both limits are L. That is,

$$\text{if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L, \text{ then } \lim_{x\to a} f(x) = L.$$

Example 2. Given $f(x) = \begin{cases} 2-x, & \text{if } x \geq 1, \\ 2x+1, & \text{if } x < 1. \end{cases}$ Find $\lim_{x\to 3} f(x)$. What is the limit $\lim_{x\to 1} f(x)$?

One-side limits

Theorem

As x approaches a, the limit of $f(x)$ is L if the limit from the left exists and the limit from the right exists and both limits are L. That is,

$$\text{if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L, \text{ then } \lim_{x\to a} f(x) = L.$$

Example 2. Given $f(x) = \begin{cases} 2-x, & \text{if } x \geq 1, \\ 2x+1, & \text{if } x < 1. \end{cases}$ Find $\lim_{x\to 3} f(x)$. What is the limit $\lim_{x\to 1} f(x)$?

Example 3. Given $g(x) = \begin{cases} 1+x, & \text{if } x \leq -1, \\ x^2, & \text{if } -1 < x < 3 \\ 9, & \text{if } x \geq 3. \end{cases}$ Find $\lim_{x\to 3} g(x)$, $\lim_{x\to 2} g(x)$ and $\lim_{x\to -1} g(x)$.

Limits involving infinity

Infinite limits

$$\lim_{x \to a} f(x) = \infty \Leftrightarrow \forall M > 0, \exists \delta > 0 \text{ such that } f(x) > M \text{ if } 0 < |x - a| < \delta.$$

Limits involving infinity

Infinite limits

$$\lim_{x \to a} f(x) = \infty \;\Leftrightarrow\; \forall M > 0, \exists \delta > 0 \text{ such that } f(x) > M \text{ if } 0 < |x - a| < \delta.$$

$$\lim_{x \to a^+} f(x) = \infty \;\Leftrightarrow\; \forall M > 0, \exists \delta > 0 \text{ such that } f(x) > M \text{ if } 0 < x - a < \delta.$$

Limits involving infinity

Infinite limits

$$\lim_{x \to a} f(x) = \infty \;\Leftrightarrow\; \forall M > 0, \exists \delta > 0 \text{ such that } f(x) > M \text{ if } 0 < |x - a| < \delta.$$

$$\lim_{x \to a^+} f(x) = \infty \;\Leftrightarrow\; \forall M > 0, \exists \delta > 0 \text{ such that } f(x) > M \text{ if } 0 < x - a < \delta.$$

Example 4. Let $f(x) = \dfrac{1}{x-3}$. Find

$$\lim_{x \to 2} f(x), \quad \lim_{x \to 3^+} f(x), \quad \lim_{x \to 3^-} f(x), \quad \lim_{x \to 3} f(x)?$$

Limits involving infinity

Infinite limits

$$\lim_{x\to a} f(x) = \infty \;\Leftrightarrow\; \forall M > 0, \exists\delta > 0 \text{ such that } f(x) > M \text{ if } 0 < |x-a| < \delta.$$

$$\lim_{x\to a^+} f(x) = \infty \;\Leftrightarrow\; \forall M > 0, \exists\delta > 0 \text{ such that } f(x) > M \text{ if } 0 < x-a < \delta.$$

Example 4. Let $f(x) = \dfrac{1}{x-3}$. Find

$$\lim_{x\to 2} f(x), \quad \lim_{x\to 3^+} f(x), \quad \lim_{x\to 3^-} f(x), \quad \lim_{x\to 3} f(x)?$$

Example 5. Find the limits

$$\lim_{x\to 0^+} (x+2)e^{1/x}, \quad \lim_{x\to 0^-} (x+2)e^{1/x}.$$

Limits involving infinity

Limits at infinity

Sometimes we need to determine limits when the inputs get larger and larger without bound, that is, as the inputs approach infinity:

$$\lim_{x\to\infty} f(x) \quad \text{or} \quad \lim_{x\to-\infty} f(x).$$

Limits at infinity

Sometimes we need to determine limits when the inputs get larger and larger without bound, that is, as the inputs approach infinity:

$$\lim_{x\to\infty} f(x) \quad \text{or} \quad \lim_{x\to-\infty} f(x).$$

Example 6. Find $\lim\limits_{x\to\infty} 2^x$, $\lim\limits_{x\to-\infty} 2^x$.

Limits at infinity

Sometimes we need to determine limits when the inputs get larger and larger without bound, that is, as the inputs approach infinity:

$$\lim_{x\to\infty} f(x) \quad \text{or} \quad \lim_{x\to-\infty} f(x).$$

Example 6. Find $\lim_{x\to\infty} 2^x$, $\lim_{x\to-\infty} 2^x$.

Example 7. Find the limits at infinity of the following functions

$$f(x) = \frac{5}{x-1}, \quad g(x) = \frac{2|x|}{x+3}, \quad h(x) = \frac{3x+1}{\sqrt{x^2+4}},$$

$$p(x) = \frac{2^x - 5}{3^x + 4}, \quad q(x) = \sin x.$$

Algebraic limits

If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$; $L, M \in \mathbb{R}$ then

- $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n.$
- $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M.$
- $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M.$
- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M},$ assuming $M \neq 0.$

Algebraic limits

If $\lim\limits_{x \to a} f(x) = L$, $\lim\limits_{x \to a} g(x) = M$; $L, M \in \mathbb{R}$ then

- $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n = L^n.$
- $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = L \pm M.$
- $\lim\limits_{x \to a} [f(x) \cdot g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x) = L \cdot M.$
- $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{L}{M},$ assuming $M \neq 0.$

Note that for any rational function f, with $a \in D_f$, then $\lim\limits_{x \to a} f(x) = f(a)$.

Algebraic limits

If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$; $L, M \in \mathbb{R}$ then

- $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n.$
- $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M.$
- $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M.$
- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M},$ assuming $M \neq 0.$

Note that for any rational function f, with $a \in D_f$, then $\lim_{x\to a} f(x) = f(a)$.

Example 8. Find the limit $\lim_{x\to 2} \dfrac{\sqrt{x^2+12}-4}{x-2}$, $\lim_{x\to 1} \dfrac{x-1}{x^n-1}$, $\lim_{x\to 0} \dfrac{\sqrt[5]{1+3x}-1}{x}$.

Algebraic limits

If $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$; $L, M \in \mathbb{R}$ then

- $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n = L^n.$
- $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M.$
- $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M.$
- $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{M},$ assuming $M \neq 0.$

Note that for any rational function f, with $a \in D_f$, then $\lim_{x \to a} f(x) = f(a)$.

Example 8. Find the limit $\lim_{x \to 2} \dfrac{\sqrt{x^2+12}-4}{x-2}$, $\lim_{x \to 1} \dfrac{x-1}{x^n-1}$, $\lim_{x \to 0} \dfrac{\sqrt[5]{1+3x}-1}{x}$.

Example 9. Given $\lim_{x \to 1} \dfrac{f(x)-5}{x-1} = 4$. Find $\lim_{x \to 1} f(x)$.

Some particular limits

Particular limits

Some particular limits

- $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e,\qquad \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e = \lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$

Particular limits

Some particular limits

- $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e,\qquad \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e = \lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$
- $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x} = 1,\qquad \lim\limits_{x\to 0}\frac{e^x-1}{x} = 1,\qquad \lim\limits_{x\to 0}\frac{a^x-1}{x} = \ln a,\ \ a>0$

Particular limits

Some particular limits

- $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e,\qquad \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e = \lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$
- $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x} = 1,\qquad \lim\limits_{x\to 0}\frac{e^x-1}{x} = 1,\qquad \lim\limits_{x\to 0}\frac{a^x-1}{x} = \ln a,\ \ a>0$

Exercise. Evaluate the limits

$$\lim_{x\to 0}\frac{9^x-5^x}{4^x-3^x},\qquad \lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{x+1},\qquad \lim_{x\to\infty}\left(\frac{x+1}{x+3}\right)^{2x}.$$

Outline

Continuity

Definition

- *A function* f *is continuous at* $x = a$ *if* $\lim_{x \to a} f(x) = f(a)$.

Continuity

Definition

- *A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.*
- *A function is continuous on an interval if and only if it is continuous at every point of the interval.*

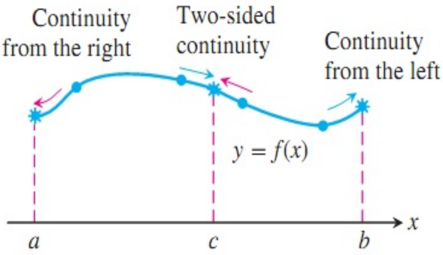
Continuity

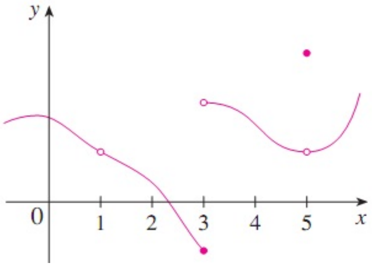
Definition

- *A function* f *is continuous at* $x = a$ *if* $\lim_{x \to a} f(x) = f(a)$.
- *A function is continuous on an interval if and only if it is continuous at every point of the interval. If* f *is not continuous at* $x = a$*, we say that* f *is discontinuous, or has a discontinuity, at* $x = a$.

Definition

- *A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.*
- *A function is continuous on an interval if and only if it is continuous at every point of the interval. If f is not continuous at $x = a$, we say that f is discontinuous, or has a discontinuity, at $x = a$.*





Continuity

Definition

- *A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.*
- *A function is continuous on an interval if and only if it is continuous at every point of the interval. If f is not continuous at $x = a$, we say that f is discontinuous, or has a discontinuity, at $x = a$.*

Exercise 1. The Copy Shoppe charges \$0.08 per copy for quantities up to and including 100 copies. For quantities above 100, the charge is \$0.06 per copy. If x represents the number of copies, the price function is $p(x) = \begin{cases} 0.08x & \text{for } x \leq 100, \\ 0.06x & \text{for } x > 100. \end{cases}$ Is $p(x)$ continuous at $x = 100$?

Continuity

Definition

- *A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.*
- *A function is continuous on an interval if and only if it is continuous at every point of the interval. If f is not continuous at $x = a$, we say that f is discontinuous, or has a discontinuity, at $x = a$.*

Exercise 1. The Copy Shoppe charges \$0.08 per copy for quantities up to and including 100 copies. For quantities above 100, the charge is \$0.06 per copy. If x represents the number of copies, the price function is $p(x) = \begin{cases} 0.08x & \text{for } x \leq 100, \\ 0.06x & \text{for } x > 100. \end{cases}$ Is $p(x)$ continuous at $x = 100$?

Exercise 2. Investigate the continuity of

a) $f(x) = \begin{cases} \dfrac{\sqrt{1+2x} - 1}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$

b) $g(x) = \begin{cases} x^3 - 3x & \text{if } x \geq 1, \\ a - 4x & \text{if } x < 1. \end{cases}$

Average rates of change

If a car travels 110 mi in 2 hr, its *average rate of change (speed)* is 110 mi/2 hr, or 55 mi/hr (55 mph). Suppose you accelerate on a freeway and, glancing at the speedometer, you see that at that instant your *instantaneous rate of change* is 55 mph. These are two quite different concepts. The first you are probably familiar with. The second involves ideas of limits and calculus.

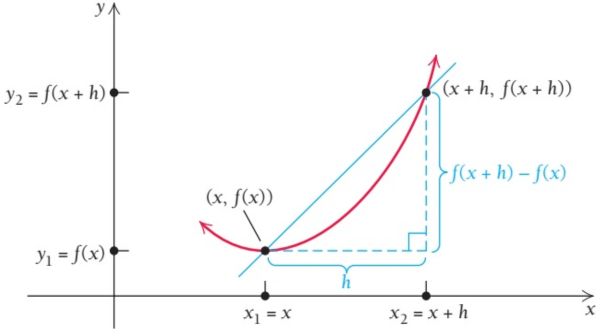
Definition

The average rate of change of y with respect to x, as x changes from x_1 to x_2, is the ratio of the change in output to the change in input:

$$\frac{y_2 - y_1}{x_2 - x_1}, \quad \text{where } x_1 \neq x_2.$$

Difference quotient as average rates of change

Let $y = f(x)$. we write $x_1 = x$ and $x_2 = x + h$



$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h}$$ difference quotient.

Example. Find the simplified difference quotient for each function listed.

a) $f(x) = x^2$ b) $f(x) = \dfrac{1}{x^2}$ c) $f(x) = \sqrt{x}$.