Differentiation
MTH 201 - Business Calculus

Phan Xuân Thành

Faculty of Mathematics and Informatics
Hanoi University of Science and Technology
e-mail: thanh.phanxuan@hust.edu.vn

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Outline

Average rates of change

If a car travels 110 mi in 2 hr, its *average rate of change (speed)* is 110 mi/2 hr, or 55 mi/hr (55 mph). Suppose you accelerate on a freeway and, glancing at the speedometer, you see that at that instant your *instantaneous rate of change* is 65 mph. These are two quite different concepts. The first you are probably familiar with. The second involves ideas of limits and calculus.

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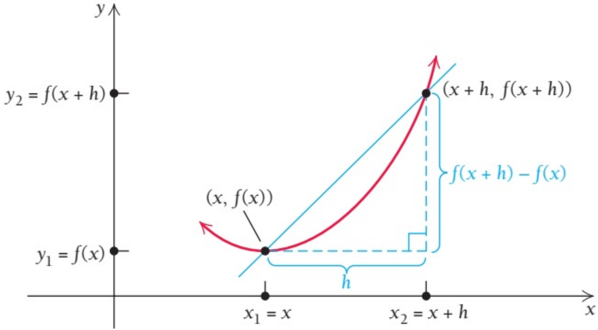
Definition

The average rate of change of y with respect to x, as x changes from x_1 to x_2, is the ratio of the change in output to the change in input:

$$\frac{y_2 - y_1}{x_2 - x_1}, \quad \textit{where } x_1 \neq x_2.$$

Let $y = f(x)$. we write $x_1 = x$ and $x_2 = x + h$



$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h}$$ difference quotient.

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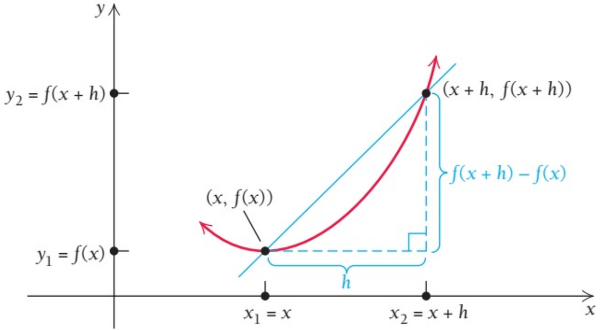

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Example 1. Find the simplified difference quotient for each function listed.

a) $f(x) = x^2$ b) $f(x) = \dfrac{1}{x^2}$ c) $f(x) = \sqrt{x}$.

Difference quotient as average rates of change

Example 2. For $f(x) = x^3$, find a simplified form of the difference quotient.

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When $x = 2$ and $h = 0.1$,

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Tangent lines

Consider secant lines through P and neighboring points Q_1, Q_2, Q_3, Q_4, and so on. As the Q's approach P, the secant lines approach line T.

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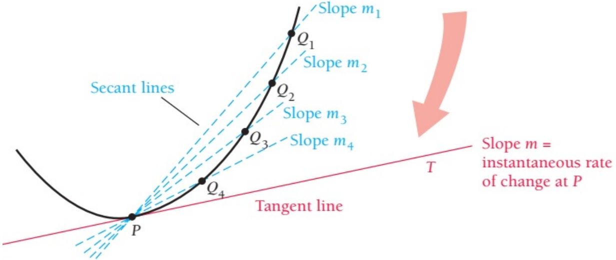

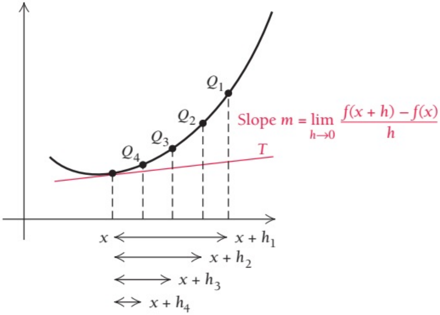

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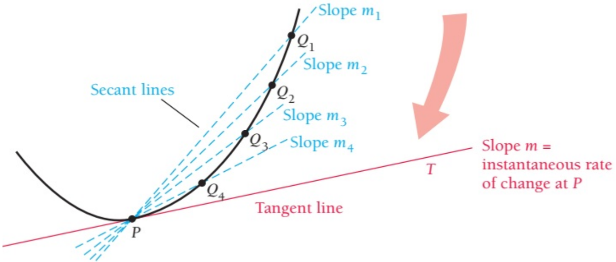

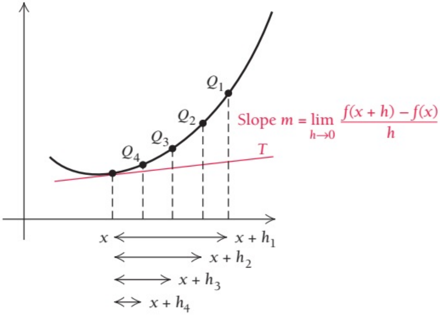

The slope of the tangent line at $(x, f(x))$ is $m = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.

Definition

For a function $y = f(x)$, its derivative at x is the function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided that the limit exists. If $f'(x)$ exists, then we say that f is differentiable at x. We sometimes call f' the derived function.

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Example. Find the derivative of $f(x) = x^3$, $g(x) = \sqrt{x}$, $p(x) = \dfrac{1}{x}$, $q(x) = \ln x$.

Exercise 1. Let f be a piecewise-defined function given by

$$f(x) = \begin{cases} x^3, & \text{for } x \leq 1, \\ 3x - 2, & \text{for } x > 1. \end{cases}$$

a) Verify that f is continuous at $x = 1$.

b) Is f differentiable at $x = 1$? Explain why or why not.

Exercise 2. Let g be a function given by

$$g(x) = \begin{cases} 2x^2 - x, & \text{for } x \leq 3, \\ mx + b, & \text{for } x > 3. \end{cases}$$

Determine the values of m and b that make $g(x)$ differentiable at $x = 3$?

Leibniz notation

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Historical Note: *The German mathematician and philosopher Gottfried Wilhelm von Leibniz (1646–1716) and the English mathematician, philosopher, and physicist Sir Isaac Newton (1642–1727) are both credited with the invention of calculus, though each worked independently. Newton used the dot notation* $\dot{y}$ *for* dy/dt*, where* y *is a function of time; this notation is still used, though it is not as common as Leibniz notation.*

Leibniz - Newton

Gottfried Wilhelm von Leibniz (1646 - 1716)

Sir Isaac Newton (1642 - 1727)

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The derivative of a constant times a function is the constant times the derivative of the function

$$\frac{d}{dx}[c\,f(x)] = c\frac{d}{dx}f(x).$$

Theorem (The sum-difference rule)

Sum. *The derivative of a sum is the sum of the derivatives*

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Differentiation techniques

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Example 2. Use the derivative to help explain why $f(x) = x^5 + x^3$ increases for all x in $(-\infty, \infty)$.

Exercise. (Growth of a baby). The median weight of a boy whose age is between 0 and 36 months is approximated by

$$w(t) = 8.15 + 1.82t - 0.0596t^2 + 0.000758t^3,$$

where t is in months and w is in pounds.

(Source: Centers for Disease Control. Developed by the National Center for Health Statistics in collaboration with the National Center for Chronic Disease Prevention and Health Promotion, 2000, Rechecked 2014)

Use this approximation to find the following:

a) The rate of change of weight with respect to time.

b) The weight of a boy at age 10 months.

c) The rate of change of a boy's weight with respect to time at age 10 months.

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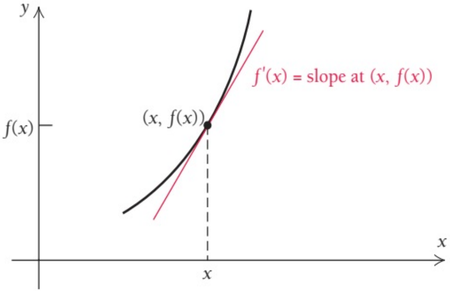

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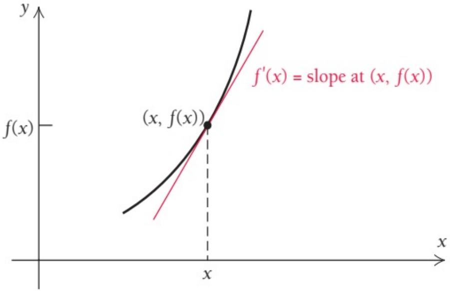

Example 1. Find the points on the graph of $f(x) = -x^3 + 6x^2$ at which

a) the tangent line is horizontal. b) the tangent line has slope 9.

Example 2. Find the equation of the tangent line to the graph of $f(x) = \sqrt[3]{x^2} + x$ at $(-1; 0)$.

Example 3. Find the equation of the tangent line to the graph of $g(x) = \dfrac{8}{\sqrt{x-2}}$ at $(6; 4)$.

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Example. Use the derivative to help explain why $f(x) = x^3 + ax$ increases for all x in $(-\infty, \infty)$ when $a \geq 0$ but not when $a < 0$.

The product rule

The product rule

Theorem

Let $F(x) = f(x).g(x)$. *Then*

$$F'(x) = \frac{d}{dx}[f(x).g(x)] = f(x)\left[\frac{d}{dx}g(x)\right] + g(x)\left[\frac{d}{dx}f(x)\right].$$

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If $Q(x) = \dfrac{N(x)}{D(x)}$, *then*

$$Q'(x) = \frac{D(x)N'(x) - N(x)D'(x)}{[D(x)]^2}.$$

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$N(x)$ numerator, $D(x)$ denominator.

Differentiation techniques

Example. Differentiate

a) $f(x) = (x^2 + 3x)\ln x$

b) $g(x) = x\sqrt{4 - x^2}$

c) $h(x) = \dfrac{x^3 + 2}{x^2 - 1}$

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If $P(x)$ is the profit from the sale of x items, then the average profit from selling x items is $\dfrac{P(x)}{x}$.

Example (Exercise 60, Textbook). Cruzin' Boards has found that the cost, in dollars, of producing x skateboards is given by $C(x) = 900 + 18x^{0.7}$. If the revenue from the sale of x skateboards is given by $R(x) = 75x^{0.8}$, find the rate at which the average profit per skateboard is changing when 20 skateboards have been built and sold.

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Theorem (The extended power rule)

Suppose that $g(x)$ is a differentiable function of x. Then, for any real number k,

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Example. Differentiate

$$f(x) = (x^3+x+1)^5, \quad g(x) = \sqrt[3]{x^2+x+1}, \quad h(x) = \left(\frac{x^2}{8}+x-\frac{1}{x}\right)^4, \quad r(x) = \sqrt[4]{\frac{x+1}{x-2}}.$$

Outline

The chain rule

Theorem (The chain rule)

The derivative of the composition $f \circ g$ *is given by*

$$\frac{d}{dx}[(f \circ g)(x)] = \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x).$$

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Example 1.

$$\begin{pmatrix} \text{Change in profits} \\ \text{with respect to time} \end{pmatrix} = \begin{pmatrix} \text{Change in profits with respect} \\ \text{to number of games produced} \end{pmatrix} \cdot \begin{pmatrix} \text{Change in number of games} \\ \text{produced with respect to time} \end{pmatrix}.$$

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Example 1. Game Boss Video's profit, in dollars, is given by $y = f(u) = 0.1u^2 - 500$, where u is the number of units sold. Its sales are given by $u = g(x) = 125x + 40$, where x is the number of days. Find the rate at which profit is changing on the 5th day.

Answer.

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Answer. We have $y = (f \circ g)(x)$... For $x = 5$, we have $\dfrac{dy}{dx} = \$16625$ per day.

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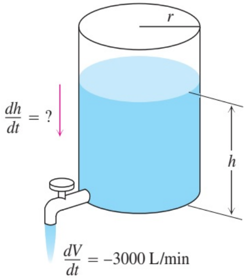

How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L$/$min?

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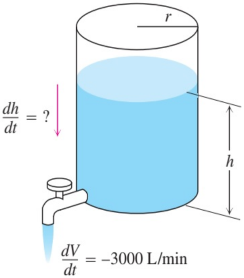

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Answer. Let h and V be the height and the volume of the fluid respectively.

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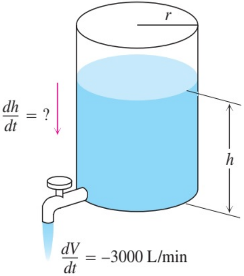
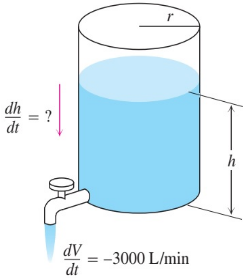


How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L/min?

Answer. Let h and V be the height and the volume of the fluid respectively. We have $V = 1000\pi r^2 h$ (because a cubic meter contains 1000L)...

Example 2. The rate of change of fluid volume in a cylindrical tank is related to the rate of change of fluid level in the tank.



How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L/min?

Answer. Let h and V be the height and the volume of the fluid respectively. We have $V = 1000\pi r^2 h$ (because a cubic meter contains 1000L)... The fluid level will drop at the rate of $3/(\pi r^2)$ m/min.

Exercise 1. (Cost, revenue, and profit) A company can manufacture x items at a cost of $c(x)$ thousand dollars, a sales revenue of $r(x)$ thousand dollars, and a profit of $p(x) = r(x) - c(x)$ thousand dollars. Find dc/dt, dr/dt, and dp/dt for the following values of x and dx/dt.

a) $r(x) = 9x$, $c(x) = x^3 - 6x^2 + 15x$, and $dx/dt = 0.1$ when $x = 2$

b) $r(x) = 70x$, $c(x) = x^3 - 6x^2 + 45/x$, and $dx/dt = 0.05$ when $x = 1.5$.

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Exercise 2. (Changing dimensions in a rectangle) The length ℓ of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $\ell = 12$ cm and $w = 5$ cm, find the rates of change of $(\mathbf{a})$ the area, $(\mathbf{b})$ the perimeter, and $(\mathbf{c})$ the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

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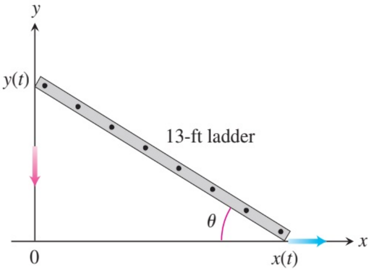
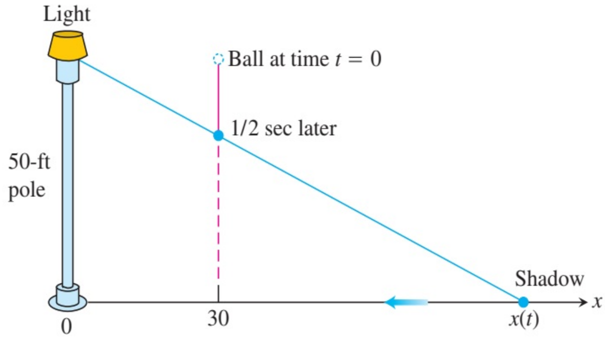
Exercise 3. (The radius of an inflating balloon) A spherical balloon is inflated with helium at the rate of 100π ft^3/min. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing?

Exercise 4. (A sliding ladder) A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.



a) How fast is the top of the ladder sliding down the wall then?
b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?
c) At what rate is the angle θ between the ladder and the ground changing then?

Exercise 5. (A moving shadow) A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light. (See accompanying figure.) How fast is the shadow of the ball moving along the ground $1/2$ sec later? (Assume the ball falls a distance $s = 16t^2$ ft in t sec.)



The chain rule

Exercise 6. For

$$y = \frac{u-1}{u+1} \quad \text{and} \quad u = \sqrt[3]{x}.$$

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$$f(u) = 2\sqrt[3]{\frac{u}{u^2+1}} \quad \text{and} \quad g(x) = u = \frac{x}{x+1}.$$

Find $(f \circ g)'(x)$.

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Exercises. Page 164: 97-109, page 170: 23-26 (see textbook).

Outline

Higher-order derivatives

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For $f(x) = x^5 + 3x^3 - 2x$, then

$$y'' = f''(x) = 20x^3 + 18x.$$

Higher-order derivatives

Continuing in this manner, we have

$$
\begin{aligned}
f'''(x) &= 60x^2 + 18, && \textit{The third derivative of } f,\\
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Note $f^{(n)}(x) = \dfrac{d^n}{dx^n} f(x)$.

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Exercise 4. If $f(x) = \dfrac{x^{50} + x^{49} + 2}{x + 1}$, evaluate $f^{(50)}(2)$. Express your answer using factorial notation.

Velocity and acceleration

Definition

The velocity *of an object that is $s(t)$ units from a starting point at time t is given by*

$$\textit{Velocity} = v(t) = s'(t) = \lim_{h\to 0} \frac{s(t+h) - s(t)}{h}.$$

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The velocity of the object is given by $v(t) = s'(t) = 9.81t$.

Definition

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Velocity, acceleration and jerk

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Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3 s}{dt^3}.$$

Example 1. Galileo's experiments with free fall

$$s(t) = \frac{1}{2}gt^2 = 4.905t^2,$$

where g is the acceleration due to Earth's gravity:

$$g = \frac{d^2}{dt^2}s(t) = 9.81m/sec^2(\text{ meters per second squared}).$$

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The jerk of gravity is zero.

Higher-order derivatives

Example 2. Assume that the function $N(t) = \dfrac{250000t^2}{(2t+1)^2}$, $t > 0$, represented the quantity sold N of a product after t weeks on the market. Its derivative is

$$N'(t) = \frac{500000t}{(2t+1)^3}.$$

$N'(t)$ represented the *rate of change* in number of units sold per week. Find $N''(t)$.