MTH 2215

Applied discrete mathematics

Chapter 2, Section 2.1
Sets

These class notes are based on material from our textbook, **Discrete Mathematics and Its Applications**, 6th ed., by Kenneth H. Rosen, published by McGraw Hill, Boston, MA, 2006. They are intended for classroom use only and are **not** a substitute for reading the textbook.

Sets

- A *set* is an unordered collection of objects.
- The set is the fundamental discrete structure on which all other discrete structures are built.
- The objects in a set are called its *elements*, or *members*.
- A set is said to *contain* its elements.

Set Notation

- We normally use upper-case letters to represent the names of sets, and lower-case letters to represent their elements.
- To denote that *a* is an element of set *S* we write: $a \in S$
- To denote that *a* is <u>not</u> an element of set *S* we write: $a \notin S$

How to Describe a Set

- We can describe a set in two ways:
  - List all of its elements
  - Give a set of rules that characterize all of the members of the set (*set builder* notation)

Listing the Elements of a Set

- To list the members of a set, we use curly braces, separating each element from the next with a comma.
- Example: the set of all vowels in the English language is the set

  $V = \{a, e, i, o, u\}$
- We can use ellipses to keep us from having to list all of the elements individually, provided the meaning is obvious: $H = \{1, 2, 3, 4, \ldots, 100\}$

Using Set Builder Notation

- Often we are dealing with sets where it is impossible to list all of their elements.
- In set builder notation, we give a rule that characterizes all members of a set.
- Example:

$$S = \{x \mid x \text{ is the square of an integer}\}$$

- This can be read, “S is the set of all x such that x is the square of an integer”.

Using Set Builder Notation

- In studying computer theory, we find it useful to remember the following sets:

$N = \{0, 1, 2, 3, \ldots\}$, the set of *natural numbers*

$Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of *integers*

$Z^+ = \{1, 2, 3, \ldots\}$, the set of *positive integers*

$Q = \{p/q \mid p \in Z, q \in Z, \text{ and } q \neq 0\}$, the set of *rational numbers*

Q^+ = the set of *positive rational numbers*

R = the set of *real numbers*

What Can Constitute a Set?

- Note that anything within a set of curly braces can be considered a set. The elements of a set don’t necessarily have to have anything to do with one another.
- Example: {Boston, “7”, iPod, 2.7, Sleepy} is a legal set.

What Can Constitute a Set?

- Two sets are *equal* if and only if they have the same elements.
- Consider sets A and B. Then A = B (A and B are equal) iff:

$$\forall x \, ((x \in A) \leftrightarrow (x \in B))$$

The Elements of a Set

The order in which elements occur in sets is irrelevant. For example, the following two sets are equal:

{a, b, c, d, e}

{c, e, a, d, b}

The Elements of a Set

It does not matter if an element of a set is listed more than once. For example,

$\{a, a, a, b, c\}$

and

$\{a, b, c\}$

are equivalent. We ignore any duplicates.

What Can Constitute a Set?

- The elements of a set can themselves be sets. For example,

  $S = \{N, Z, Q, R\}$

- Question: Does set S have any duplicate elements?
- Answer: No. Set S has only 4 elements, none of which is equivalent to any of the others.

What Can Constitute a Set?

- A set that has no elements is called the *empty set* or *null set*.
- Yes, it is still considered a real set, even though it has no elements.
- It is denoted by $\varnothing$, or by $\{\ \}$.
- Since the empty set is a set, another set can contain the empty set as one of its elements:

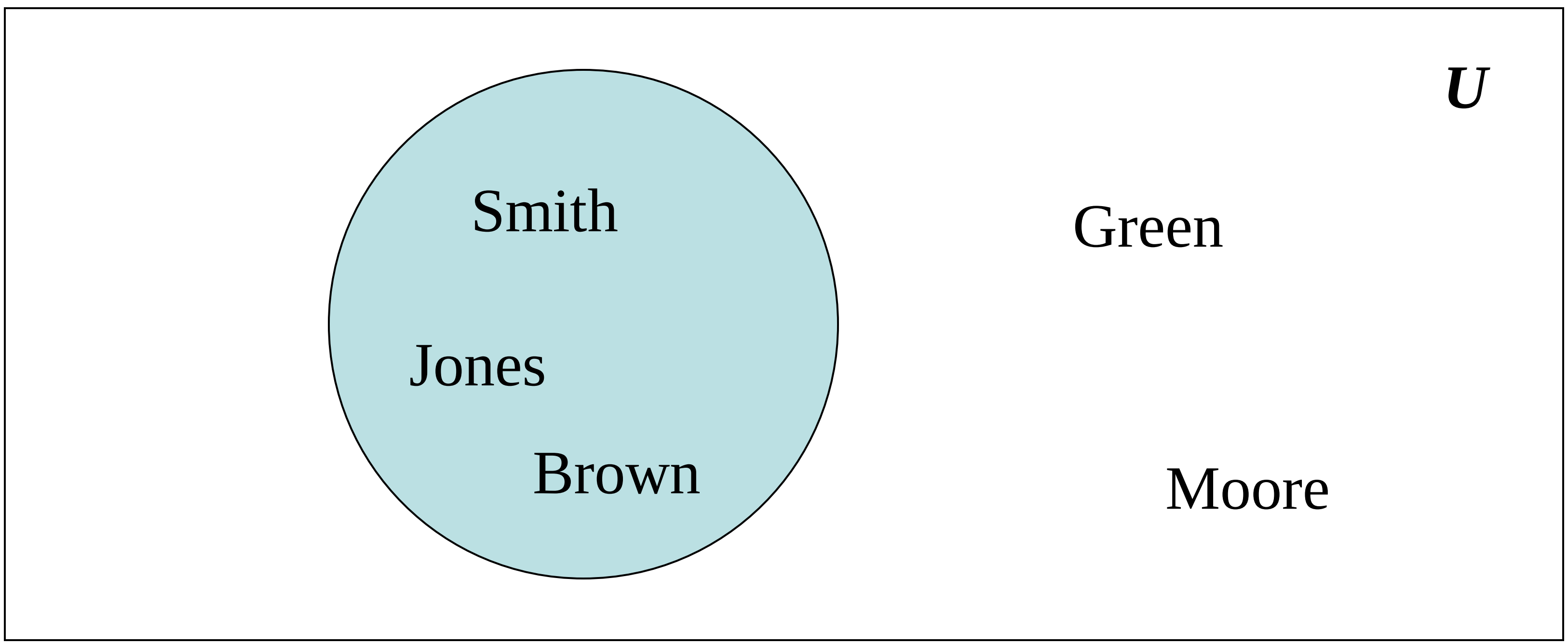
$A = \{\varnothing, a\}$	This set has 2 elements
$B = \{\varnothing\}$	This set has 1 element
$C = \varnothing$	This set has 0 elements

Venn Diagrams

- Sets can be represented graphically using Venn diagrams.
- In Venn diagrams:
  - A rectangle represents the universal set (universe of discourse)
  - Circles (and other geometric figures) represents sets
  - Points (or words) represent elements

Venn Diagrams

- Assistant professors at Blivet State University who have taught MTH 2215:



Subset

The set A is said to be a *subset* of set B if and only if every element of set A is also an element of set B.

We use this notation: $A \subseteq B$

$A \subseteq B$ is true if and only if the following quantification is true:

$$\forall x\, ((x \in A) \to (x \in B))$$

Subset

Obviously, according to the preceding definition, if $A = B$, it must be true that:

$A \subseteq B$, and

$B \subseteq A$.

Moreover, it should be self-evident that every set is a subset of itself. That is:

$A \subseteq A$

Proper Subset

However, if all of A's elements are also in B, *but* B has some elements in it that A does not have (that is, $A \neq B$), then we can be a more precise and say that A is a *proper subset* of B.

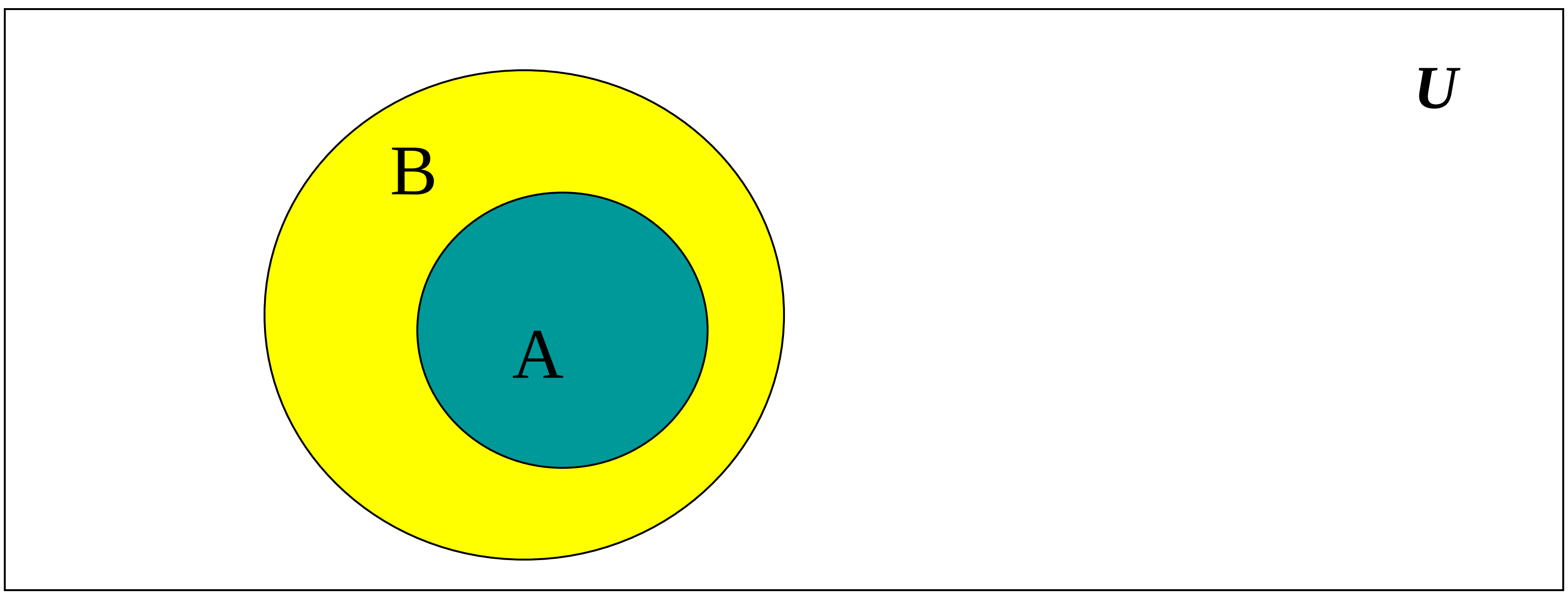
We use this notation: $A \subset B$

A is a proper subset of B iff:

$$\forall x\,(x \in A \rightarrow x \in B) \land \exists x(x \in B \land x \notin A)$$

Proper Subset

We can represent the subset relationship using a Venn diagram. The following diagram represents A ⊂ B:



Subset

Interestingly enough, the empty set, $\emptyset$, is a subset of every other set (or, more precisely, every *nonempty* set).

Your book gives a formal proof, but you might think of it this way instead: if set S = {a, b}, then it has 4 subsets –

$$\{\{a\}, \{b\}, \{a, b\}, \emptyset\}$$

Properties of Sets

One way to show that two sets are equal is to show that each set is a subset of the other.

The Cardinality of a Set

Given a set S, and $n \in$ N (that is, n is an element of the set of natural numbers -- the integers from 0 on up),

if there are exactly n distinct elements in S, then:

S is a *finite set*, and

n is the *cardinality* of S

The cardinality of S is represented by |S|.

Properties of Sets

We now can see that every nonempty set S must have at least two subsets:

$\varnothing$ and S

Theorem 1 in section 2.1 of your textbook says:

> For every set S,
>
> $\varnothing \subseteq S$
>
> $S \subseteq S$

Powerset

The *powerset* of S is the set of all subsets of S.

The powerset of S is represented by $P(S)$, or by the symbol 2^S

For example, if $S = \{a, b\}$, then:

$P(S) = \{\{a\}, \{b\}, \{a, b\}, \emptyset\}$

Powerset

Remember that $|S|$ represents the cardinality of S (the number of elements in S).

Here S has two elements, *a* and *b*. So $2^{|S|}$ can be understood as 2^2, which is 4.

And 4 is the number of subsets of S, or the *cardinality of the powerset* of S.

Powerset

The powerset of the empty set is a special case.

The powerset of the empty set is:

$P(\emptyset) = \{\emptyset\}$

Note $\quad P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

Cartesian Product

The Cartesian product, or cross product, of two sets is the set of *ordered pairs* of elements of the two sets. To represent the cross product of sets A and B we use the symbol ×, as in A × B.

For example, given:

set A = {a, b} and

set Y = {x, y}

The Cartesian product A × Y = {(a,x), (a,y), (b,x), (b,y)}

Set Notation with Quantifiers

$\forall x \in S(P(x))$ means " for all x that are elements of S, P(x) is true". This is referred to as the *universal quantification of P(x) over all elements in the set S*. It is shorthand for: $\forall x(x \in S \rightarrow P(x))$

$\exists\, x \in S(P(x))$ is the *existential quantification of P(x) over all elements in the set S*. It is shorthand for: $\exists x(x \in S \wedge P(x))$

Truth Sets of Quantifiers

Given a predicate, P, and a domain, D, the *truth set* of P is defined as the set of elements in D for which P(x) is true.

The truth set of P(x) is is denoted by $\{x \in D \mid P(x)\}$: “those elements of domain D such that P(x) is true”

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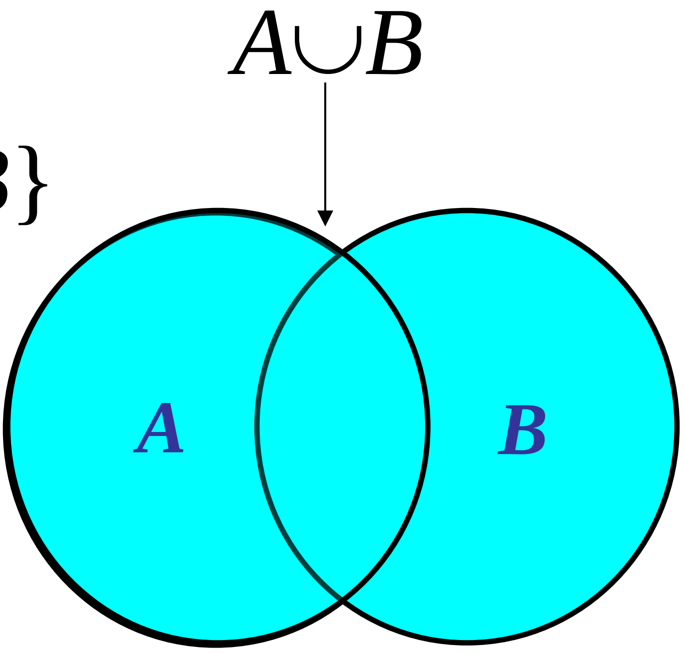
Chapter 2, Section 2.2
Set Operations

Set Union

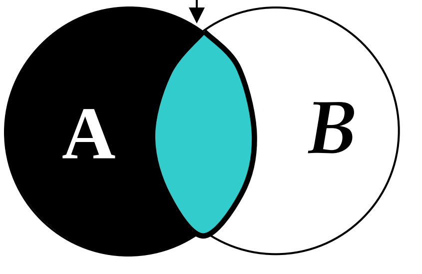
- Union of two sets A and B is denoted by $A \cup B$
- $A \cup B$ contains elements that are either in A or in B or in both.
- $A \cup B = \{x \mid x \in A \lor x \in B\}$
- $A = \{1,3,5\}$, $B = \{2,3,4\}$
- $A \cup B = \{1, 2, 3, 4, 5\}$



Set Intersection

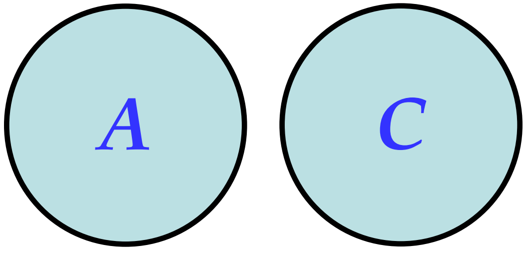
- Intersection of two sets A and B is denoted by $A \cap B$
- $A \cap B$ contains elements that are in both A and B
- $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- $A = \{1,3,5\}$, $B = \{1,2,3\}$
- $A \cap B = \{1, 3\}$

$A \cap B$



Disjoint Sets

- Two sets are called disjoint if their intersection is the empty set.
- $A = \{1,3,5\}$, $B = \{1,2,3\}$, $C = \{6,7,8\}$
- Are A and B disjoint? NO
- Are A and C are disjoint? YES



Cardinality of the Union of Sets

How many elements does $A \cup B$ have?

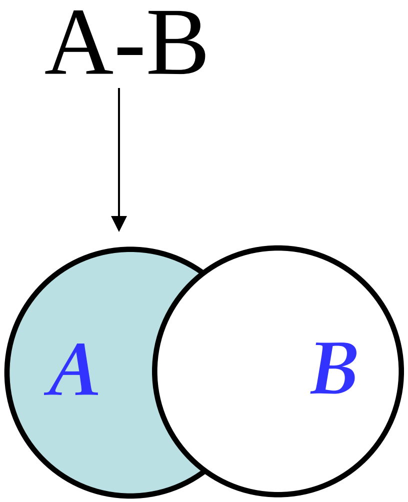
The number of elements in A plus the number of elements in B, minus the number of elements in both sets.

This can be written:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

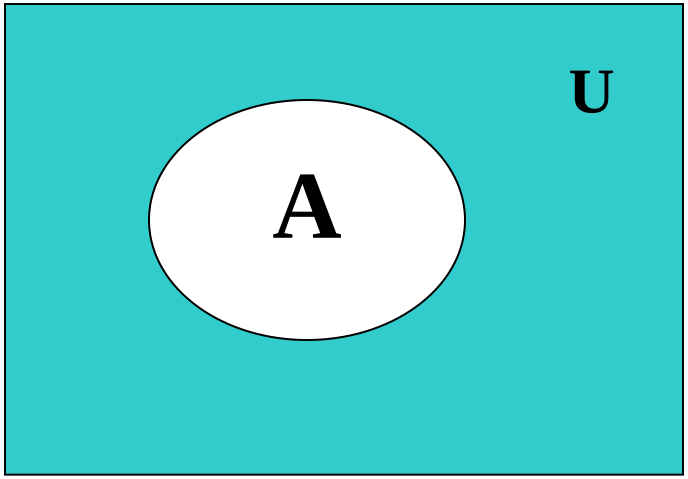
Set Difference

- Difference of two sets A and B is denoted by $A–B$
- $A–B$ contains elements that are in A but not in B.
- $A–B = \{x \mid x \in A \wedge x \notin B\}$
- $A = \{1,3,5\}$, $B = \{1,2,3\}$
- $A–B = \{5\}$



Complement of a Set

- Complement of a set A is denoted by $\overline{A}$
- Done with respect to a Universal set U
- $\overline{A}$ contains elements which are not in A, but are in U.
- $\overline{A} = U - A$
- $\overline{A} = \{x \mid x \in U \wedge x \notin A\}$



Set Identities

$A \cup \varnothing = A$	Identity
$A \cap U = A$	
$A \cup U = U$	Domination
$A \cap \varnothing = \varnothing$	
$A \cup A = A$	Idempotent
$A \cap A = A$	
$\overline{\overline{A}} = A$	Double Complement

Set Identities (Cont.)

$A \cup B = B \cup A$	Commutative
$A \cap B = B \cap A$	
$A \cup (B \cup C) = (A \cup B) \cup C$	Associative
$A \cap (B \cap C) = (A \cap B) \cap C$	
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	
$A \cup (A \cap B) = A$	Absorption
$A \cap (A \cup B) = A$	
$\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's
$\overline{A \cap B} = \overline{A} \cup \overline{B}$	

Examples

- Use set builder notation to prove that:

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Use set identities to prove that

$$\overline{\overline{(A \cup B) \cap C} \cup \overline{B}} = B \cap C$$

More Exercises

- Describe the following sets using the set builder notation:
  1. The set of all positive integers between 1 and 99.
  2. $\overline{A \cup B}$
  3. $\overline{A} \cap \overline{B}$
  4. $A \oplus B$
  5. $(A \cup B) \cap C$
- Use set builder notation to prove .

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

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Chapter 2, Section 2.3
Functions

Recap 2.1

- Set – an <u>**UNORDERED**</u> collection of objects
  - *Element* /*member* - an object in a set
  - Notation - {a,b,c,d}
- Cardinality
  - The number of distinct elements in a set
- Power Set
  - The set of all subsets of a set
- Cartesian product of two sets A and B $A \times B$

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Recap 2.2

- Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$
- Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- Difference: $A - B = \{x \mid x \in A \wedge x \notin B\}$
- Complement: $\bar{A} = U - A$
- Identities similar to those from logic, e.g.

Definitions

- Let *A* and *B* be sets. A *function f* from *A* to *B* is an assignment of *exactly* one element of *B* to *each* element of *A*.
- We write $\boldsymbol{f(a) = b}$ if *b* is the unique element of *B* assigned by the function *f* to the element of *A*.
- If $\boldsymbol{f}$ is a function from $\boldsymbol{A}$ to $\boldsymbol{B}$, we write

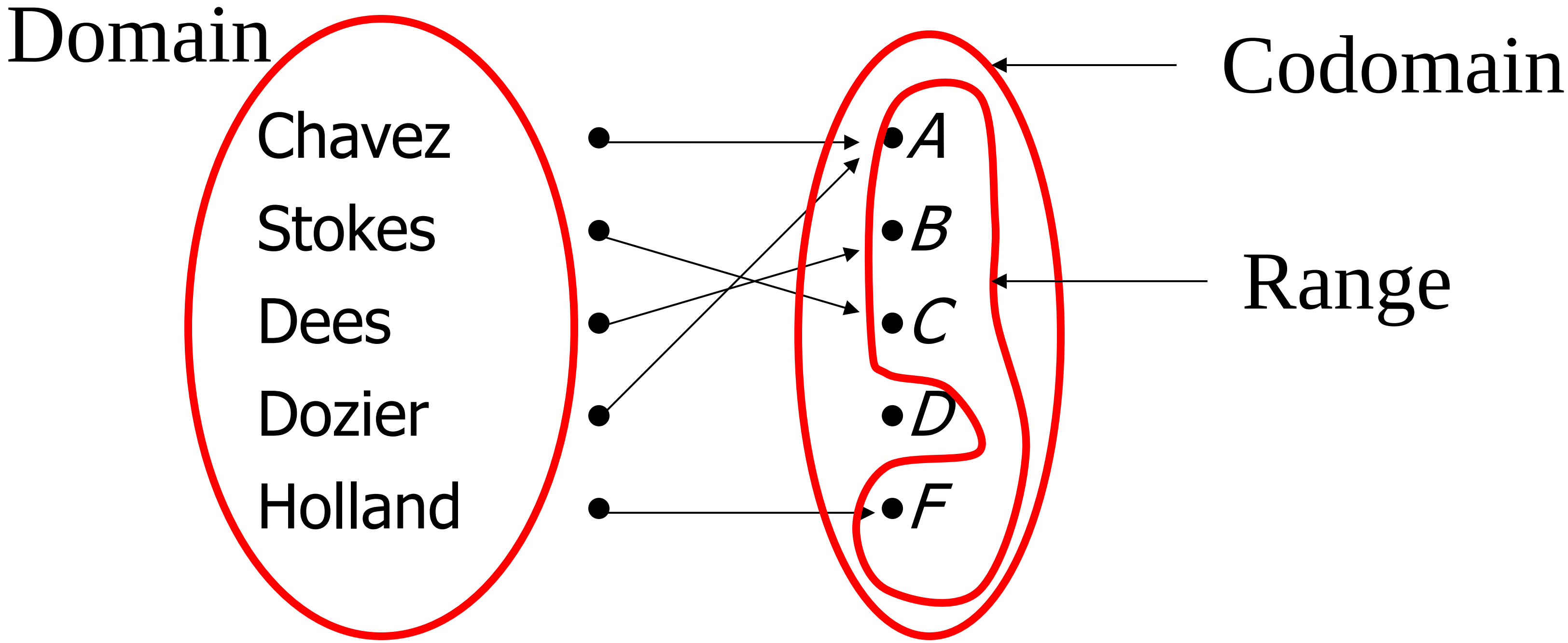
$$\boldsymbol{f : A \to B}$$

Definitions

- If $\boldsymbol{f : A \rightarrow B}$, we say that *A* is the *domain* of *f* and *B* is the *codomain* of *f*.
- If $\boldsymbol{f(a) = b}$, we say that *b* is the *image* of *a*.
- The *range* of *f* is the set of all images of elements of *A*.

Example

- Suppose that each student in a class is assigned a letter grade from the set {***A*, *B*, *C*, *D*, *F***}. Let ***g*** be the function that assigns a grade to a student.



Example

- Consider a function $\boldsymbol{f}$ **:** $\mathbf{Z} \rightarrow \mathbf{Z}$ that assigns the square of an integer to this integer.
- How can you write this function?

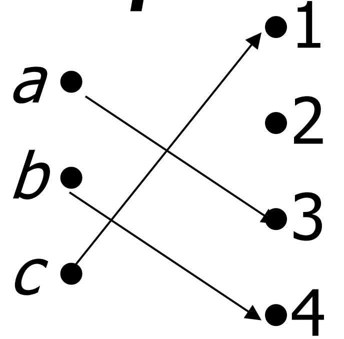
$$\boldsymbol{f(x) = x^2}$$

- What is the domain of $\boldsymbol{f}$? The integers
- What is the codomain of $\boldsymbol{f}$? The integers
- What is the range of $\boldsymbol{f}$?

  The nonnegative integers {0,1,4,9,..}

One-to-One Functions (injective)

- No value in the range is used by more than one value in the domain.
- If $\boldsymbol{f(x) = f(y)}$, then $\boldsymbol{x = y}$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$ in the domain of $\boldsymbol{f}$.



- In other words $\forall x\ \forall y\ (f(x) = f(y) \rightarrow x = y)$,

or using the contrapositive

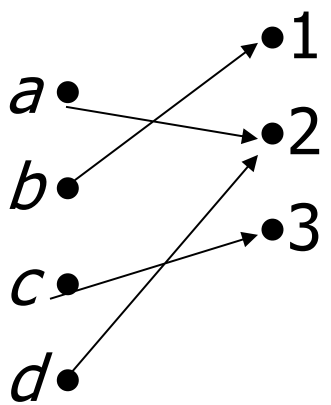
$$\forall x\ \forall y\ (x \neq y \rightarrow f(x) \neq f(y))$$

One-to-One Functions

- Is the function $\boldsymbol{f(x) = x^2}$ from the set of integers to the set of integers one-to-one?
    - $\forall x\ \forall y\ (x^2 = y^2 \rightarrow x = y)$?
    - $1^2 = (-1)^2$ but $1 \neq -1$
    - NO
- Is the function $\boldsymbol{f(x) = x + 1}$ one-to-one?
    - $\forall x\ \forall y\ (x + 1 = y + 1 \rightarrow x = y)$?
    - $(x + 1) \neq (y + 1)$ only when $x \neq y$
    - YES

Onto Functions (surjective)

- For every value in the codomain, there is a value in the domain that is mapped to it.



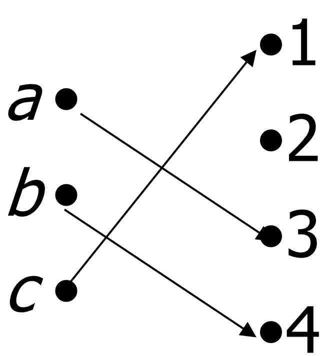
- In other words, $\forall y\ \exists x\ (f(x) = y)$
- Codomain = range!

Onto Functions

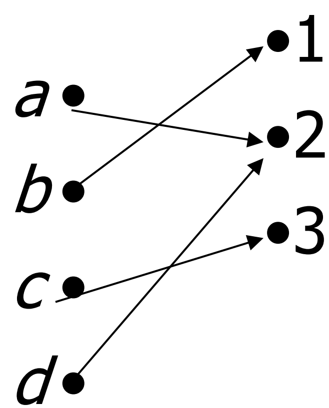
- Is the function $\boldsymbol{f(x) = x^2}$ from the set of integers to the set of integers onto?
  - Is it true that $\forall y\ \exists x\ (x^2 = y)$?
  - -1 is one of the possible values of y, but there does not exists an x such that $x^2 = -1$
  - NO
- Is the function $\boldsymbol{f(x) = x + 1}$ onto?
  - Is it true that $\forall y\ \exists x\ (x + 1 = y)$?
  - For every y, some x exists such that $x = y - 1$.
  - YES

One-to-One Correspondence (bijection)

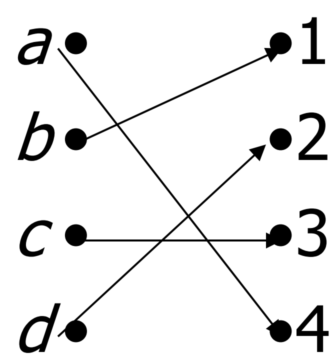
- If a function ***f*** is both one-to-one and onto, then it is a one-to-one correspondence.



One-to-One
but not Onto



Onto, but
Not One-to-One



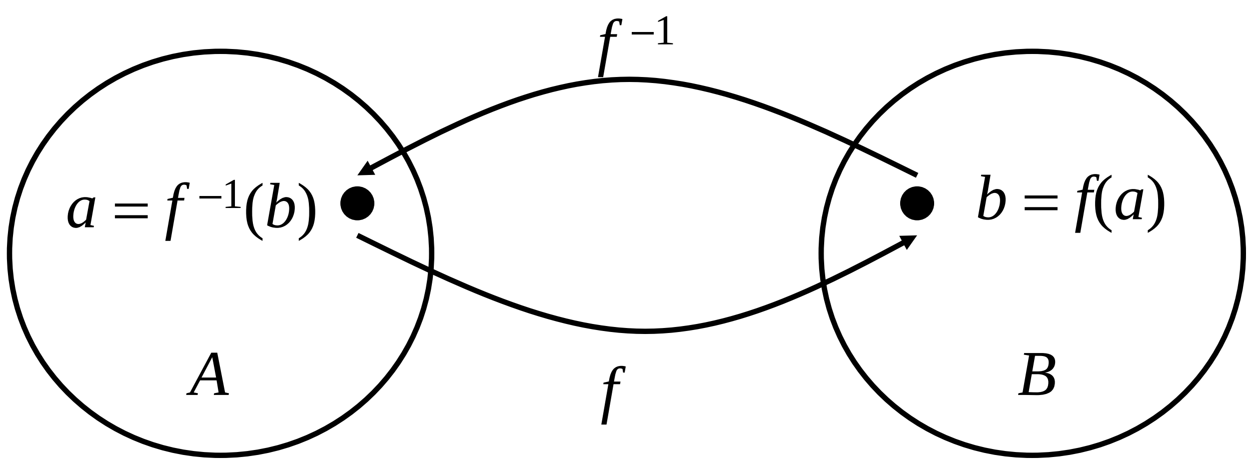
One-to-One
Correspondence

Monotonic Functions

- A monotonic function is
  - either *monotonically* (*strictly*) *increasing*
  - or *monotonically* (*strictly*) *decreasing*
- Consider a function $\boldsymbol{f} : \mathbf{R} \rightarrow \mathbf{R}$
- $\boldsymbol{f}$ is monotonically increasing
  - if $\boldsymbol{f(x) \leq f(y)}$ whenever $\boldsymbol{x} < \boldsymbol{y}$
- $\boldsymbol{f}$ is monotonically decreasing
  - if $\boldsymbol{f(x) \geq f(y)}$ whenever $\boldsymbol{x} < \boldsymbol{y}$

Inverse Functions

- Let $\boldsymbol{f : A \rightarrow B}$ be <u>one-to-one correspondence</u> such that $\boldsymbol{f(a) = b}$.
- The inverse of the function $\boldsymbol{f}$ is denoted by $\boldsymbol{f^{-1}(b) = a}$.



F needs to be bijection

- If f is not a bijection (not one-to-one correspondence), then
  - f is not injective (not one-to-one)
  - f is not surjective (not onto)
- Why can't we invert such a function?

  We cannot assign to each element b in the codomain a unique element a in the domain such that $f(a) = b$, because:
  - For some b there is either
    - More than one a
    - No such a

Inverse Functions

- Let $\boldsymbol{f : \mathbf{Z} \to \mathbf{Z}}$ be a function with $\boldsymbol{f(x) = x + 1}$
- Is f invertible? Is f a bijection?
  - Is f one-to-one? YES
  - Is f onto? YES
  - So f is a one-to-one correspondence and is therefore invertible.
- Then, what is its inverse?

  f(y) = y - 1

Inverse Functions

- Let $\boldsymbol{f} : \mathbf{Z} \rightarrow \mathbf{Z}$ be a function with $\boldsymbol{f(x) = x^2}$.
- Is *f* invertible?
- Is *f* a one-to-one correspondence. NO
- So *f* is not a one-to-one, and
- therefore, *f* is not invertible.

Compositions of Functions

- Let $\boldsymbol{g} : \boldsymbol{A} \rightarrow \boldsymbol{B}$ and $\boldsymbol{f} : \boldsymbol{B} \rightarrow \boldsymbol{C}$.
- The *composition* of the functions $\boldsymbol{f}$ and $\boldsymbol{g}$, denoted by $\boldsymbol{f} \circ \boldsymbol{g}$, is defined by:

$$\boldsymbol{f} \circ \boldsymbol{g}(\boldsymbol{a}) = \boldsymbol{f}\,(\boldsymbol{g}(\boldsymbol{a}))$$

- $\boldsymbol{f} \circ \boldsymbol{g}$ can't be defined unless the range of $\boldsymbol{g}$ is a subset of the domain of $\boldsymbol{f}$.

Example

- Let:

  $f(x) = 2x + 3$

  $g(x) = 3x + 2$

- Find $f \circ g(x)$:

  2(3x +2) + 3

- Find $g \circ f(x)$:

  3(2x + 3) + 2

Let:

$f(a) = b$, so

$f^{-1}(b) = a$

- Find $f^{-1} \circ f\,(a)$:

  a

- $f \circ f^{-1}\,(b)$

  b

Important functions – Floor

- Let $\boldsymbol{x}$ be a real number. The *floor function* is the closest integer less than or equal to $\boldsymbol{x}$.
- Examples:

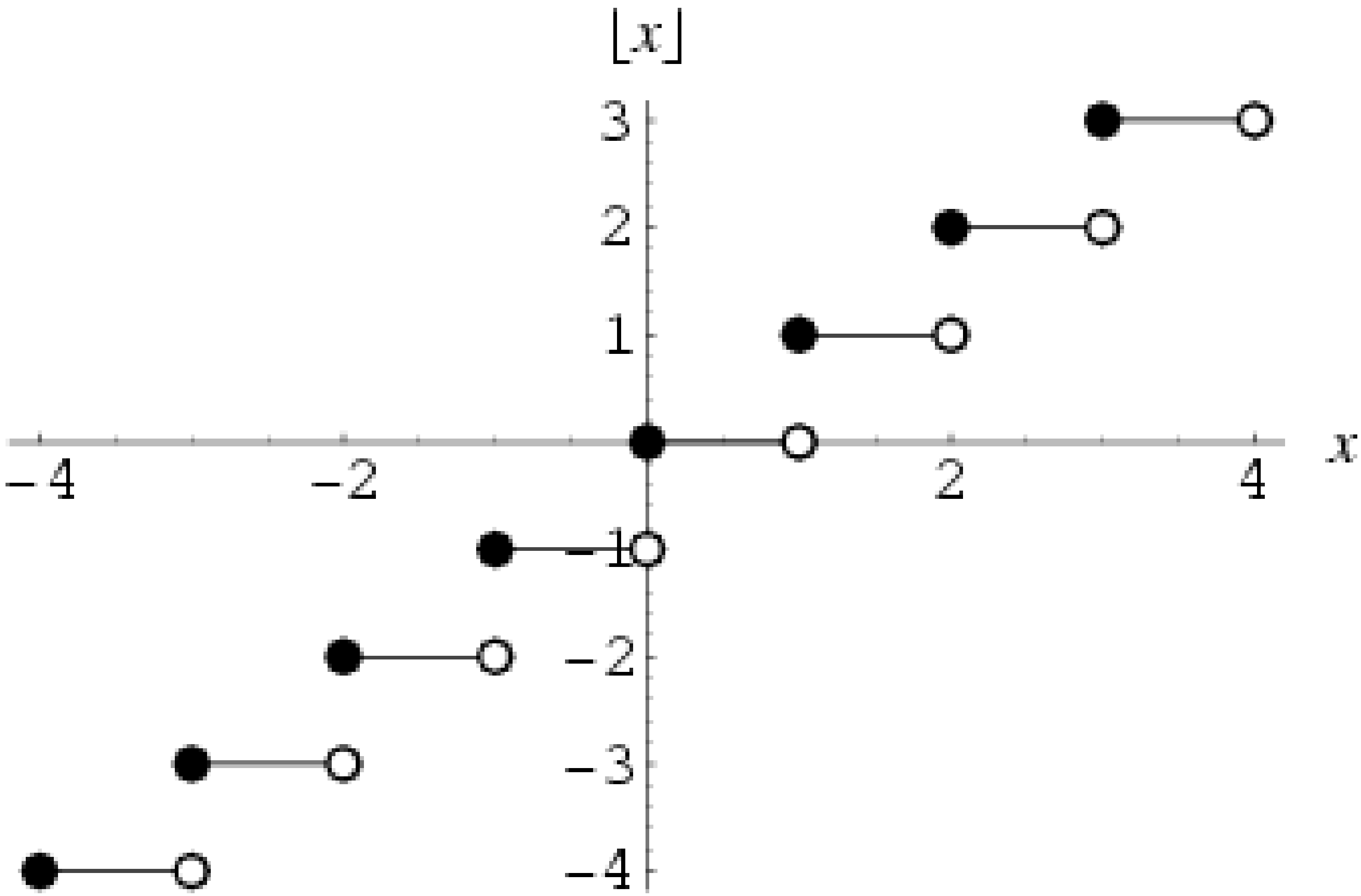
$$\lfloor \tfrac{1}{2} \rfloor = 0$$

$$\lfloor -\tfrac{1}{2} \rfloor = ?$$

$$\lfloor 3.1 \rfloor = ?$$

$$\lfloor 7 \rfloor = ?$$

Floor



http://mathworld.wolfram.com/FloorFunction.html

Important functions – Ceiling

- Let ***x*** be a real number. The *ceiling function* is the closest integer greater than or equal to ***x***.
- Examples:

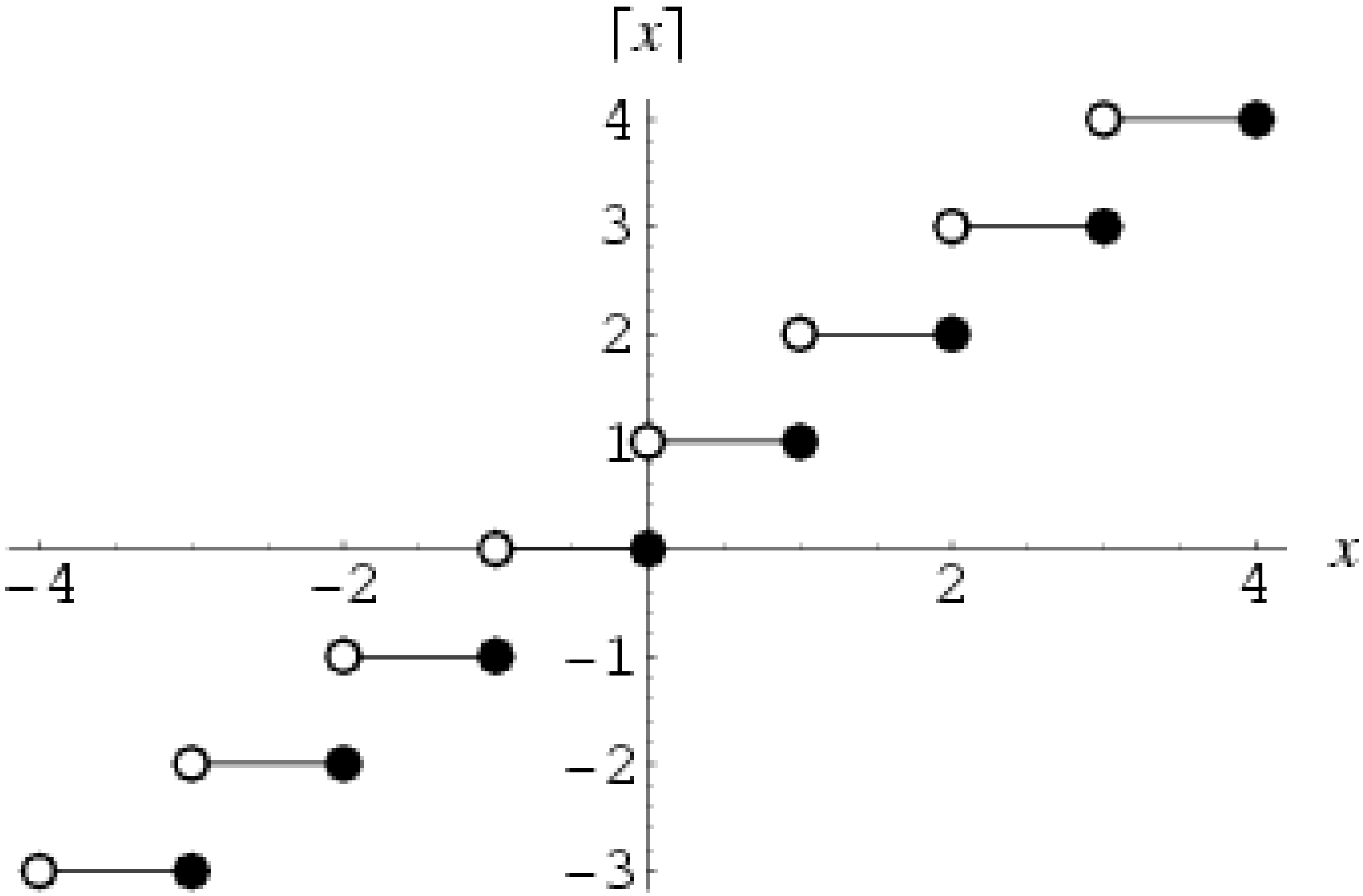
$$\lceil 1/2 \rceil = 1$$

$$\lceil -1/2 \rceil = ?$$

$$\lceil 3.1 \rceil = ?$$

$$\lceil 7 \rceil = ?$$

Ceiling



http://mathworld.wolfram.com/CeilingFunction.html

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Applied discrete mathematics

Chapter 2, Section 2.4
Sequences and Summations

Sequence

- If the domain of a function is restricted to integers, the function is called a *sequence*.
- The domain is specifically the set **N** or the set $\mathbf{Z}^+$.
- a_n denotes the image of n
  - called a *term* of the sequence
- Notation for whole sequence: $\{a_n\}$

Example

- Let $\{a_n\}$ be a sequence, where

$$a_n = 1/n \quad \text{and} \quad n \in \mathbf{Z}^+$$

- What are the *terms* of the sequence?

$$a_1 = 1$$

$$a_2 = 1/2$$

$$a_3 = 1/3$$

$$a_4 = 1/4$$

… … …

Sequence Notation

- Unless stated to the contrary, we will assume the domain of a sequence to be the set of all positive integers.
- a_n is called the n^{th} term or general term.

Geometric/Arithmetic Progression

- Geometric Progression:
  - A sequence of the form $a, ar, ar^2, \ldots, ar^n$
  - $a \in \mathbf{R}$ and $r \in \mathbf{R}$
  - a is the *initial term* and r is the *common ratio*
- Arithmetic Progression:
  - A sequence of the form $a, a+d, a+2d, \ldots, a+nd$
  - $a \in \mathbf{R}$ and $d \in \mathbf{R}$
  - a is the *initial term* and d is the *common difference*

Example

- Let $\{b_n\}$ be a sequence, where $b_n = (-1)^n$
  - What type of progression is this?
    (Geometric)
  - What is the initial term?
    (-1)
  - What is the common ratio/difference?
    (-1)
  - What are the terms of the sequence?
    (-1,1, -1, 1, -1, 1, …)

Example

- Let $\{d_n\}$ be a sequence, where $d_n = 6 \cdot (1/3)^n$
  - What type of progression is this?

    (Geometric)
  - What is the initial term?

    (2)
  - What is the common ratio/difference?

    (1/3)
  - What are the terms of the sequence?

    (2, 2/3, 2/9, 2/27, …)

Example

- Let $\{s_n\}$ be a sequence, where $s_n = -1 + 4n$
  - What type of progression is this?

    (Arithmetic)
  - What is the initial term?

    (3)
  - What is the common ratio/difference?

    (4)
  - What are the terms of the sequence?

    (3, 7, 11,15, …)

Example

- Let $\{t_n\}$ be a sequence, where $t_n = 7 - 3n$
  - What type of progression is this?

    (Arithmetic)
  - What is the initial term?

    (4)
  - What is the common ratio/difference?

    (-3)
  - What are the terms of the sequence?

    (4, 1, -2, -5, …)

Example

- Find a formula for this sequence:

  1, 1/2, 1/3, 1/4, 1/5, ...

  What is the formula?

  ($a_n = 1/n$)

Summations

- A summation denotes the sum of the terms of a sequence.
- Example:

Upper limit

Index of Summation

Lower limit

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

$$= 1 + 4 + 9 + 16 + 25$$

$$= 55$$

Geometric Series

- The sum of a geometric progression is called a geometric series
- Commonly used

$$S = \sum_{j=0}^{n} ar^{j} = ar^{0} + ar^{1} + ar^{2} + \ldots + ar^{n}$$

Double Summation

$$\sum_{i=1}^{3}\sum_{j=1}^{2}(i-j)$$

$$\sum_{j=1}^{2}(i-j)=(i-1)+(i-2)=2i-3$$

$$\sum_{i=1}^{3}(2i-3)=(2\cdot 1-3)+(2\cdot 2-3)+(2\cdot 3-3)$$

$$=-1+1+3=3$$

Useful Summation Formulae

TABLE 2 **Some Useful Summation Formulae.**

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \; (r \neq 0)$	$\dfrac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\dfrac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\dfrac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\dfrac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \; \lvert x \rvert < 1$	$\dfrac{1}{1 - x}$
$\sum_{k=1}^{\infty}, kx^{k-1}, \; \lvert x \rvert < 1$	$\dfrac{1}{(1 - x)^2}$

Cardinality of Infinite Sets

- A finite set is obviously countable. How about an infinite set?
- Remember that sets A and B have the same cardinality (number of elements) iff there is a one-to-one correspondence between A and B.
- We say that a set S is *countable* iff there is a one-to-one correspondence between S and Z^+, the set of positive integers.
- A set that is not countable is called *uncountable*.

Cardinality of Infinite Sets

- The set of all integers is countable.
- This means that there is a one-to-one correspondence between the set of *all* integers and the set of *positive* integers.
- We establish the one-to-one correspondence as follows:

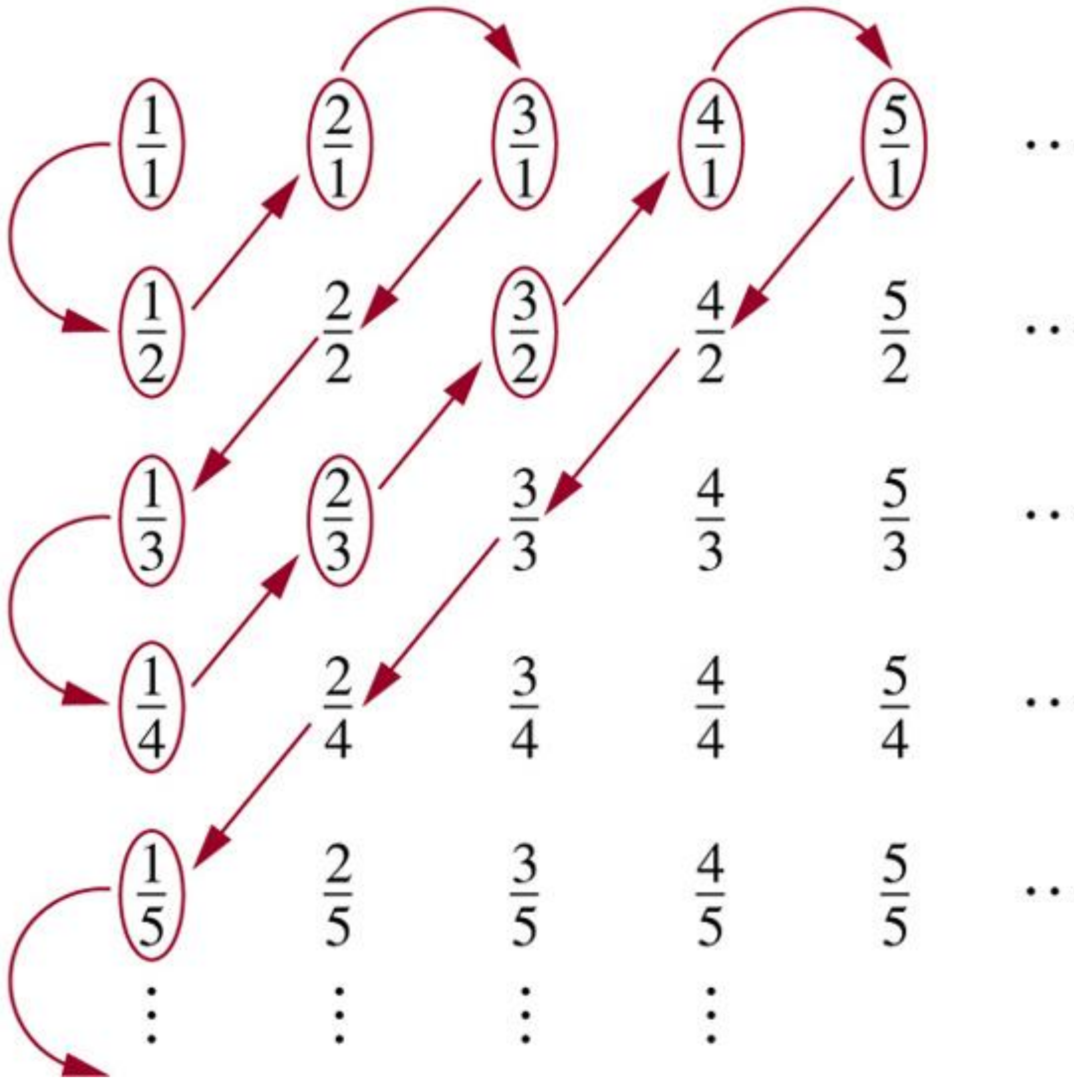
…	***11***	***9***	***7***	***5***	***3***	***1***	***2***	***4***	***6***	***8***	***10***	…
…	**−5**	**−4**	**−3**	**−2**	**−1**	**0**	**1**	**2**	**3**	**4**	**5**	…

Cardinality of Infinite Sets

- The set of positive rational numbers is countable.
- To prove this we set up the postive rational numbers in a 2-D matrix in which the numerators increase as you move to the right in a row, and the denominators increase as you go down a column.
- You count them by moving along the diagonals of the matrix, skipping any rational numbers that we have already counted previously.

Cardinality of Infinite Sets

Terms not circled are not listed because they repeat previously listed terms



Cardinality of Infinite Sets

- Can we establish a one-to-one correspondence between Z^+ and R, the set of real numbers?

- No. Goerg Cantor showed that we can't, using the Cantor diagonalization proof.

- Basically, this proof assumes that we can, and shows that this implies that we can put all the real numbers into a sequence, in which we can specify for any real number *x* what the "next" real number *y* is.

- However, for any two real numbers, no matter how close they are, we can always find another real number between them. Thus, we have a contradiction.

Cardinality of Infinite Sets

- We denote the cardinality of any infinite countable set by the symbol $\aleph_0$, pronounced "aleph null".
- We denote the cardinality of any uncountable set by the symbol $\aleph_1$, pronounced "aleph one".

Cardinality of Infinite Sets

- Is the set of all possible computer programs countable?
- Any given computer program can be represented in binary form, as a finite sequence of 0’s and 1’s.
- Any finite sequence of 0’s and 1’s can be interpreted as an integer.
- The set of integers is countable.
- Therefore, the set of all possible computer programs is countable.

Cardinality of Infinite Sets

- Is the set of all functions countable?
- Let us assume that each different function returns a different subset of R, the set of real numbers.
- There are $2^{|R|}$ different subsets of R. Since R is uncountable, $2^{|R|}$ is certainly also uncountable.
- So the set of all functions is uncountable.
- But the set of computer programs is countable.
- So there are fewer computer programs than there are functions that we might want to compute.

Conclusion

- In this chapter we have covered:
  - Introduction to sets
  - Set operations
  - Functions
  - Sequences and summations
  - Cardinality of infinite sets

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- In this chapter we have covered:
  - Introduction to sets
  - Set operations
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  - Sequences and Summations