PART A: Multiple choice or Short answer

(5 mark for each correct answer/choice)

Problem A1. Find the least positive integer n such that the function $f(x) = 2x^3 - 4x \ln x + 3x - 3$ is $O(x^n)$.

Circle the best answer.

A. 2 B. 3 C. 4 D. A, B, C are not correct.

Problem A2. How many positive integers less than 1000 are divisible by both 6 and 9?

Write your answer here:

Problem A3. Find the integer a such that

$$a \equiv 42 \ (\mathbf{mod}\ 23), \quad -22 \leq a \leq 0.$$

(fill the gap...)

Your answer is $a =$

Problem A4. Assume that there are five possible final grades in this class: A, B, C, D, and F. What is the minimum number of students required to be sure that at least 9 will receive the same grade?

Your answer: The least number of students is .. *(fill the gap...)*

Problem A5. Find -2022 **mod** 5

Circle the best answer.

A. 2 B. 3 C. 4 D. A,B,C are not correct.

Problem A6. Let $a = 16.27.343.121,\ b = 32.49.3^7.11.29^3$. Find gcd($a$,$b$)?

Circle the best answer.

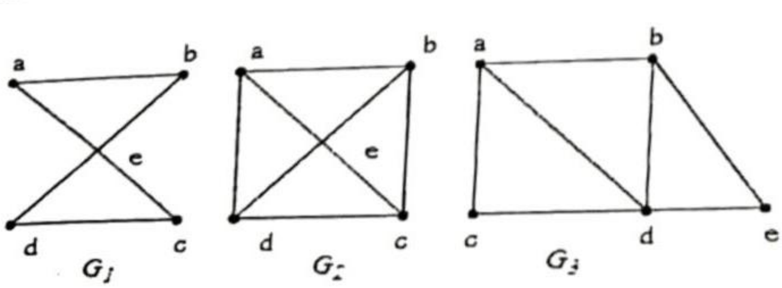
A. $2^4.7^2.3^2.11^2.29$ B. $2^5.7^3.3^7.11$ C. $2^4.7^2.3^3.11$ D. A,B,C are not correct.

Problem A7. What is the coefficient of $x^{1000}y^{1025}$ in the expansion of $(x-3y)^{2025}$?

Write your answer here:

(the answer must be written in the form $C(n,k)3^m$ with specific n,k,m and sign)

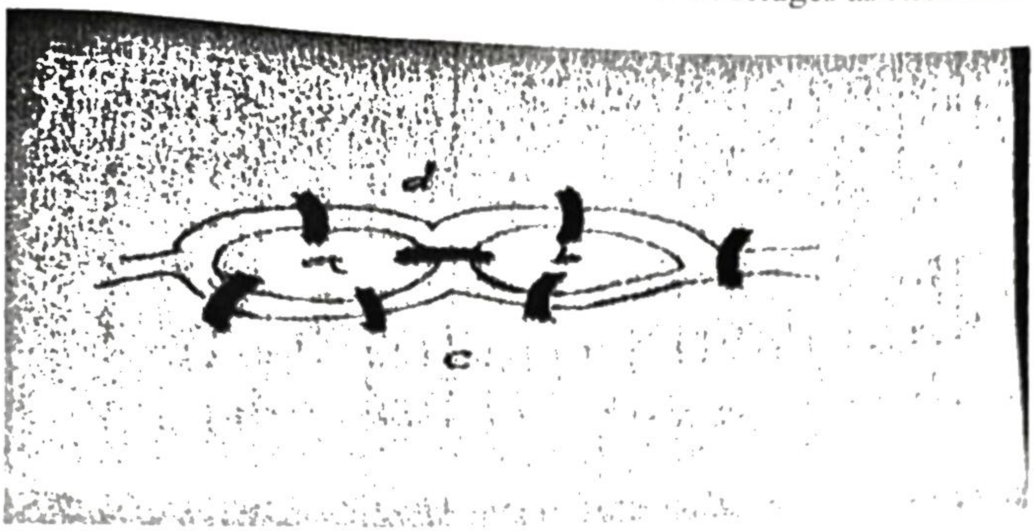
Problem A8. Let m,n,k be the number of cut vertices of the graphs G_1, G_2, G_3, respectively. Find the sum $m+n+k$?



Write your answer here:

PART B: WRITE YOUR FULL ANSWERS.

Problem B1. Suppose that in a city, there are seven bridges as shown in the picture below.



a. [*3 marks*] Can someone cross all seven bridges in the city exactly once? If yes, find that route, otherwise, explain why?

b. [*3 marks*] Can someone cross all seven bridges exactly once and return to the starting point? If yes, find that route, otherwise, explain why?

c. If we are allowed to build one more bridge linking region *b* so that someone can cross all eight bridges exactly once and return to the starting point, where can we place that bridge? Justify your answer.

Problem B2. A pair of dice is loaded. The probability that a 1 appears on the first die is 1/11, and the probability that a 3 appears on the second die is 1/11. Other outcomes for each die appear with probability 2/11.

a) What is the probability of 7 appearing as the sum of the numbers when the two dice are rolled?

b) Let X be the sum of the numbers when the two dice are rolled. Find the distibution of X.

c) Find the distribution function of X.

d) Find the expectation and the variance of X.

Problem B3. Use mathematical induction to prove that $n^5 - n$ is divisible by 5 whenever n is a nonnegative integer.

Problem B4. Given the linear recurrence relation

$$a_n = 6a_{n-2} + a_{n-1}, \qquad a_0 = 1, a_1 = -3.$$

a. [*2 marks*] Find the value of a_5?

b. Solve the recurrence relation together with the initial conditions.

Problem B5. How many permutations of the letters *ABCDEFGH* contain

a)[*2 marks*] the string *EF*?

b)[*3 marks*] the strings AB and *FGH*?

c)[5 *marks*] the strings ABC and $CGHF$? List all such permutations.