Chapter 3: Exponential and Logarithmic Functions
MTH 201 - Business Calculus

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Outline

1. **Exponential Functions**
2. Logarithmic Functions
3. Applications: Uninhibited and limited growth models

Exponential Functions

Definition

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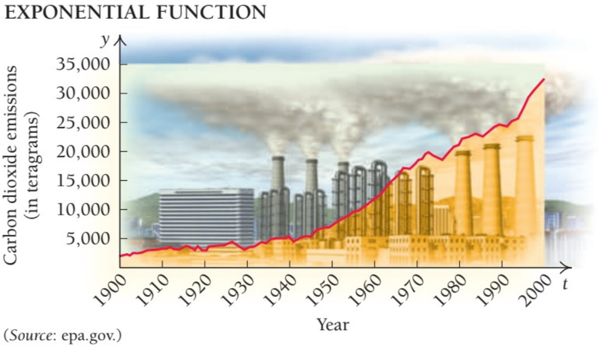

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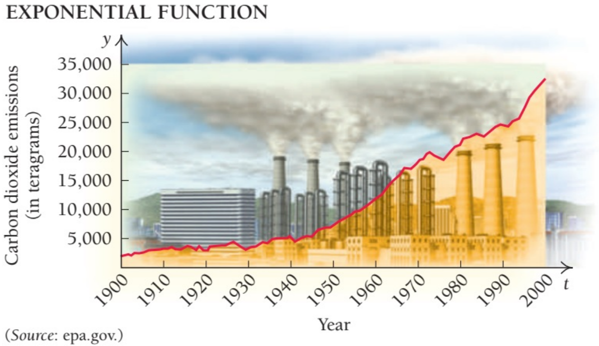

Example. The functions $f(x) = 10^x$, $g(x) = e^x$.

Exponential Functions

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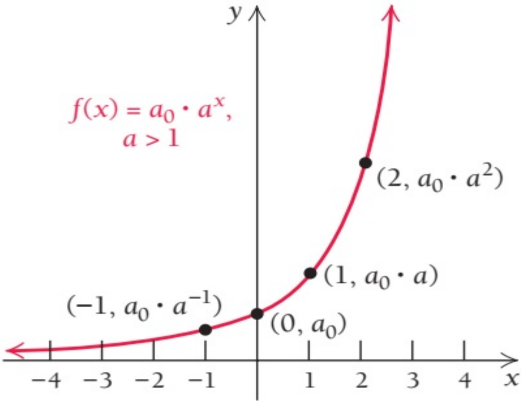

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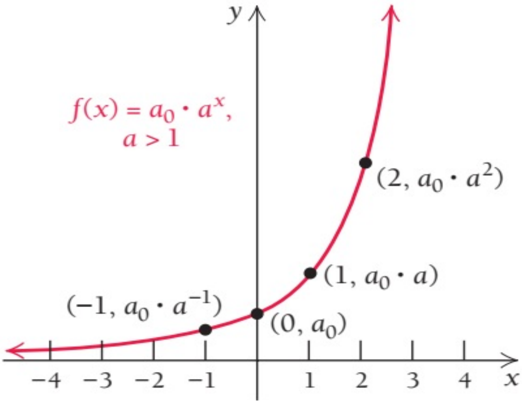

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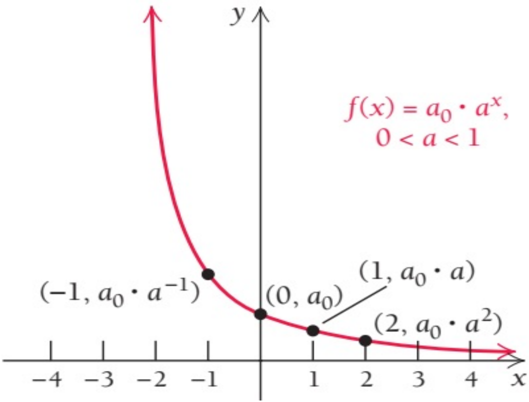

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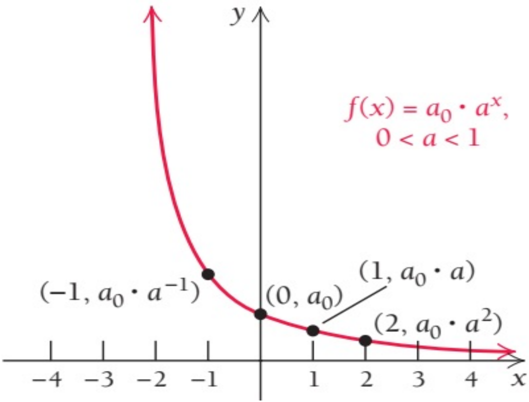

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Answer. After 3.564 years.

Example 4. A country's annual GNP (gross national product), currently at \$25000 million, is predicted to grow by 3.5% each year. The population is expected to increase by 2% a year from its current level of 40 million. After how many years will GNP per capita (that is, GNP per head of population) reach \$700?

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Answer. $r = 7.67\%$.

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The number is denoted e, in honor of Leonhard Euler (pronounced "Oiler"), the great Swiss mathematician (1707–1783) who did groundbreaking work with it.

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Example. Find dy/dx

a) $y = e^{x^2}$ b) $y = (x^3 + 2)e^{2x}$ c) $y = \dfrac{e^{3x}}{x^6}$ d) $y = (5x^2 - 8x)e^{x^2 - x}$.

Exponential Functions

Exercise 1. Perriot's Restaurant purchased kitchen equipment on January 1, 2014. The value of the equipment decreases by 15% every year. On January 1, 2016, the value was \$14450.

a) Find an exponential model for the value, V, of the equipment, in dollars, t years after January 1, 2016.

b) What is the rate of change in the value of the equipment on January 1, 2016?

c) What was the original value of the equipment on January 1, 2014?

d) How many years after January 1, 2014 will the value of the equipment have decreased by half?

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Exercise 2. The concentration C, in parts per million, of a medication in the body t hours after ingestion is given by the function $C(t) = 10t^2e^{-t}$.

- **a)** Find the concentration after 0 hr, 1 hr, 2 hr, 3 hr, and 10 hr.
- **b)** Sketch a graph of the function for $0 \leq t \leq 10$.
- **c)** Find the rate of change of the concentration, $C'(t)$.
- **d)** Find the maximum value of the concentration and the time at which it occurs.

Exercise 3. Find local maximum and local minimum of the function $f(x) = (x^2 - 3)e^x$.

Exercise 4. Find the minimum value of $f(x) = xe^x$ over $[-2, 0]$. (Textbook, page 322)

Exercise 5. Find the relative extrema of $f(x) = (x+1)e^{-x}$. State where the function is increasing or decreasing, where its graph is concave up or concave down.

Exercise 6. The demand function of a good is given by $D(x) = 1000e^{-0.2x}$, where x is the price in dollars. If fixed costs are 100 and the variable costs are 2 per unit, show that the profit function is given by

$$p(x) = 1000xe^{-0.2x} - 2000e^{-0.2x} - 100.$$

Find the price needed to maximize profit.

Exercise 7. The value of a second-hand car reduces exponentially with age, so that its value $\$V$ after t years can be modelled by the formula $V = Ae^{-at}$.
If the car was $\$50000$ when new and was worth $\$38000$ after 2 years, find the values of A and a, correct to 3 decimal places. Use this model to predict the value of the car
a) when the car is 5 years old
b) in the long run.

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Logarithmic Functions

Definition

A logarithm is defined as follows:

$$\log_a x = y \qquad \textit{means} \qquad a^y = x,\ a > 0,\ a \neq 1.$$

The number $\log_a x$ is the power y to which we raise a to get x. The number a is called the logarithmic base. We read $\log_a x$ as "the logarithm, base a, of x."

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We can obtain the graph of $g(x) = \log_a x$ by reflecting the graph of $f(x) = a^x$ across the line $y = x$. The function $g(x) = \log_a x$ is the inverse function of $f(x) = a^x$.

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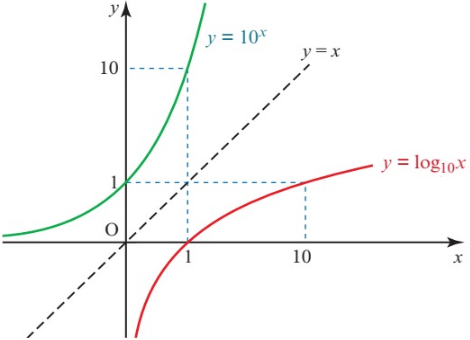

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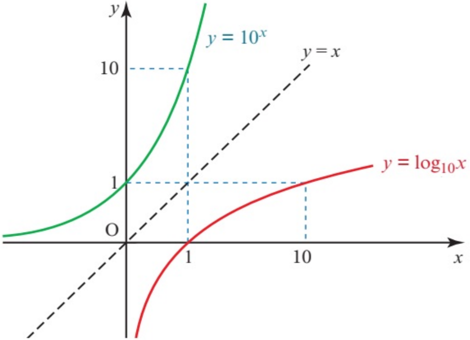

Exercise. Solve each of these equations, giving your answers correct to 3 significant figures.

a) $10^x = 200$ b) $10^x = 0.04$ c) $\log_{10} x = 1.25$ d) $\log_{10} x = -2.5$.

Basic properties of logarithms

Theorem (Properties of logarithms)

For any positive numbers M, N, a and b with $a, b \neq 1$, and any real number k:

P1. $\log_a(MN) = \log_a M + \log_a N$

P2. $\log_a \dfrac{M}{N} = \log_a M - \log_a N$

P3. $\log_a(M^k) = k \log_a M$

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Proof. Multiplication

$$a^{\log_a(MN)} = MN = a^{\log_a M} a^{\log_a N} = a^{\log_a M + \log_a N}$$

Consequently $\log_a(MN) = \log_a M + \log_a N$.

Common and natural logarithms

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Theorem

$\ln x$ exists only for positive numbers x. The domain of the function $f(x) = \ln x$ is $(0, \infty)$.

- *$\ln x < 0$ for $0 < x < 1$.*
- *$\ln x = 0$ when $x = 1$.*
- *$\ln x > 0$ for $x > 1$.*

The function given by $f(x) = \ln x$ is always increasing. The range is the entire real line, $(-\infty, \infty)$, or the set of real numbers, $\mathbb{R}$.

Theorem

$$\frac{d}{dx}a^x = (\ln a)a^x, \quad \frac{d}{dx}a^{f(x)} = (\ln a)a^{f(x)}f'(x), \quad \frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

In particular, we have

$$\frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x), \quad \frac{d}{dx}\ln x = \frac{1}{x}, \quad \frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}.$$

Example 1. Differentiate

a) $f(x) = x^5e^x$ b) $f(x) = 4^{x^3+1}$ c) $f(x) = e^x\ln(x^2)$ d) $f(x) = \sqrt{x^3+x}\log_5 x$.

The derivatives of exponential and logarithmic functions

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Example 2. Find the equation of the line tangent to the graph of $y = e^{2x}\ln x$ at $x = 1$.

Exercise 1. The demand for a new computer game can be modeled by $p(x) = 53.5 - 8\ln x$, where $p(x)$ is the price consumers will pay, in dollars, and x is the number of games sold, in thousands. Recall that total revenue is given by $R(x) = xp(x)$.

a) Find $R(x)$.

b) Find the marginal revenue, $R'(x)$.

c) Is there any price at which revenue will be maximized? Why or why not?

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Exercise 2. The profit, in thousands of dollars, from the sale of x thousand candles can be estimated by $P(x) = 2x - 0.3x\ln x$.

a) Find the marginal profit, $P'(x)$.

b) Find $P'(150)$, and explain what this number represents.

c) How many thousands of candles should be sold to maximize profit?

(see textbook, page 336)

1 Exponential Functions

2 Logarithmic Functions

3 Applications: Uninhibited and limited growth models

Exponential growth

Problem. Find functions $P(t)$ that satisfy $dP/dt = kP$ for a given number k.

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Example. Solve the differential equation $\dfrac{dP}{dt} = 5P$.

Uninhibited Population Growth

The equationd $\frac{dP}{dt} = kP$ or $P'(t) = kP(t)$, with $k > 0$, is the basic model of uninhibited (unrestrained) population growth, whether the population is comprised of humans, bacteria in a culture, or dollars invested with interest compounded continuously.

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Note that $k =$ (Birth rate) $-$ (Death rate). When the birth rate is less than the death rate, k is negative, and the population is decreasing, or "decaying," at a rate proportional to its size.

Example 1. (Life Science: Population Growth in China). In 2006, the population of China was 1.314 billion, and the exponential growth rate was 0.6% per year. Thus, $\dfrac{dP}{dt} = 0.006P$, where t is the time, in years, after 2006. Assume uninhibited growth. *(Source: Time Almanac, 2007.)*

a) Find the function that satisfies the equation. Assume $P_0 = 1.314$ and $k = 0.006$.

b) Estimate the population of China at the beginning of 2020 (2024).

c) After what period of time will the population be double that in 2006? *(see textbook, page 341)*

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c) After what period of time will the population be double that in 2006? *(see textbook, page 341)*

Example 2. (Bottled water sales). Since 2000, sales of bottled water have increased at the rate of approximately 9.3% per year. That is, the volume of bottled water sold, G, in billions of gallons, t years after 2000 is growing at the rate given by $\dfrac{dP}{dt} = 0.093G$.

a) Find the function that satisfies the equation, given that approximately 4.7 billion gallons of bottled water were sold in 2000.

b) Predict the number of gallons of water sold in 2025.

c) What is the doubling time for $G(t)$? *(see textbook, page 348)*

Exercise 1. (Decline in beef consumption). Annual consumption of beef per person was about 64.6 lb in 2000 and about 61.2 lb in 2008. Assuming that $B(t)$, the annual beef consumption t years after 2000, is decreasing according to the exponential decay model.

a) Find the value of k, and write the equation.

b) Estimate the consumption of beef in 2015.

c) In what year (theoretically) will the consumption of beef be 20 lb per person?

Applications: Decay

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Exercise 2. (Population decrease of Russia). The population of Russia dropped from 150 million in 1995 to 142.5 million in 2013. (*Source: CIA–The World Factbook.*) Assume that $P(t)$, the population, in millions, t years after 1995, is decreasing according to the exponential decay model.

a) Find the value of k, and write the equation.

b) Estimate the population of Russia in 2018.

c) When will the population of Russia be 100 million?

(see textbook, page 362)

Exercise 3. (Population decrease of Ukraine). The population of Ukraine dropped from 51.9 million in 1995 to 44.5 million in 2013. *(Source: CIA–The World Factbook.)* Assume that $P(t)$, the population, in millions, t years after 1995, is decreasing according to the exponential decay model.

a) Find the value of k, and write the equation.

b) Estimate the population of Ukraine in 2018.

c) In what year will the population of Ukraine be 40 million, according to this model?

(see textbook, page 362)

Models of limited growth

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$$P(t) = \frac{L}{1 + be^{-kt}} \quad \text{for} \quad k > 0,$$

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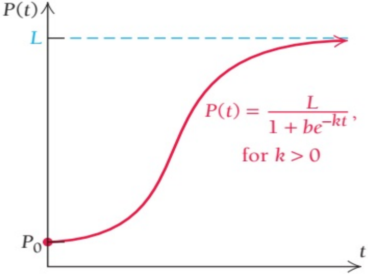

Example. (Life Science: Spread of an Epidemic). In a town whose population is 3500, an epidemic of a disease occurs. The number of people N infected t days after the disease first appears is given by

$$N(t) = \frac{3500}{1 + 19e^{-0.6t}}.$$

a) How many people are initially infected with the disease ($t = 0$)?
b) Find the number infected after 2 days, 8 days, and 18 days.
c) Graph the equation.
d) Find the rate at which the disease is spreading after 16 days.
e) Will all 3500 residents ever be infected? Why or why not? *(see textbook, page 345)*

Models of limited growth

Exercise. (Effect of advertising). Suppose that SpryBorg Inc. introduces a new computer game in Houston using television advertisements. Surveys show that $P\%$ of the target audience buys the game after x ads are broadcast, satisfying

$$P(x) = \frac{100}{1 + 49e^{-0.13x}}.$$

a) What percentage buys the game without seeing a TV ad $(x = 0)$?
b) What percentage buys the game after the ad is run 5 times? 10 times? 20 times? 30 times? 50 times? 60 times?
c) Find the rate of change, $P'(x)$.
d) Sketch a graph of the function. *(see textbook, page 350)*

Business application: Annuities

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If we let p represent the amount paid into the annuity on a regular basis, r the annual interest rate, n the number of compounding periods per year (which is also the number of payments made per year), and t the time in years, then the future value of the annuity, A, is given by

$$A(t) = \frac{p\left[\left(1+\dfrac{r}{n}\right)^{nt} - 1\right]}{\dfrac{r}{n}}.$$

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Example. An individual saves $\$1000$ in a bank account at the end of each year. The bank offers a return of 6% compounded annually.

a) Determine the amount saved after 10 years.

b) After how many years does the amount saved first exceed $\$20000$?

The future value of the annuity

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Exercise 1. Yukiko opens a savings account to pay for her new baby's college education. She deposits \$200 every month into the account at an annual interest rate of 4.2%, compounded monthly.

a) Find $A(t)$, the value of Yukiko's account after t years.

b) What is the value of Yukiko's account after 18 yr?

c) What is the rate of change in the value of Yukiko's account after 8 yr? *(see textbook, page 369)*

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Exercise 2. Tom opens a retirement savings account, and he deposits $\$1000$ every 4 months into the account, which has an annual interest rate of 3.8%, compounded every 4 months. Find the value of Tom's account after 12 yr? *(see textbook, page 369)*