MTH 2215
Applied discrete mathematics

Chapter 5, Section 5.1
The Basics of Counting

These class notes are based on material from our textbook, **Discrete Mathematics and Its Applications**, 6^{th} ed., by Kenneth H. Rosen, published by McGraw Hill, Boston, MA, 2006. They are intended for classroom use only and are **not** a substitute for reading the textbook.

The Sum Rule

- Suppose a task can be done by doing *either* (but not both) of two subtasks:
  - The first subtask can be done in n_1 ways
  - The second subtask can be done in n_2 ways
  - These subtasks cannot be done at the same time, nor are they done in sequence; one and only one subtask must be chosen.
- Then there are $(n_1 + n_2)$ ways to do the task.

Example

- Suppose that you have to select a representative to a university committee
  - Either a member of the mathematics faculty or a mathematics major student can be selected.
  - There are 37 member of the mathematics faculty and 83 mathematics major students.
- How many different choices are there for selecting the representative?

Example

How many different choices are there for selecting the representative?

Since a student can’t also be a faculty member (or vice versa) we can use the Sum Rule and simply add 37 and 83:

$37 + 83 = 120$

Example

- A student has to do a computer project
  - He can choose a project from one of three lists.
  - The three lists contain 23, 15, and 19 possible projects, respectively.
- How many possible projects are there to choose from?

Example

- How many possible projects are there to choose from?
- No project is on more than one list, so we can use the Sum Rule:

$$23 + 15 + 19 = 57$$

The Product Rule

- Suppose that a procedure can be broken down into two tasks such that both tasks must be done, in sequence:
  - The first task can be done in n_1 ways
  - The second task can be done in n_2 ways after the first task has been done
- Then there are $n_1 n_2$ ways to do the procedure.

Example

- The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100.
- What is the largest number of chairs that can be labeled differently?

Example

- What is the largest number of chairs that can be labeled differently?
- We can think of this problem as involving a sequence of two tasks:
  - Assign a letter between A and Z
  - Assign a number between 1 and 100
- The Product Rule says that there are 26 * 100 = 2600 ways to do this.
- So we can label 2600 chairs.

Example

- Will it make a difference in how many chair we can label if we assign the *number* first and then the *letter*?
- No; in either case we can label 2600 chairs.

Example

- How many different bit strings are there of length seven?
- You probably already know it is 2^7.
- Think of this as:

$$2\,(2\,(2\,(2\,(2\,(2 * 2)))))$$

Example

- How many different bit strings are there of length 1? Only 2: 0 or 1
- How many different bit strings are there of length 2? There are 4: 00, 01, 10, 11
- How many different bit strings are there of length 3? There are 8: 000, 001, 010, 011, 100, 101, 110, 111
- You can see the pattern here ….

Example

- How many different license plates are available if each plate contains a sequence of three letters followed by three digits?

26 choices for each letter

10 choices for each digit

Total of: $26 * 26 * 26 * 10 * 10 * 10$

Example

- Telephone Numbering Plan:
  - Area Code - Office Code - Station Code
  - X: 0-9, N: 2-9, Y: 0-1
  - Old Plan: NYX, NNX, XXXX
  - New Plan: NXX, NXX, XXXX

Example

- Telephone Numbering Plan:

  8 * 2* 10 area codes with NYX

  8 * 10 * 10 area codes with NXX

  8 * 8 *10 office codes with NNX

  8 * 10 * 10 office codes with NXX

  10 * 10 * 10 * 10 station codes with XXXX

  So:

  old plan → 160 * 640 * 10,000

  new plan → 800 * 800 * 10,000

Combining Sum Rule and Product Rule

- Used in complex counting problems
- Example:
  - A programming language can have 1 or 2 character case-insensitive variable names.
  - Each variable name must start with a letter.
  - There are five 2-character reserved words that cannot be used.
  - How many different variable names are there?

Combining Sum Rule and Product Rule

- There are 26 different 1-character variable names.
- The 2-character variable names must:
  - Start with one of 26 letters
  - End with one of 26 letters or 10 numerals
- So by the Product Rule there are 26 * 36 = 936 two-character variable names. Subtract the 5 reserved words and you get 931.
- By the Sum Rule there are 26 + 931 = 957 different variable names in this language.

Example

- Each user on a computer system has a password
  - Each password is six to eight characters long
  - Each character is an uppercase letter or a digit
  - Each password must contain at least one digit
- How many possible passwords are there?

Example

- There are 26 letters and 10 decimal digits = 36 characters that we can use to form passwords.
- For P_6 (6-character) passwords, the Product Rule says there are 36^6 potential passwords.
- But passwords that are all letters are prohibited. There are 26^6 of these.
- So there are $36^6 – 26^6$ P_6 passwords.
- Similarly, for P_7 and P_8 passwords.
- Total passwords = $P_6 + P_7 + P_8$

The Inclusion-Exclusion Principle

- Question: What happens when two tasks can be done at the same time, but we want to count the number of ways to do one of the two tasks?
- Answer: First, we add the number of ways to do the first task to the number of ways to do the second task. Then we subtract the number of ways to do both tasks. (Why? Because otherwise they get counted twice.)

Example

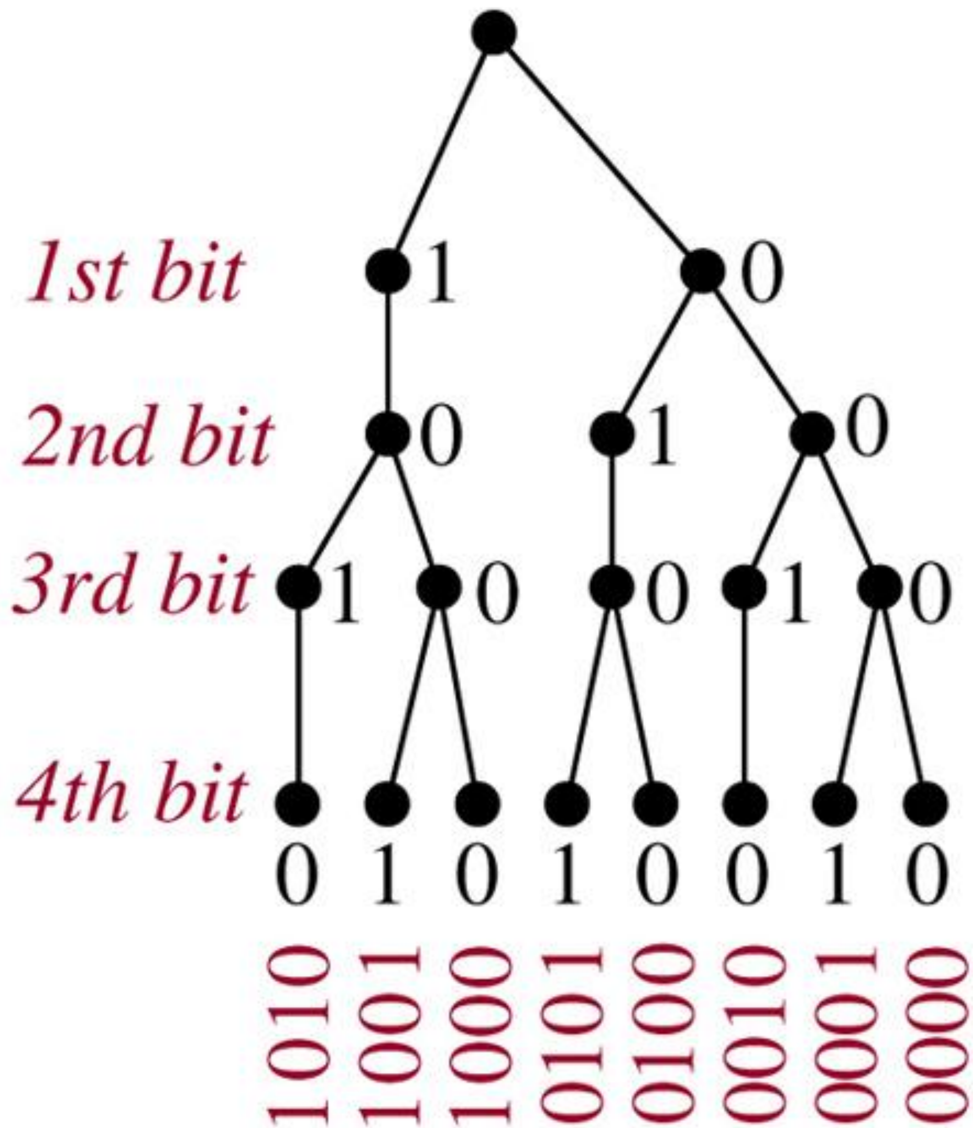
- How many bit strings of length eight either start with a 1 or end with the two bits 00?
  - 1st Task: Construct a string beginning with a 1.
  - 2nd Task: Construct a string ending with 00.
  - Both tasks: Construct a string that begins with a 1 and ends with 00.
- 1^{st}: There are 2^8 ways to construct a binary string of 8 bits, but half of these start with a 0, so there are 2^7 ways to construct an 8-bit binary string starting with 1. (The product rule says there is 1 way to chose the first bit and 2 ways to choose each of the other 7 bits.)

Example

- 2nd: Construct a string ending with 00.
  - The product rule says there are 2 ways to choose the first 6 bits and 1 way to chose the last 2 bits, so there are 2^6 ways to construct this string.
- Both: Construct a string that begins with 1 and ends with 00.
  - The product rule says there is 1 way to choose the first bit, 2 ways to chose the middle 5 bits, and 1 way to chose the last 2 bits, so there are 2^5 ways to construct this string.
- Total = $(2^7 + 2^6) – 2^5 = 160$

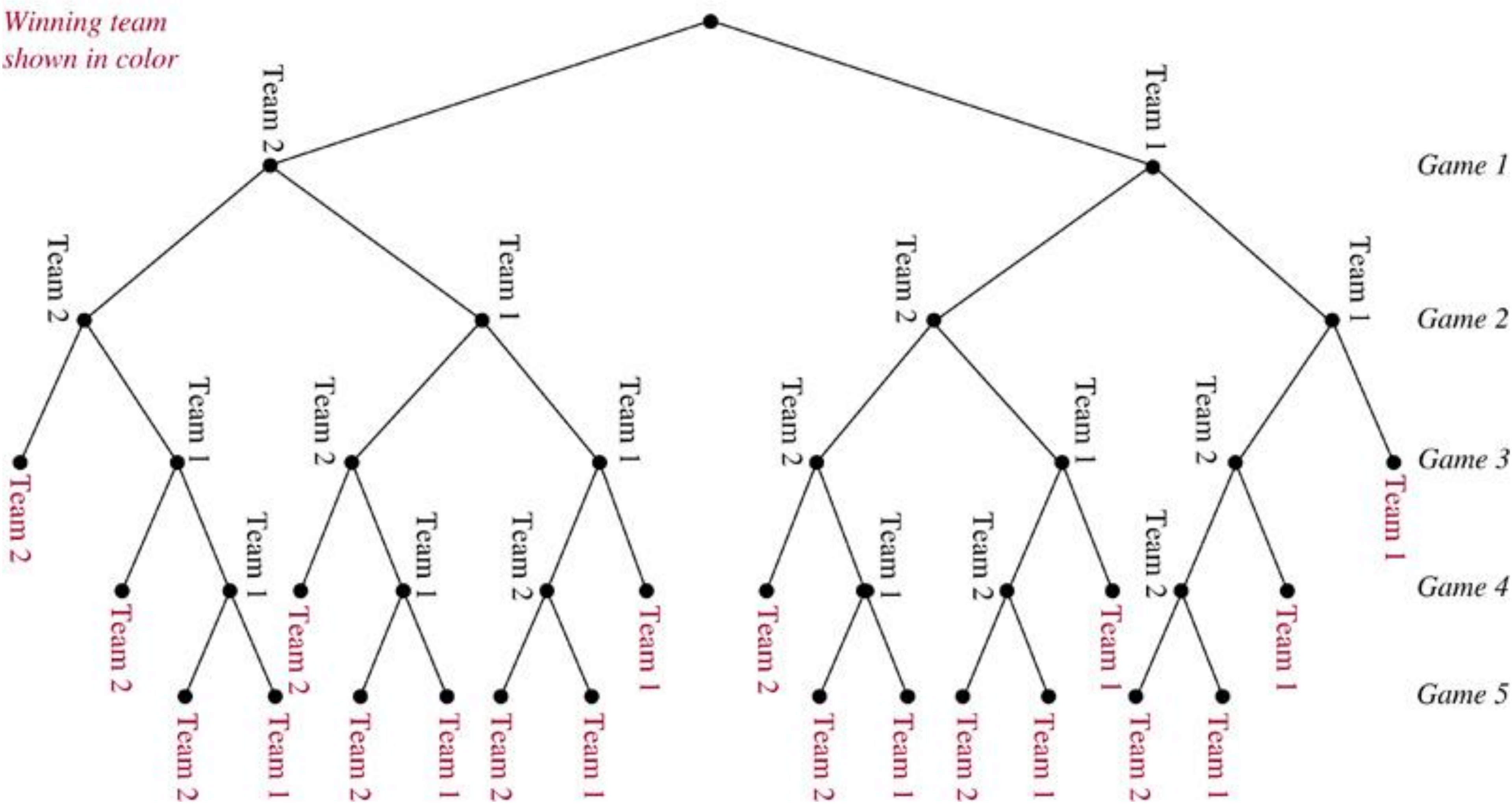
Tree Diagrams

- Sometimes we can use *tree diagrams* to solve counting problems.
- In a tree diagram, we use a *branch* to represent each possible choice. The possible outcomes are the *leaves* (terminal endpoints of the branches).
- Problem: How many bit strings of length 4 do not have two consecutive 1's?
- See the tree diagram on the next slide.
- There are 8 bit strings of length 4 that do not have two consecutive 1's.


1st bit
2nd bit
3rd bit
4th bit
1
0
0
1
0
1
0
0
1
0
0 1 0 1 0 0 1 0
1010
1001
1000
0101
0100
0010
0001
0000

Tree Diagrams

- Suppose that we are having a playoff between two baseball teams, and the first team that wins three games (out of 5) wins the playoff.
- In how many possible ways can the playoff occur?
- See the next slide.
- The playoff tree has 20 leaf nodes, each of which represents a different possible winning sequence.


Winning team
shown in color
Game 1
Game 2
Game 3
Game 4
Game 5
Team 1
Team 2

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Chapter 5, Section 5.2
The Pigeonhole Principle

The Pigeonhole Principle

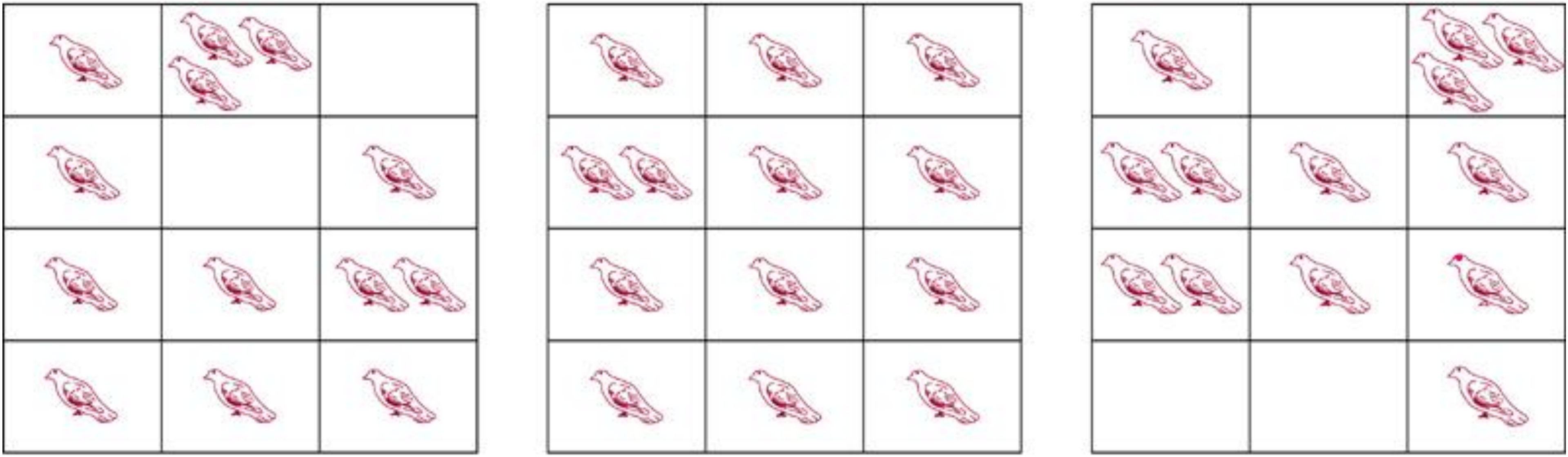
- The Pigeonhole Principle says:

  If k+1 or more objects are placed into k boxes then there is at least one box containing two or more of the objects.

The Pigeonhole Principle

- Suppose we have 12 pigeonholes, but 13 pigeons to put into them.
- Then at least one pigeonhole *must* contain at least two pigeons.
- See next slide:

The Pigeonhole Principle



(a) (b) (c)

The Pigeonhole Principle

- Proof by contradiction:
  - Given that we have k boxes and $k + 1$ objects.
  - Assume that, after distributing the objects into the k boxes, none of the boxes contains more than one object.
  - Therefore, the maximum total number of objects in the boxes would be $1 * k = k$.
  - But this contradicts the fact that we have $k + 1$ objects.
  - Therefore, after distributing the objects into the k boxes, at least one of the boxes must contain more than one object.

Example

- Among any group of 367 people, there *must* be at least two with the same birthday.
- Why?
- Because there are only 366 possible birthdays.

Example

- How many students must be in a class to *guarantee* that at least two students receive the same score on the final exam if the exam is graded on a scale from 0 to 100 points?

The Generalized Pigeonhole Principle

- If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.
- Example: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Example

- Assume that there are five possible final grades in this class: A, B, C, D, and F.
- What is the minimum number of students required to be sure that at least six will receive the same grade?

Example

- *Assume that there are five possible final grades in this class: A, B, C, D, and F.*
- *What is the minimum number of students required to be sure that at least six will receive the same grade?*
- The solution is the smallest integer N such that $\lceil N/5 \rceil = 6$.
- Obviously, N = 25 is too small.
- The ceilings of 26/5, 27/5, 28/5, 29/5, and 30/5 all = 6, and the smallest integer N is 26.

Example

- Assume telephone numbers are of the form *NXX*-*NXX*-*XXXX*, where:
  - the first three digits form the area code,
  - *N* represent a digit from 2 to 9 inclusive, and
  - *X* represents any digit.
- What is the least number of area codes to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers?

Example

- There are 8 million telephone numbers of the form *NXX-XXXX*.
- For 25 million phones to be assigned distinct 10-digit telephone numbers we would need $\lceil 25{,}000{,}000 / 8000{,}000 \rceil = 4$ different area codes.

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Chapter 5, Section 5.3

Permutations and Combinations

Permutations

- Permutation: A set of distinct objects in an *ordered arrangement* of these objects.
- *r*-permutation: An ordered arrangement of *r* elements of a set.
- Example: Consider the set $S = \{1,2,3\}$.
  - What are the 3-permutations of S ?
  - What are the 2-permutations of S ?

Permutations

- The number of r-permutations of an n-element set is:

$$P(n,r) = n(n-1)(n-2) \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

if n and r are integers with $0 \leq r \leq n$

Example

- Suppose that there are eight runners in a race. Gold, silver and bronze medals are given. How many different ways are there to award these medals, if all possible outcomes of the race can occur?
- Think about it this way: gold is for first place, silver for second, and bronze for third.

Example

- The *ordered set* {Smith, Jones, Green} is saying that Smith won the gold, Jones the silver, and Green the bronze.
- A *permutation* is simply an ordered set or subset.
- So the number of different ways to award the medals is the number of 3-permutations of a set with 8 elements.

Example

- Here is our formula for permutations:

$$P(n,r)=\frac{n!}{(n-r)!}$$

- So our formula for this problem would be:

P(8, 3) = 8! / (8 – 3)! = 8! / 5! = 336

Example

- Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the sales woman use when visiting these cities?

Example

- The starting city is predetermined, so it is not part of the permutation.
- There are 7 other cities that she can visit in any order.
- Therefore, this is a permutation problem, and the formula is:
- $P(7, 7) = 7! / (7 – 7)! = 7! / 1 = 5040$
- Trying to determine the best of all of the possible paths is called the *Traveling Salesman Problem*.

Combinations

- *r*-combination: An *unordered selection* of *r* elements of a set.
- Example: Consider the set $S = \{1, 2, 3\}$.
  - What are the 2-combinations of S ?

    $\{1, 2\}, \{1, 3\}, \{2, 3\}$
- Note that, because *combinations are unordered*, $\{1, 2\}$ is the same as $\{2, 1\}$.

Combinations

- ***Theorem 2:*** The number of *r*-combinations of an *n*-element set (read “n choose r”), where $0 \le r \le n$, is:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

or, equivalently,

$$C(n,r) = \frac{P(n,r)}{r!}$$

Combinations

- ***Corollary 2:*** Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n\text{-}r)$.

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

and,

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!}$$

and these are obviously equivalent.

Example

- How many 2-combinations are there of a set with three elements, {a, b, c}?
- Here is our formula:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

So:

```
C(3, 2) = 3!                / (2! •       (3 - 2)!
C(3, 2) = (3 • 2 • 1) / ((2 • 1) • (1))
C(3, 2) = (6)               / ((2)       • (1))
C(3, 2) = (6)               / ((2))       = 3
```

Example

- What is the value of $C(7, 3)$?
- Here is our formula:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

So:

$$\frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = \frac{7 \cdot 5}{1} = 35$$

Example

- How many ways are there to select 5 players from a 10 member tennis team to make a trip to a match at another school?
- Well, how many 5-combinations are there in a set with 10 elements? Here is the formula:

$$C(n,r)=\frac{n!}{r!(n-r)!}$$

- C(10,5) = 10! / ((5! • (10 – 5)!)
- C(10,5) = 10! / (5! • 5!)
- C(10,5) = 252

Example

- A CSE senior needs 5 CSE courses to graduate. There are 12 different computer science courses that she can take next semester that would count toward her degree. How many different sets of 5 courses could she take?
- This is a "12 choose 5" problem; how many 5-combinations are there in a set with 12 elements? Here is the formula:

- $C(12,5) = 12! / ((5! \cdot (12 - 5)!)$
- $C(12,5) = 12! / (5! \cdot 7!)$
- $C(12,5) = 792$

Example

- A committee has to be selected to develop a discrete mathematics course at a school.
  - The committee is to consist of 3 faculty members from the mathematics department and 4 from the computer science department.
  - There are 9 faculty members in the mathematics department and 11 in the computer science department.
- How many ways are there to select the committee?

Example

- By the Product Rule, the answer is the product of:
  - the number of 3-combinations of a set with 9 elements, and
  - the number of 4-combinations of a set with 11 elements
- This is $C(9, 3) \cdot C(11, 4)$, which equals:

$[9! / (3! \cdot 6!)] \cdot [11! / (4! \cdot 7!)] =$

$84 \cdot 330 = 27,720$

Example

- For you poker players out there: the total number of five-card poker hands from a deck of playing cards is “52 choose 5”, or

$$C(52,5)=\frac{52!}{5!(52-5)!}$$

52! / (5!(52 – 5)!) = 2,598,960

Example

- What are the odds of being dealt a hand of 5-card stud containing two pair?
- First, let's look at the steps in the process of forming a 5-card hand containing two pairs, say 2 eights and 2 aces:
  a) Choose two face values out of 13 possible face values for the pairs (eights and aces):
     13 choose 2 = 78
  b) Choose two cards from the smaller face value (8♣, 8♠):
     4 choose 2 = 6
  c) Choose two cards from the larger face value (A♠, A♣):
     4 choose 2 = 6
  d) Choose one card from those remaining (9♦):
     44 choose 1 = 44 (eliminating the 8 face cards already considered: 52 – 8 = 44)
- By the Product Rule, H = 78 • 6 • 6 • 44 = 123,552.

Example

- H = 123,552 = the number of different ways we can form a five-card poker hand containing 2 pair.
- T = 2,598,960 = the total number of different five-card poker hands that can be dealt from a standard deck of 52 playing cards.
- So, the probability of obtaining a hand with two pairs is H / T = 123,552 / 2,598,960 or 4.75%.

Summary

Order Relevant	Repetitions Allowed	Type of Result	Formula
Yes	Yes	Arrangement	n^r $n, r \geq 0$
Yes	No	Permutation	$P(n, r) = n!/(n\text{-}r)!$ $0 \leq r \leq n$
No	No	Combination	$C(n, r) = n!/[r!(n\text{-}r)!]$ $0 \leq r \leq n$

Homework Exercise

- In how many different ways can we select 4 different players from 10 players on a team to play four tennis matches, where the matches are ordered?

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Chapter 5, Section 5.4
Binomial Coefficients

Binomial Coefficients

- This number represents the *binomial coefficient*:

$$\binom{n}{r}$$

(read this as “n choose r”)

- Why? Because these numbers occur as coefficients in the expansion of powers of binomial expression such as $(a+b)^n$

Example

$$(x+y)^0 = 1$$

$$(x+y)^1 = x + y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Example

Instead of multiplying, the expansion of $(x+y)^3$ can be found by combinatorial reasoning:

$$(x+y)^3 = (x+y) \cdot (x+y) \cdot (x+y)$$

first sum second sum third sum

Let's put subscripts on each of the x's and y's so we can tell which sum they belong to:

$$(x+y)^3 = (x_1+y_1) \cdot (x_2+y_2) \cdot (x_3+y_3)$$

Example

$(x+y)^3$ = the sum of all products resulting from choosing a term in the first sum, a term in the second sum, and a term in the third sum.

From the expansion of $(x+y)^3$ the following terms will be produced, and it will be our job to determine the coefficients of each term: x^3, x^2y, xy^2, and y^3.

Example

In how many ways can we form x^3? Obviously, there is only one way: $x_1 \bullet x_2 \bullet x_3$

In how many ways can we form x^2y? We can chose an x from any two sums, and a y from the third sum, as follows:

$x_1x_2 \bullet y_3$ or $x_2x_3 \bullet y_1$ or $x_1x_3 \bullet y_2$

So there are $3 = \binom{3}{2}$ 2-combinations of 3 variables here. Thus, the coefficient for the x^2y term is 3.

Example

Similarly, there will be 3 2-combinations of 3 variables for the xy^2 term, giving a coefficient for the xy^2 term of 3.

And finally, there will be only one way to form y^3.

Now we know how to expand $(x+y)^3$ without having to multiply out the terms; just use the Binomial Theorem to tell us what the coefficient of each term will be:

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^2$$

Binomial Theorem

- Gives the coefficients of the expansion of powers of binomial expressions.
- Let $x \in \mathbf{R}$, $y \in \mathbf{R}$, and $n \in \mathbf{N}$.

$$(x+y)^n = \sum_{j=0}^{n} C(n, j) x^{n-j} y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Example

Expand $(x + y)^4$.

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^{j}$$

$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

$$= x^2 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Example

- What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?
- Note that the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$:
- The binomial theorem tells us that:

$$\binom{25}{13} = \frac{25!}{13!\,(25-13)!} = \frac{25!}{13!\cdot 12!} = 5{,}200{,}300$$

Example

- What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x\text{-}3y)^{25}$?
- Note that $(2x\text{-}3y)^{25} = (2x + (\text{-}3y))^{25}$
- According to the binomial theorem:

$$\left(2x+\left(-3y\right)\right)^{25} = \sum_{j=0}^{25} \binom{25}{j} \left(2x\right)^{25-j}\left(-3y\right)^{j}$$

- The coefficient of $x^{12}y^{13}$ is obtained when j = 13:

$$\binom{25}{13} 2^{12}\left(-3\right)^{13} = -\frac{25!}{13!\left(25-13\right)!} 2^{12}3^{13} = -\frac{25!}{13!\cdot 12!} 2^{12}3^{13}$$

Corollary 1

- Let *n* be a non-negative integer. Then:

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Proof Using Subsets

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

A set with n elements has 2^n different subsets.

Each subset consists of 0 elements, 1 element, 2 elements, … or n elements.

There is only 1 subset with 0 elements, the empty set.

There are n subsets with a single element.

There are $\frac{n!}{2!(n-2)!}$ subsets with 2 elements.

…

There are n subsets with n-1 elements.

There is only 1 subset with n elements.

Proof Using Combinations

$$\sum_{k=0}^{n}\binom{n}{k}=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n-1}+\binom{n}{n}=2^{n}$$

A set with *n* elements has 2^n different subsets.

Each subset consists of 0 elements, 1 element, 2 elements, … or *n* elements.

There are $\binom{n}{0}$ subsets with 0 elements, $\binom{n}{1}$ subsets with 1 elements,

$\binom{n}{2}$ subsets with 2 elements,... and $\binom{n}{n}$ subsets with n elements.

Therefore, $\sum_{k=0}^{\mathrm{n}}\binom{\mathrm{n}}{k}$ counts the total number of subsets.

Example

- How many subsets are there of the set $S = \{a, b, c\}$?
- $C(3,0) = 1$, and there is one set with 0 elements: $\varnothing$
- $C(3,1) = 3$, and there are three sets with 1 element: $\{a\}, \{b\}, \{c\}$
- $C(3,2) = 3$, and there are three sets with two elements: $\{a, b\}, \{a, c\}, \{b, c\}$
- $C(3,3) = 1$, and there is one set with three elements: $\{a, b, c\}$
- So the total number of subsets $= 8 = 2^3$

Pascal's Triangle

- Geometric arrangement of binomial coefficients
- Based on Pascal's Identity

Pascal's Identity

- Let $n \in \mathbf{Z}^+$ and $k \in \mathbf{Z}^+$ with $n \geq k$.
- Then,

$$C(n+1, k) = C(n, k-1) + C(n, k)$$

- When two adjacent binomial coefficients in this Pascal's triangle are added, the binomial coefficient in the next row between these two coefficients is produced.
- *Example:* $C(7,5) = C(6,4) + C(6,5)$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$

$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$

$$1 \quad 1$$

$$1 \quad 2 \quad 1$$

$$1 \quad 3 \quad 3 \quad 1$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$

$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

(b)