1. (50 points)
   a) Alice picks 31 integers from 21, 22, ..., 97, 98. Prove that she must have picked three whose digits sum up to the same amount.
   b) Put 20 balls into 10 box at random. Assume that the 10 boxes are distinguishable, and each of the 20 balls independently chooses one of the 10 boxes with probability 1/10. What is the probability that there exists at least one box containing exactly 10 balls. (Write your answer using binomial coefficients is fine)

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2. (40 points)
   A ternary digit string is a string in which each digit can be 0, 1, or 2.
   (a) How many ternary digit strings contain exactly n digits?
   (b) How many ternary digit strings contain exactly n digits in which the first and the last digit are the same.
   (c) How many ternary digit strings contain exactly n digits and exactly $(n-2)$ digits are 2's.
   (d) How many ternary digit strings contain exactly n digits and digit 1 appears at least once.

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3. (50 points)
   A robotics club has six kinds of spare parts: sensors, motors, wheels, batteries, cameras, controllers. Assume each kind is available in unlimited quantity, and parts of the same kind are identical.
   How many ways are there to choose:
   a) 12 parts?
   b) 24 parts with at least 2 of each kind?
   c) 24 parts with no more than 2 controllers?
   d) 24 parts with at least 5 motors and at least 3 cameras?
   e) 24 parts with at least 1 sensor, at least 2 motors, at least 3 wheels, at least 1 battery, at least 2 cameras, and no more than 3 controllers?

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4. (50 points)
   a) Give a formula for the coefficient of x^k in the expansion of $(x - \frac{2}{x})^{200}$ for any integer k.
   b) Let $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}, n \geq 2$. Prove that $L_n \leq 2^n$ for all $n \geq 1$.

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5. (40 points) Which of the following is a degree sequence of a simple graph? If it is, draw the graph. If it is not, explain why not.
   a) 5, 4, 4, 2, 2, 2

b) 5, 5, 4, 4, 3, 1
c) 5, 3, 3, 2, 2, 1
d) 5, 5, 4, 2, 2, 2

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6. (40 points)
   a) For which values of c, d the sequence $a_n = c3^n + d, n = 0, 1, 2, \ldots$ is a solution of the following recurrence relation:
   $a_n = 3a_{n-1} - 2a_{n-2} + 3^n, n \geq 2$.
   b) Find the solution of the following recurrence relation:
   $a_n = 3a_{n-1} - 2a_{n-2} + 3^n, n \geq 2$, with $a_0 = 1, a_1 = 2$.

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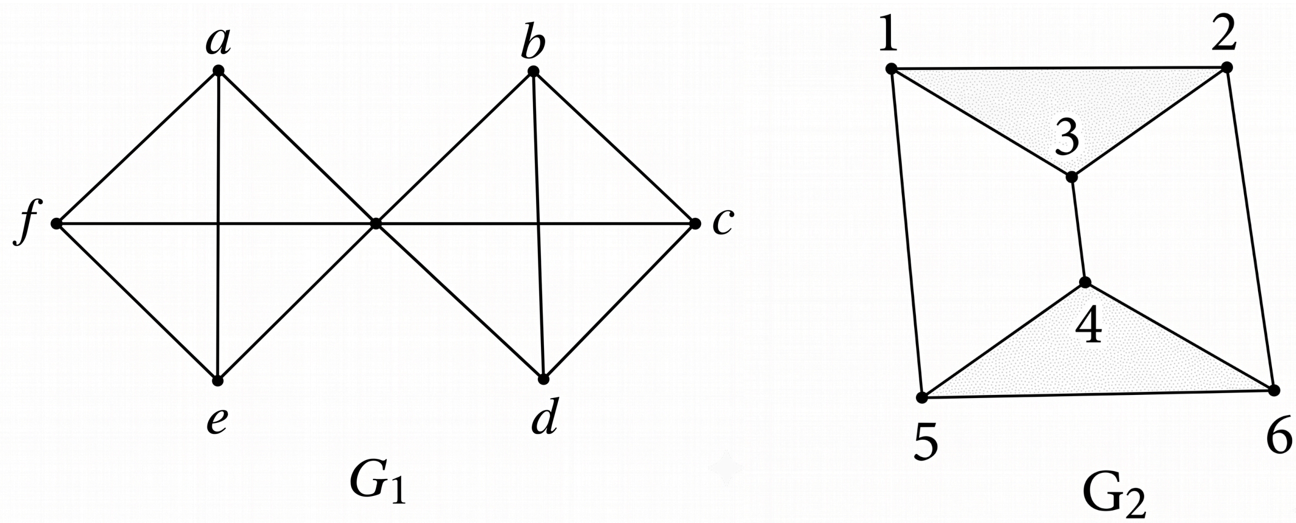
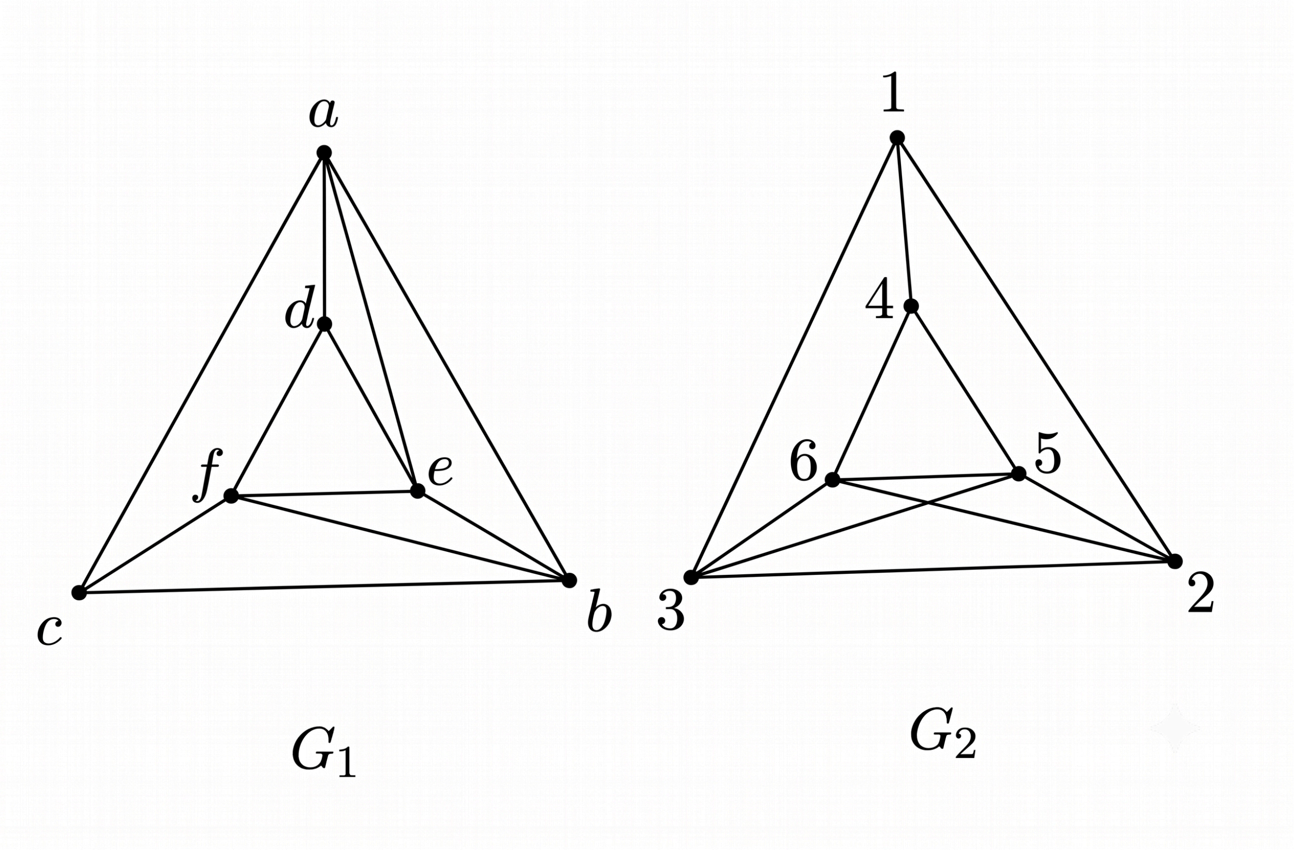
7. (40 points) Are the following pairs of graphs isomorphic? If they are, state the correspondence of the vertices. If they are not, explain why not.
   a)

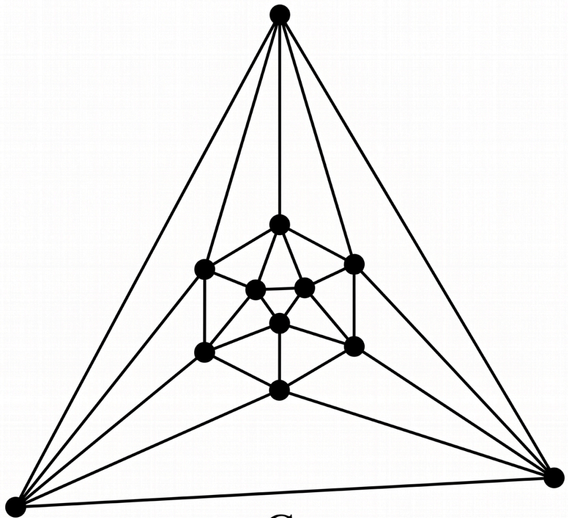


b)



8. (40 points) Let $G = (V, E)$ be a graph such that $|V| = 1000$ and for every edge (A, B) at least one of two vertices A, B has degree no more than 10. What is the greatest possible value of $|E|$? give an example where the greatst value attains.

9. (50 points)
   i) Which of the following graphs have a Hamilton circuit? If they do, find it. If they do not, explain why not.
   ii) Which of the following graphs have a Hamilton path? If they do, find it. If they do not, explain why not.



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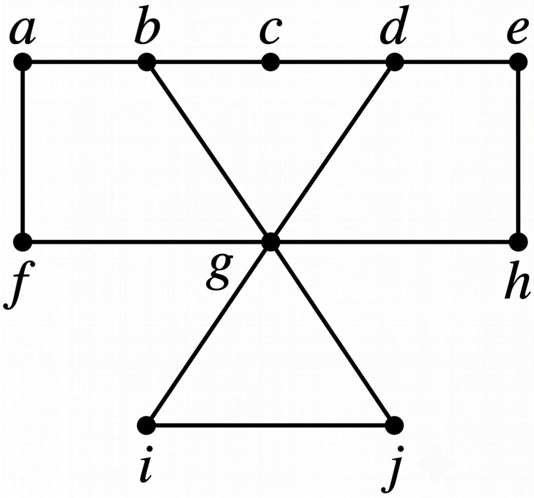

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