MTH 2215
APPLIED DISCRETE MATHEMATICS

Chapter 4, Section 4.1
Mathematical Induction

These class notes are based on material from our textbook, **Discrete Mathematics and Its Applications**, 6th ed., by Kenneth H. Rosen, published by McGraw Hill, Boston, MA, 2006. They are intended for classroom use only and are **not** a substitute for reading the textbook.

Mathematical Induction

- Used to prove propositions of the form $\forall n P(n)$, where $n \in \mathbf{Z}^+$
- Can be used only to prove results originally obtained in some other way --
- Not a tool for discovering new theorems

Steps

- A proof by mathematical induction that $P(n)$ is true for every positive integer n consists of two steps:
  - *Basis step*: The proposition $P(1)$ is shown to be true.
  - *Inductive step*: The implication $P(k) \to P(k + 1)$ is shown to be true for every positive integer k.

Mathematical Induction

- Expressed as a rule of inference, this proof technique can be stated as

$$[P(1) \wedge \forall k(P(k) \rightarrow P(k+1))] \rightarrow \forall n P(n)$$

- It is not assumed that $P(k)$ is true for all positive integers!
- It is only shown that if it is assumed that $P(k)$ is true for some individual (but unspecified) k , then $P(k + 1)$ is also true.
  - not a case of circular reasoning

Mathematical Induction

- When we use mathematical induction to prove a theorem, we first show that $P(1)$ is true. Then we know:

$P(1) \rightarrow P(2)$	Therefore, $P(2)$ is true
$P(2) \rightarrow P(3)$	Therefore, $P(3)$ is true
$P(3) \rightarrow P(4)$	Therefore, $P(4)$ is true
………………	
$P(l\text{-}2) \rightarrow P(l\text{-}1)$	Therefore, $P(l\text{-}1)$ is true
$P(l\text{-}1) \rightarrow P(l)$	Therefore, $P(l)$ is true

Examples

- Assume that you live in the1950’s, when postage for a first-class letter weighing 1 ounce or less cost 8¢.
- Heavier letters require more postage: an additional 1¢ for each extra ounce.
- You have a supply of 3¢ and 5¢ stamps.
- Theorem: Using 3¢ anf 5¢ stamps, you can put the correct postage on a letter of any weight.

Three cases

- Base case: A letter weighing ≤ 1 ounce requires 8¢ postage. One 3¢ and one 5¢ stamp = 8¢.
- A letter weighing up to 1 once more than our base case (i.e., 2 ounces) requires 9¢ postage. We observe that: (a) 9¢ is 1¢ more than 8¢, and (b) 6¢ is 1¢ more than 5¢. Since 6¢ is 3¢ + 3¢, we can replace the 5¢ stamp in the base case with two 3¢ stamps: 3¢ + 3¢ + 3¢ = 9¢.
- A letter weighing 3 ounces requires 10¢ postage. We know: (a) 10¢ is 1¢ more than 9¢, and (b) two 5¢ stamps are 1¢ more than three 3¢ stamps. So we replace the three 3¢ stamps in our previous case with two 5¢ stamps. 5¢ + 5¢ = 10¢.

Induction step

- Can we devise a general rule based on these cases? Yes:
- Suppose we have the correct postage for a letter that requires *k*¢ worth of stamps, and the stamps include at least one 5¢ stamp. In that case we can supply the correct postage for a letter requiring (*k*+1)¢ worth of stamps by replacing a 5¢ stamp with two 3¢ stamps.
- Suppose we have the correct postage for a letter that requires *k*¢ worth of stamps, and the stamps include at least three 3¢ stamps. In that case we can supply the correct postage for a letter requiring (*k*+1)¢ worth of stamps by replacing three 3¢ stamps with two 5¢ stamps.

Does this work for <u>any</u> *k*?

- Yes, it does.
- k = 8¢: use one 3¢ and one 5¢ stamp (base case)
- k = 9¢: does k-1 have at least one 5¢ stamp? Yes; then replace 5¢ stamp with two 3¢ stamps.
- k = 10¢: does k-1 have at least one 5¢ stamp? No. Does k-1 have at least three 3¢ stamps? Yes; then replace three 3¢ stamps with two 5¢ stamps.
- k = 11¢: does k-1 have at least one 5¢ stamp? Yes; then replace with two 3¢ stamps.
- k = 12¢: does k-1 have at least one 5¢ stamp? No. Does k-1 have at least three 3¢ stamps? Yes; then replace three 3¢ stamps with two 5¢ stamps.
- etc. ...

Example

- Let S_n denote the sum of the first *n* positive integers. Using inductive proof, we want to show that for any $n \geq 1$, $S_n = (n\ (n + 1))\ /\ 2$.
- Every inductive proof has the same pattern:
  - (1) we establish that some statement S(*k*) is true for some particular value of *k* [the *basis*], and then
  - (2) we prove that, **if** S(*n*) is true for *n* [the *inductive hypothesis*], it **must** be true for *n* + 1.

Example

- When we ask, “What is the *basis* for this proof?” we are asking, “what do we already know, or could show by demonstration if asked to do so.”
- Since the proof involves positive integers and the condition is that $n \geq 1$, we start with $n = 1$:

$$S_1 = (1\ (1 + 1)) / 2 = 2 / 2 = 1$$

Example

- What is the *inductive hypothesis* for this proof?
- We know that S_k is true for some k, namely, $k = 1$. Our inductive hypothesis is that S_k is true for any $k < (n + 1)$, that is:

  $$S_n = (n\ (n + 1))\ /\ 2$$

- Our job will be to prove that S_{n+1} is also true. That is, **we must prove that**:

  $$S_{n+1} = ((n + 1)\ ((n + 1) + 1))\ /\ 2$$

  (This is our *goal*; we got this by substituting $n + 1$ for n in our inductive hypothesis.)

Example

Give the proof:

1) $S_{n+1} = 1 + 2 + 3 + \ldots + n + (n + 1)$		definition
2) $= S_n$	$+ (n + 1)$	substitution
3) $= n (n + 1) / 2$	$+ (n + 1)$	ind. hyp. + sub.
4) $= (n2 + n) / 2$	$+ (n + 1)$	distribution
5) $= (n2 + n) / 2$	$+ (2n + 2)/2$	mult. by 2/2
6) $= (n2 + 3n + 2) / 2$		addition
7) $= (n + 1) (n + 2) / 2$		factoring
8) $= ((n + 1) ((n + 1) + 1)) / 2$		$2 = 1 + 1$
9) Q.E.D.		

Example

In every inductive proof we need to start from something we already know, and derive the goal statement, using the inductive hypothesis somewhere in the proof.

Homework Exercises

- Use mathematical induction to show that

  $a + ar + ar^2 + \ldots + ar^n = (ar^{n+1} - a)/(r - 1)$

  for all non-negative integers n.
- Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

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Chapter 4, Section 4.2
Strong Induction and Well-Ordering

Mathematical Induction (Recap)

- A proof by mathematical induction that $P(n)$ is true for every positive integer n consists of two steps:
  - *Basis step*: The proposition $P(1)$ is shown to be true.
  - *Inductive step*: The implication $P(k) \rightarrow P(k + 1)$ is shown to be true for every positive integer k.

Mathematical Induction (Recap)

- The inductive step can be thought of as also consisting of two parts:
  - Assume P(k)
  - Prove P(k + 1)

Example

- Use mathematical induction to prove that $2^k > k$ for all positive integers k.

Example

- What is the *basis* for the proof?
- For this problem, the basis is when $k = 1$, so P(1) is the assertion that $2^1 > 1$.
- This is obviously true.

Example

- The inductive step is to:
  - Assume P(k)
  - Prove P(k + 1)
- P(k) is the assertion that $2^k > k$. We can assume that this is true.
- Now we try to prove that P(k + 1) is also true. P(k + 1) is the assertion that $2^{k+1} > k + 1$.

Example

- The inductive step is to:
  - Assume P(k)
  - Prove P(k + 1)
- P(k) is the assertion that $2^k > k$. We can assume that this is true.
- Now we try to prove that P(k + 1) is also true. P(k + 1) is the assertion that $2^{k+1} > k + 1$.

Example

- Work through the proof:

$2^{k+1} = 2^k \cdot 2$	Def. of 2^{k+1}
$2^k > k$	Inductive hypothesis
$2^k \cdot 2 > k \cdot 2$	Mult. both sides by 2
$2^{k+1} > k \cdot 2$	Substitution
$k \cdot 2 = k + k$	Def. of "times 2"
$2^{k+1} > k + k$	Substitution
$k + k \geq k + 1$	True if k is a pos. int.
$2^{k+1} > k + 1$	Substitution

Q.E.D.

Example 2

- Use mathematical induction to prove that $k^3 - k$ is divisible by 3 whenever k is a positive integer.

Example 2

- What is the *basis* for the proof?
- For this problem, the basis is when $k = 1$, so P(1) is the assertion that $1^3 – 1$ is divisible by 3.
- $1^3 – 1 = 0$, which is divisible by 3 (with no remainer). This is obviously true.

Example 2

- The inductive step is to:
  - Assume P(k)
  - Prove P(k + 1)
- P(k) is the assertion that $k^3 - k$ is divisible by 3. We can assume that this is true.
- Now we try to prove that P(k + 1) is also true. P(k + 1) is the assertion that $(k + 1)^3 - (k + 1)$ is divisible by 3.

Example 2

$(k + 1)^3 - (k + 1)$	P(k + 1)
$= (k^3 + 3k^2 + 3k + 1) - (k + 1)$	Carrying out the mult.
$= (k^3 + 3k^2 + 3k + 1) - k - 1$	Distrib. the -
$= (k^3 - k) + 3(k^2 + k)$	Rearrange & subtract
$(k^3 - k)$ is divisible by 3	Ind. hyp.
$3(k^2 + k)$	Def. of "3 *"

$\therefore$ $(k^3 - k) + 3(k^2 + k)$ is divisible by 3

Q.E.D.

Well-Ordered Sets

- The validity of mathematical induction follows from a fundamental axiom about the set of integers called the *Well-Ordering Property:*
  - Every nonempty set of nonnegative integers has a *least element*

Why Mathematical Induction Is Valid

- Suppose we know that $P(1)$ is true and that the proposition $P(k) \to P(k + 1)$ is true for all positive integers k.
- To show that $P(n)$ must be true for all positive integers, assume that there is at least one positive integer for which $P(n)$ is false.
- Then the set ***S*** of positive integers for which $P(n)$ is false is nonempty.

Why Mathematical Induction Is Valid

- By the well-ordering property, ***S*** has a least element, which will be denoted by m. We know that m cannot be 1, since $P(1)$ is true.
- Since $m > 1$, $m - 1$ is a positive integer.
- Furthermore, since $m - 1$ is less than m, it is not in ***S***, so $P(m - 1)$ must be true.

Why Mathematical Induction Is Valid

- Since the implication $P(m - 1) \rightarrow P(m)$ is also true, it must be the case that $P(m)$ is true.
- This contradicts the choice of m.
- Hence, $P(n)$ must be true for every positive integer n.

Strong Induction

- To prove that $P(n)$ is true for every positive integer n consists of two steps:
  - *Basis step*: The proposition $P(1)$ is shown to be true.
  - *Inductive step*: The implication $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k + 1)$ is shown to be true for every positive integer k.

Strong Induction

- We can express the principle of strong induction in a slightly different way that may be a little clearer: The following two statements:
- P(1) is true
- $[(\forall k)P(r)$ is true for all r, where $1 \leq r \leq k], \rightarrow P(k + 1)$ is true

mean that

- P(n) must be true for every positive integer k.

Example

- Use Strong Induction to show that if n is an integer greater than 1, then *n* can be written as the product of primes.

Example

- Let P(*n*) mean "*n* can be written as the product of primes"
- Can we find a basis step? Yes; 2 is the product of one prime, itself. So P(2) is true.
- What is the inductive hypothesis. Since this is a strong induction proof, we assume that P(*j*) is true for all positive integers *j* with $j \leq k$.

Example

- This means that we can assume that j can be written as the product of primes whenever j is a positive integer ≥ 2 and $j \leq k$.
- What do we need to prove? That $P(k+1)$ is also true; that is, that $k + 1$ can also be written as the product of primes.

Example

- Proof, case 1: Consider the case that $k + 1$ is prime. Then $(k + 1)$ can be written as the product of one prime, itself. Q.E.D.
- Case 2: Consider the case that $k + 1$ is not prime. Then $k + 1$ must be a *composite* positive integer which, by definition, can be written as the product of two positive integers a and b, with $2 \leq a \leq b \leq k$.

Example

- Remember that our inductive hypothesis says that we can assume that a (or b) can be written as the product of primes whenever a (or b) is a positive integer ≥ 2 and a (or b) $\leq k$.
- Therefore, $k + 1 =$ (the primes in the prime factorization of a) • (the primes in the prime factorization of b). Q.E.D.

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Chapter 4, Section 4.3
Recursive Definitions

Recursion

- We can define a sequence, series, or function in terms of itself. This process is called *recursion*.
- To define a function with the set of nonnegative integers as its domain:
  - Specify the value of the function at zero
  - Give a rule for finding its value at an integer from its values at smaller integers

Example

- The sequence of powers of 2 can be written explicitly as

$$a_n = 2^n$$

- However, this sequence can also be defined using recursions as

$$a_0 = 1, \quad a_{n+1} = 2a_n$$

Example

- Suppose that f is defined recursively by

$$f(0) = 3$$

$$f(n + 1) = 2f(n) + 3$$

- Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$

Example

- If f is defined recursively by

$$f(0) = 3$$
$$f(n + 1) = 2f(n) + 3$$

- Then:

$$f(1) = 2f(0) + 3 = 2 \cdot 3 + 3 = 9$$
$$f(2) = 2f(1) + 3 = 2 \cdot 9 + 3 = 21$$
$$f(3) = 2f(2) + 3 = 2 \cdot 21 + 3 = 45$$
$$f(4) = 2f(3) + 3 = 2 \cdot 45 + 3 = 93$$

Example

- Give a recursive definition of a^n where a is a nonzero real number and n is a nonnegative integer.
- First, define the base case a^0 as $a^0 = 1$.
- Next, define a rule for finding a^{n+1} from a^n for all $n \geq 0$: $a^{n+1} = a \bullet a^n$
- Now we know how to find $a^0, a^1, a^2, a^3, \ldots$

Example

- Give a recursive definition of the factorial function $F(n) = n!$
- First, define the base case $F(0) = 1$
- Next, define a rule for finding $F(n+1)$ from $F(n)$ for all $n \geq 0$: $F(n+1) = (n+1)F(n)$
- Now, to find $F(3)$ we just compute $(3)F(2)$. To find $F(2)$ we just compute $(2)F(1)$. To find $F(1)$ we just compute $(1)F(0)$.

Example

- We know that $F(0) = 1$.
- So:

$F(3) = (3)\ (2)\ (1)F(0)$, or $3 \cdot 2 \cdot 1 \cdot 1 = 6$

Example

- Give a recursive definition for the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21 ...
- First, define the base case. For the Fibonacci numbers there are two base cases: $f(0) = 0$ and $f(1) = 1$
- Next, define a rule for finding $f(n+1)$ from $f(n)$ for all $n \geq 1$: $f(n+1) = f(n) + f(n-1)$

Example

- So:

$f(0) = 0$

$f(1) = 1$

$f(2) = f(1) + f(0) = 1 + 0 = 1$

$f(3) = f(2) + f(1) = 1 + 1 = 2$

$f(4) = f(3) + f(2) = 2 + 1 = 3$

$f(5) = f(4) + f(3) = 3 + 2 = 5$

$f(6) = f(5) + f(4) = 5 + 3 = 8 \quad \ldots$

Conclusion

- In this chapter we have covered:
- Mathematical induction
- Strong induction
- Well-ordering
- Recursive definitions