MTH 2215
Applied Discrete Mathematics

Chapter 7.1
Recurrence Relations

These class notes are based on material from our textbook, **Discrete Mathematics and Its Applications**, 6^{th} ed., by Kenneth H. Rosen, published by McGraw Hill, Boston, MA, 2006. They are intended for classroom use only and are **not** a substitute for reading the textbook.

Definition

- A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.
- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

Recurrence Relations vs. Recursive Definitions

- *Recursive definitions* can be used to solve counting problems. When they are used in this way, the rule for finding terms from those that precede them is called a *recurrence relation*.

Example

- Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

  $$a_n = a_{n-1} - a_{n-2} \quad \text{for } n = 2, 3, 4, \ldots$$

  Suppose that $a_0 = 3$ and $a_1 = 5$.
- What are a_2 and a_3?

Example

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

Suppose that $a_0 = 3$ and $a_1 = 5$.

- For a_2 , $n = 2$, so $n - 1 = 1$ and $n - 2 = 0$.
- So $a_2 = a_1 - a_0$
- Therefore, $a_2 = 5 - 3 = 2$

Example

- How about $a_{3?}$
- Here is our recurrence relation:

  $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

  We know that $a_0 = 3$, $a_1 = 5$, and $a_2 = 2$
- For a_3 , $n = 3$, so $n - 1 = 2$ and $n - 2 = 1$.
- So $a_3 = a_2 - a_1$
- Therefore, $a_3 = 2 - 5 = -3$

Example

- Consider the recurrence relation:

$$a_n = 2a_{n-1} - a_{n-2} \quad \text{for } n = 2, 3, 4, \ldots$$

- Show whether each of the following is a solution of this recurrence relation:

$$a_n = 3n$$

$$a_n = 2^n$$

$$a_n = 5$$

Example

Consider the recurrence relation:

$$a_n = 2a_{n\text{-}1} - a_{n\text{-}2} \quad \text{for } n = 2, 3, 4, \ldots$$

- Is $a_n = 3n$ a solution of this recurrence relation? Let's check:

$a_n = 2a_{n\text{-}1} - a_{n\text{-}2} = 2[(3(\text{n}–1)] – 3(\text{n}–2)$

$= 2[(3\text{n}–3)] – 3\text{n}–6) = 6\text{n}–6 – (3\text{n}–6)$

$= 6\text{n} – 6 – 3\text{n} + 6 = \underline{3\text{n}}$

Example

Consider the recurrence relation:

$$a_n = 2a_{n-1} - a_{n-2} \quad \text{for } n = 2, 3, 4, \dots$$

- Is $a_n = 2^n$ a solution of this recurrence relation? Assume that it is; then:

  $a_0 = 2^0 = 1$; $a_1 = 2^1 = 2$; $a_2 = 2^2 = \underline{4}$

- But our recurrence relation says that

  $a_2 = 2a_1 - a_0 = 4 - 1 = \underline{3}$

- So, no.

Example

Consider the recurrence relation:

$$a_n = 2a_{n\text{-}1} - a_{n\text{-}2} \quad \text{for } n = 2, 3, 4, \ldots$$

- Is $a_n = 5$ a solution of this recurrence relation? Let's check:

$$a_n = 2a_{n\text{-}1} - a_{n\text{-}2} = 2(5) - 5$$

$$= 10 - 5 = \underline{5}$$

Modeling with Recurrence Relations

- A person deposits $10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Interest Compounded Annually

- What is the recurrence?
- Look at the figures for the first year:
  - Starting amount = $10,000
  - Interest rate = .11
  - 1^{st} year's interest = $1,100
  - Total after 1 year = 11,100
- Now for the second year:
  - Starting amount = $11,100

Interest Compounded Annually

- So the process for the second year is exactly the same as for the first year, except that the starting amount for the second year is the total amount in the account at the end of the first year.
- The third year is the same as the second, except that the starting amount for the third year is the total amount in the account at the end of the second year.
- And so on ….

Interest Compounded Annually

- So the recurrence is:

  $$a_{\mathrm{n}} = a_{\mathrm{n-1}} + 0.11 * a_{\mathrm{n-1}}$$

  with an initial condition of

  $$a_0 = \$10{,}000.00$$

- We can see that

  $$a_{\mathrm{n}} = a_{\mathrm{n-1}} + 0.11 * a_{\mathrm{n-1}}$$

  reduces to

  $$a_{\mathrm{n}} = 1.11 * a_{\mathrm{n-1}}$$

Interest Compounded Annually

- Note that:

  $a_0 = \$10,000.00$

  $a_1 = 1.11 * a_0$

  $a_2 = 1.11 * a_1 = (1.11)^2 * a_0$

  $a_3 = 1.11 * a_2 = (1.11)^3 * a_0$

  . . .

  $a_n = 1.11 * a_{n-1} = (1.11)^n * a_0$

- So:

  $a_n = (1.11)^n * \$10,000$

Interest Compounded Annually

Given our formula:

$$a_n = (1.11)^n * \$10,000$$

Then for n = 30, at the end of 30 years the account contains:

$$a_{30} = (1.11)^{30} * \$10,000$$

or

$228,922.97

Such are the wonders of compound interest!

Rabbits and the Fibonacci Sequence

- A young pair of rabbits (one of each sex) is placed on an island.
  - A pair does not breed until they are 2 months old.
  - After they are 2 months old, each pair produces another pair each month.
- Find a recurrence relation for the number of ***pairs*** of rabbits on the island after n months, assuming that no rabbits ever die.

Rabbits and the Fibonacci Sequence

- Let f_n = number of pairs of rabbits on the island at the end of n months.
- We know that $f_1 = 1$ and $f_2 = 1$ (because the rabbits don't breed until they are two months old).
- At the end of the third month, the first pair has 2 baby bunnies. So $f_3 = 2$.

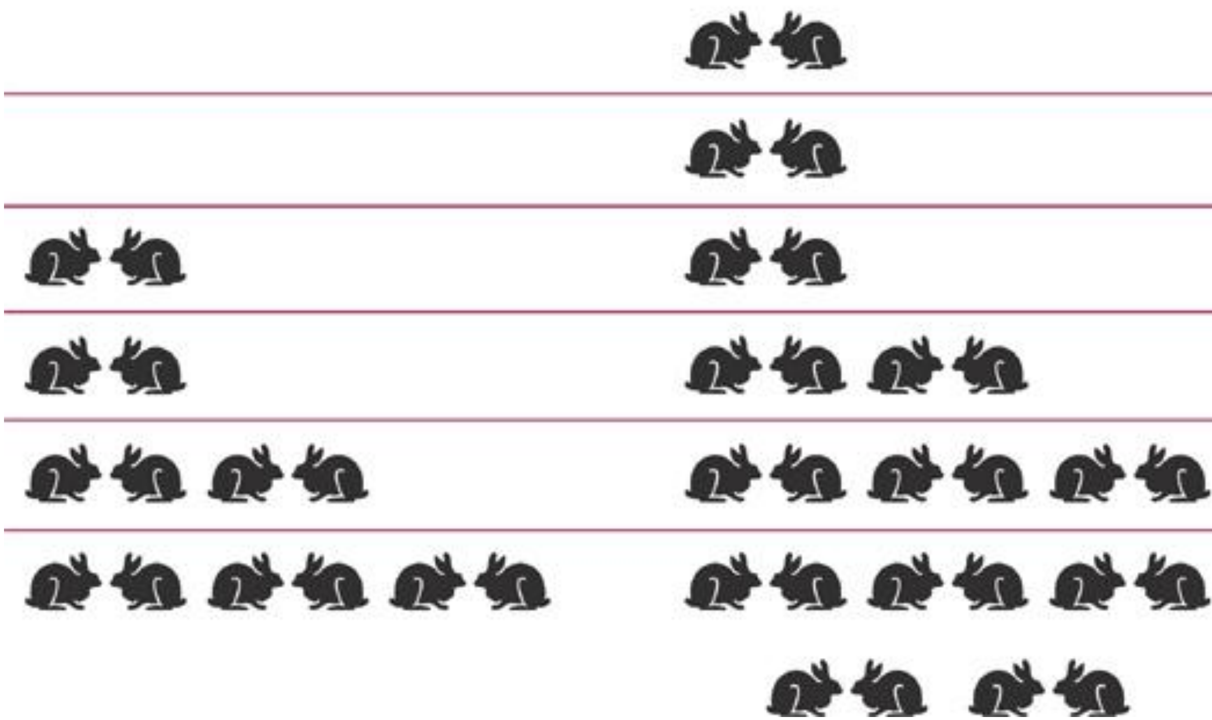
Rabbits and the Fibonacci Sequence

- At the end of the fourth month, the first pair has 2 more baby bunnies. The second pair is still too young to have bunnies, so $f_4 = 3$.
- At the end of the fifth month, the first pair has 2 more baby bunnies, and the second pair has its first 2 baby bunnies. The third pair is still too young to have bunnies, so $f_5 = 5$.

Rabbits and the Fibonacci Sequence

Month	Reproducing Pairs	Young Pairs	Total Pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

Rabbits and the Fibonacci Sequence



Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

Rabbits and the Fibonacci Sequence

We know that $f_1 = 1$ and $f_2 = 1$ (because the rabbits don't breed until they are two months old).

- At the end of the n^{th} month, the number of pairs = the number of pairs the previous month (which is f_{n-1}) + the number of newborn pairs (which is f_{n-2}, because any pair 2 months old or older will produce a new pair).

Rabbits and the Fibonacci Sequence

So our recurrence is:

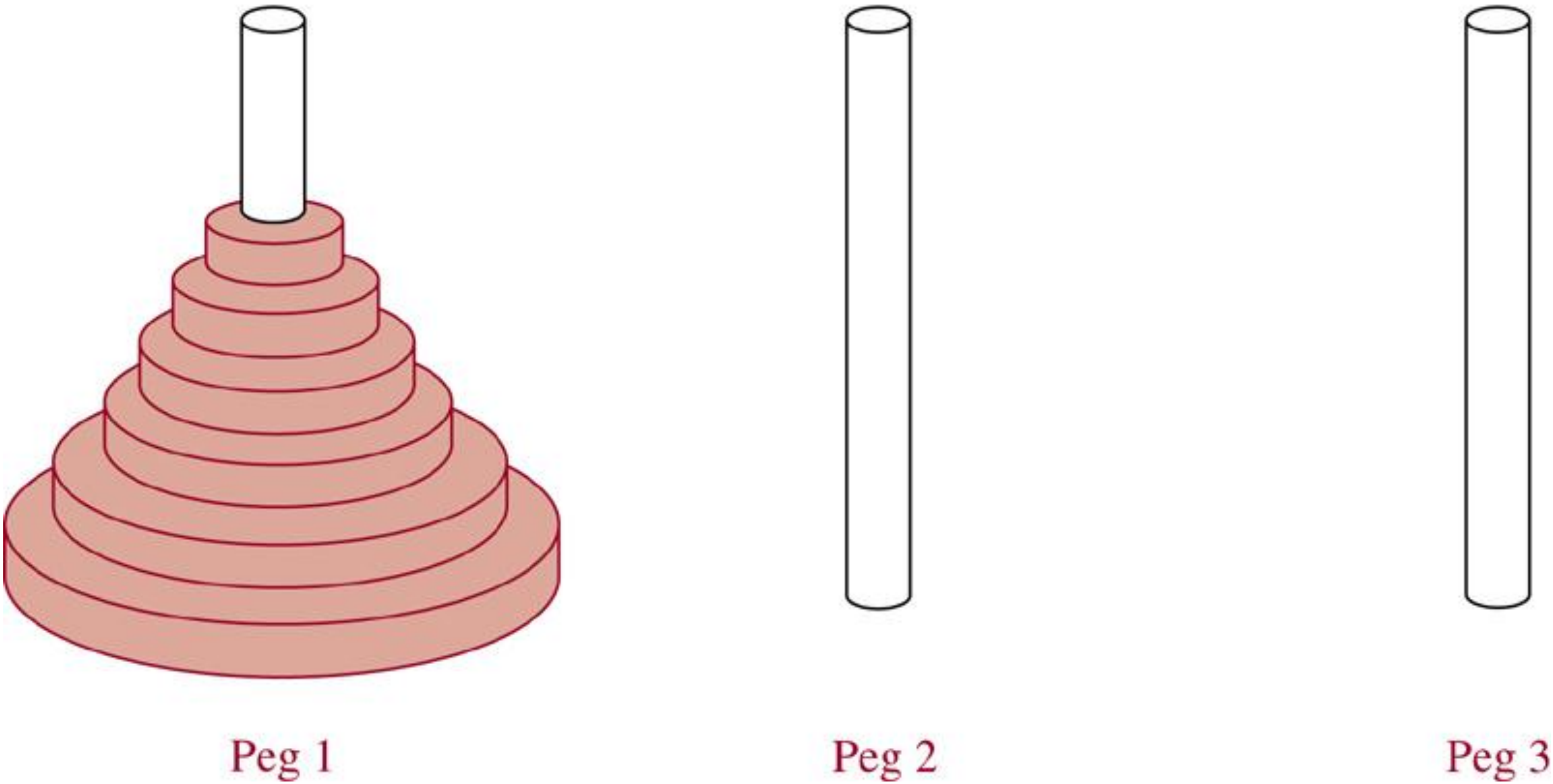
$f_1 = 1$

$f_2 = 1$

$f_n = f_{n-1} + f_{n-2}$, for $n \geq 3$.

This is the Fibonacci Sequence.

The Tower of Hanoi



The Tower of Hanoi

- Find a recurrence relation to find the number of moves needed to solve the *Tower of Hanoi* problem with n disks.

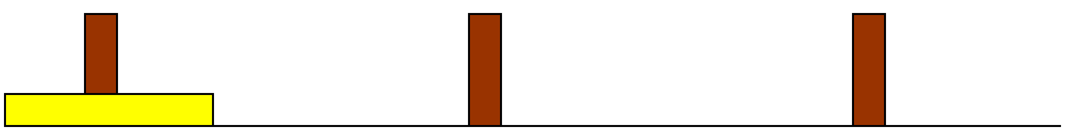
The Tower of Hanoi

- An old legend states that there is a monastery in Hanoi containing three golden pegs with 64 gold disks.
- Originally, all 64 disks were on one peg, arranged so that the largest disk was on the bottom and so on up to the top disk, which was the smallest.
- The monks' task is to move all of the disks from one peg to another.
- When they finish, the world will end!

The Tower of Hanoi

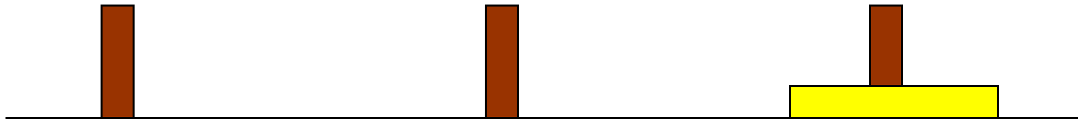
- However, the monks have to follow two rules:
  - Only one disk can be moved at a time.
  - No larger disk may be placed on a smaller disk.

Consider the following smaller version of the problem:



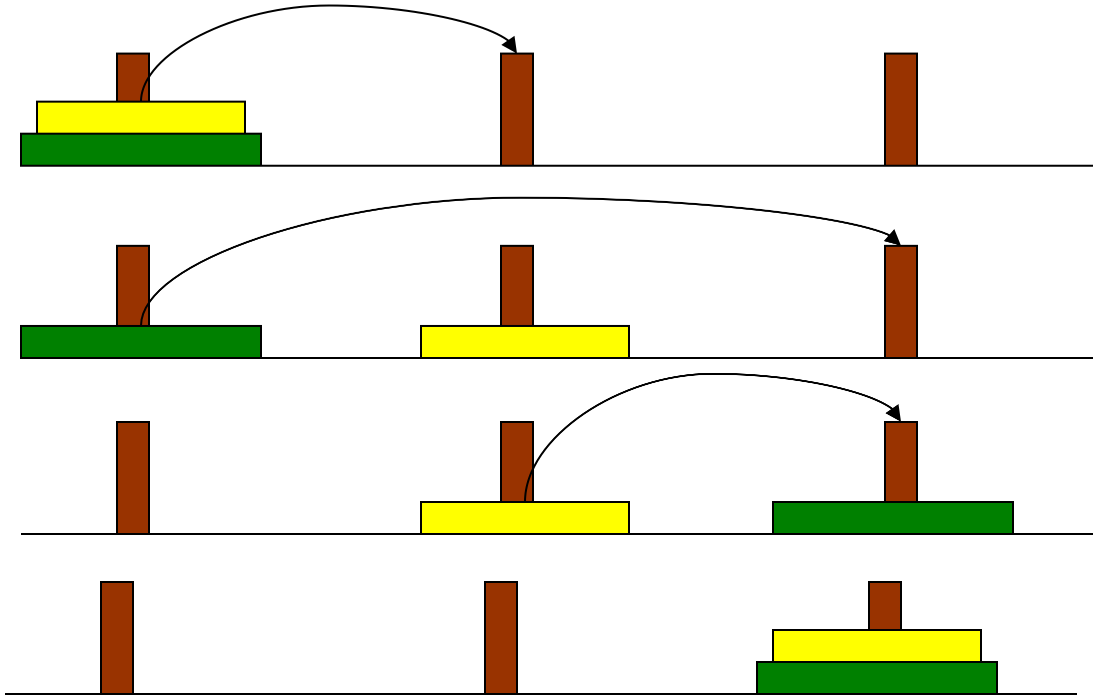
The Tower of Hanoi

- How many moves does it take to transfer all 1 of the disks to peg 3?
- Obviously, just 1.



The Tower of Hanoi

- Now consider the following slightly larger version of the problem. How many moves does it take to to transfer all 2 of the disks to peg 3?



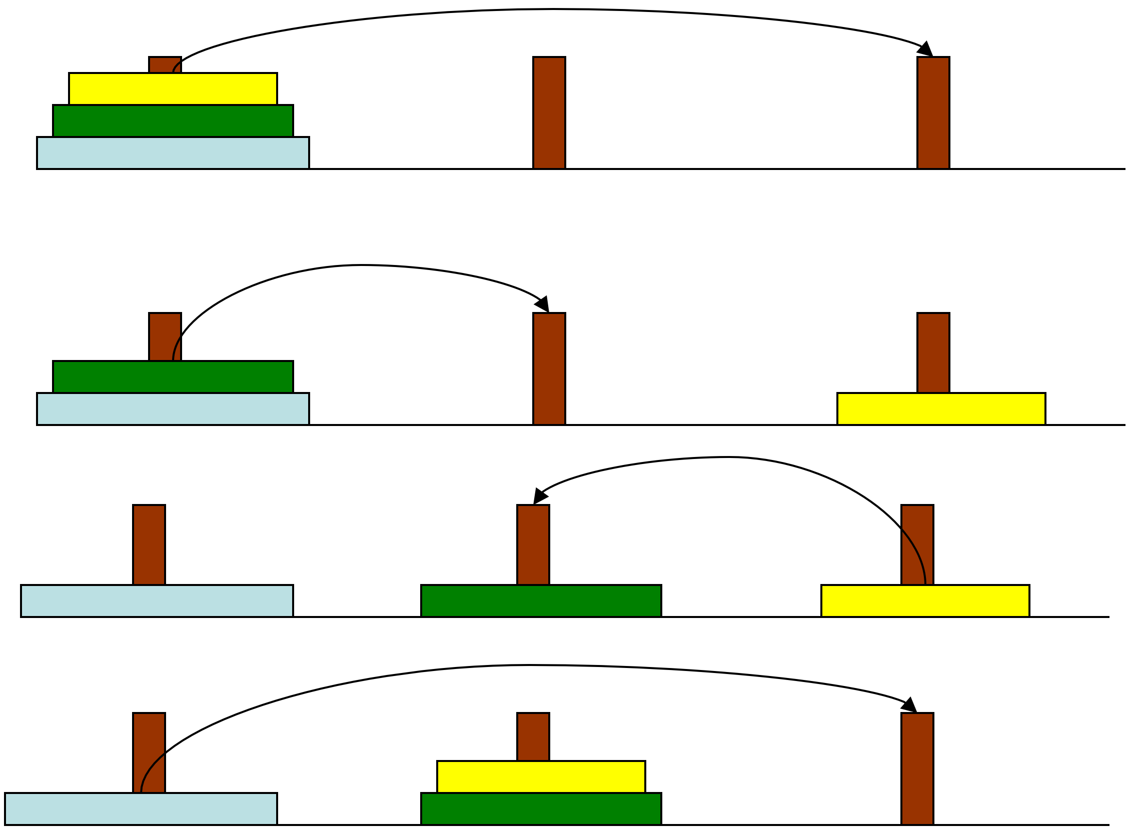
After move 1.

After move 2

After move 3.

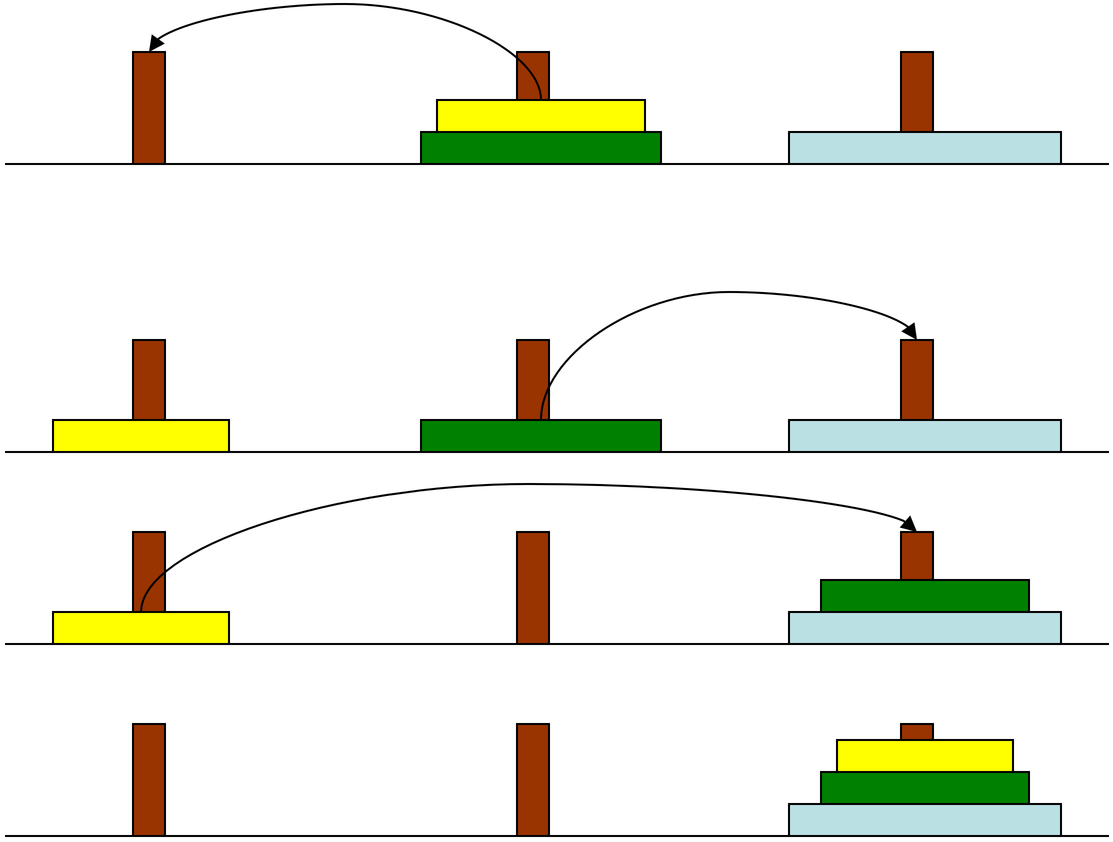
The Tower of Hanoi

What is the shortest legal sequence of moves necessary to move all 3 disks to peg 3?



The Tower of Hanoi

What is the shortest legal sequence of moves necessary to move the three disks to another peg?



It takes 7 moves.

The Tower of Hanoi

- Note that in order to move one disk, it took us 1 move.
- In order to move a stack of two disks, it took us 3 moves.
- In order to move a stack of three disks, it took us 7 moves.
- Looks like a pattern is developing, doesn’t it?

The Tower of Hanoi

- Note that in order to move a single disk, all we had to do was to move it, at a cost of 1.
- It looks as if we have a base case, whose cost is not given in terms of the other disks.
- So:

$H_1 = 1$

$H_2 = 3$

$H_3 = 7$ … and the pattern is

$H_n = 2H_{n-1} + 1$ for all $n > 1$

The Tower of Hanoi

- In order to transfer a stack of disks:
  - First we have to transfer the stack above the bottom disk (by legally stacking the disks above it somewhere else),
  - Then we move the bottom disk
  - Then we rebuild the stack on top of the bottom disk.
- It is this unstacking/restacking process that gives us the factor of 2 in our recurrence equation.

The Tower of Hanoi

We can solve this recurrence relation and remove the reference to the previous condition.

We get:

$$H_n = 2^n - 1$$

So it will take the monks $2^{64} - 1$ moves to solve the Tower of Hanoi puzzle. That's a BIG number; there are (very roughly) 10^{64} atoms in the Milky Way galaxy. So don't worry about the monks finishing any time soon ….

Catalan Numbers

Find a recurrence relation for C_n, the number of ways to parenthesize the product of $n+1$ numbers, $x_0, x_1, x_2, \ldots, x_n$, to specify the order of multiplication.

Catalan Numbers

Example: $C_3 = 5$ because there are 5 ways to parenthesize x_0, x_1, x_2, and x_3 to determine the order of multiplication:

$((x_0 \bullet x_1) \bullet x_2) \bullet x_3$

$(x_0 \bullet (x_1 \bullet x_2)_{)} \bullet x_3$

$(x_0 \bullet x_1) \bullet (x_2 \bullet x_3)$

$x_0 \bullet ((x_1 \bullet x_2) \bullet x_3)$

$x_0 \bullet (x_1 \bullet (x_2 \bullet x_3))$

Catalan Numbers

The base cases here are:

$C_0 = 1$

$C_1 = 1$

and the recurrence relation is:

$C_n = C_0C_{n-1} + C_1C_{n-2} + \ldots\ C_{n-2}C_1 + C_{n-1}C_0$

which is:

$$\sum_{k=0}^{n-1} C_k C_{n-k-1}$$

Example

How many ways are there to parenthesize an expression with 4 terms? It can be shown that:

$$C_n = \frac{C(2n, n)}{n+1}$$

C_3 represents the Catalan number with 4 terms, n_0 through n_3. So:

$$C_3 = \frac{C(6,3)}{3+1} = \frac{6!/(3!\cdot(6-3)!)}{3+1} = \frac{720/36}{4} = 5$$

Example

The expression

$$C_n = \frac{C(2n, n)}{n+1}$$

representsa solution to the recurrencerelation. The solution allows us to provide a value for any term in the sequence without referenceto any previous term.

Homework Exercise

- Find a recurrence relation for the number of bit strings of length n that contain two consecutive 0s.

CSE 2813
Applied discrete mathematics

Chapter 7.2
Solving Recurrence Relations

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Recurrence Relation (Review)

- A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.
- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

Degree of a Recurrence Relation

The *degree* of a recurrence relation is k if the sequence $\{a_n\}$ is expressed in terms of the previous k terms:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $c_k \neq 0$.

Degree of a Recurrence Relation

What is the degree of $a_n = 2a_{n\text{-}1} + a_{n\text{-}2}$?

2, because a_n is expressed in terms of the 2 previous terms of the sequence

Degree of a Recurrence Relation

What is the degree of $a_n = a_{n\text{-}2} + 3a_{n\text{-}3}$?

3, because a_n is expressed in terms of the 3 previous terms of the sequence
(with 0 * $a_{n\text{-}1}$)

Degree of a Recurrence Relation

What is the degree of $a_n = 3a_{n\text{-}4}$?

4, because a_n is expressed in terms of the 4 previous terms of the sequence
(with $0 * a_{n\text{-}1}$, $0 * a_{n\text{-}2}$, $0 * a_{n\text{-}3}$)

Linear Recurrence Relations

- A recurrence relation is *linear* when a_n is a sum of multiples of the previous terms in the sequence
- Is $a_n = a_{n\text{-}1} + a_{n\text{-}2}$ linear?

  yes

- Is $a_n = a_{n\text{-}1} + a^2_{n\text{-}2}$ linear?

  no, because $a^2_{n\text{-}2}$ is not a <u>multiple</u> of the previous term

Homogeneous Recurrence Relations

- A recurrence relation is *homogeneous* when a_n depends only on multiples of previous terms.
- Is $a_n = a_{n\text{-}1} + a_{n\text{-}2}$ homogeneous?

  yes
- Is $P_n = (1.11)P_{n\text{-}1}$ homogeneous?

  yes
- Is $H_n = 2H_{n\text{-}1} + 1$ homogeneous?

  no, because the "+ 1" term is not a multiple of H_j

Recurrence Relations w/Constant Coefficients

- A recurrence relation has constant coefficients when the coefficients of the terms of the sequence are all *constants*, instead of functions that depend upon n.
- Does $P_n = (1.11)P_{n-1}$ have constant coefficients?

  yes
- Does $B_n = nB_{n-1}$ have constant coefficients?

  no, because the coefficient of the "nB_{n-1}" term is a function of n.

Solving Recurrence Relations

- Solving 1st Order Linear Homogeneous Recurrence Relations with Constant Coefficients (LHRRCC)
  - Derive the first few terms of the sequence using iteration
  - Notice the general pattern involved in the iteration step
  - Derive the general formula
  - Now test the general formula on some previously calculated (by iteration) terms

Solving 2nd Order LHRRCC

- Form: $a_n = c_1 a_{n\text{-}1} + c_2 a_{n\text{-}2} + \ldots + c_k a_{n\text{-}k}$ with some constant values for a_0 and a_1
- Assume that the solution is $a_n = r^n$, where r is a constant and $r \neq 0$

Solving 2nd Order LHRRCC

- $a_n = r^n$ is a solution of the recurrence relation

$$a_n = c_1 a_{n\text{-}1} + c_2 a_{n\text{-}2} + \ldots + c_\mathrm{k} a_{n\text{-}\mathrm{k}}$$

if and only if

$$r^\mathrm{n} = c_1 r^{\mathrm{n}\text{-}1} + c_2 r^{\mathrm{n}\text{-}2} + \ldots + c_\mathrm{k} r^{\mathrm{n}\text{-}\mathrm{k}}$$

Solving 2nd Order LHRRCC

- *Given:* $r^{n} = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$
- Dividing both sides by r^{n-k} and subtracting the right side from the left, we get:

$$r^{k} - c_1 r^{k-1} - c_2 r^{k-2} - \ldots\ c_{k-1} r - c_k = 0$$

- This is called the *characteristic equation* of the recurrence relation

Step 1

Solve the characteristic quadratic equation

$$r^2 - c_1 r - c_2 = 0$$

to find the characteristic roots r_1 and r_2

$$r_{1,2} = \frac{c_1 \pm \sqrt{c_1^{\;2} + 4c_2}}{2}$$

Step 2

- Case I: The roots are not equal

$$a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n$$

- Case II: The roots are equal ($r_1 = r_2 = r_0$)

$$a_n = \alpha_1 r_0{}^n + \alpha_2 n r_0{}^n$$

Step 3

- Apply the initial conditions to the equations derived in the previous step.
  - Case I: The roots are not equal

$$a_0 = \alpha_1 r_1^{\,0} + \alpha_2 r_2^{\,0} = \alpha_1 + \alpha_2$$

$$a_1 = \alpha_1 r_1^{\,1} + \alpha_2 r_2^{\,1} = \alpha_1 r_1 + \alpha_2 r_2$$

  - Case II: The roots are equal

$$a_0 = \alpha_1 r_0^{\,0} + \alpha_2 \cdot 0 \cdot r_0^{\,0} = \alpha_1$$

$$a_1 = \alpha_1 r_0^{\,1} + \alpha_2 \cdot 1 \cdot r_0^{\,1} = (\alpha_1 + \alpha_2) r_0$$

Step 4

- Solve the appropriate pair of equations for α_1 and α_2.

Step 5

- Substitute the values of α_1, α_2, and the root(s) into the appropriate equation in step 2 to find the explicit formula for a_n.

Example

- Solve the recurrence relation:

$$a_n = a_{n-1} + 2a_{n-2}$$

 where $a_0 = 2$ and $a_1 = 7$
- The characteristic quadratic equation of the recurrence relation is $r^2 - r - 2 = 0$.
- Its roots are $r = 2$ and $r = -1$, so the roots are not equal: use Case I.

Example

- The sequence $\{a_n\}$ is a solution to the recurrence relation iff

  $$a_n = \alpha_1 2^n + \alpha_2(-1)^n$$

  for some constants α_1 and α_2.
- Since $a_0 = 2 = \alpha_1 + \alpha_2$

  and $a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$

  we can find $\alpha_1 = 3$ and $\alpha_2 = -1$.

Plugging these values back into our formula we get:

$$a_n = 3 \cdot 2^n + -1(-1)^n \;=\; 3 \cdot 2^n - (-1)^n$$

Example

- Solve the recurrence relation:

$$f_n = f_{n\text{-}1} + f_{n\text{-}2}$$

where $f_0 = 0$ and $f_1 = 1$

- The characteristic quadratic equation of the recurrence relation is $r^2 – r – 1 = 0$.

Its roots are $r_1 = \dfrac{1+\sqrt{5}}{2}$ and $r_2 = \dfrac{1-\sqrt{5}}{2}$

Since the roots are not equal, use Case I.

Example

- The sequence $\{f_n\}$ is a solution to the recurrence relation iff

  $$f_n = \alpha_1((1+\sqrt{5})/2)^n + \alpha_2((1-\sqrt{5})/2)^n$$

  for some constants α_1 and α_2.
- Since $f_0 = 0 = \alpha_1 + \alpha_2$
  and $f_1 = 1 = \alpha_1((1+\sqrt{5})/2) + \alpha_2((1-\sqrt{5})/2)$
  we can find $\alpha_1 = 1/\sqrt{5}$ and $\alpha_2 = -1/\sqrt{5}$.

Plugging these values back into our formula we get:

$$f_n = (1/\sqrt{5})((1+\sqrt{5})/2)^n - (1/\sqrt{5})((1-\sqrt{5})/2)^n$$

Example

- Solve the recurrence relation:

$$a_n = 6a_{n-1} - 9a_{n-2}$$

where $a_0 = 1$ and $a_1 = 6$

- The characteristic quadratic equation of the recurrence relation is $r^2 - 6r + 9 = 0$.
- Its root(s) is/are $r = 3$ and $r = 3$, so the roots are equal: use Case II.

Case II

- The sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_0^n + \alpha_2 \cdot n \cdot r_0^n$ for $n = 0, 1, 2 \ldots$ where α_1 and α_2 are constants.
- So, substituting 3 for r, the solution to this recurrence is $a_n = \alpha_1 3^n + \alpha_2 \cdot n \cdot 3^n$

Case II

- We know that $a_0 = 1$ and, substituting 0 for n,
  $\alpha_1 \cdot 3^0 + \alpha_2 \cdot 0 \cdot 3^0 = \alpha_1 \cdot 1 + \alpha_2 \cdot 0 = \alpha_1$
- Therefore, $\alpha_1 = 1$

  We know that $a_1 = 6$ and, substituting 1 for n,
  $\alpha_1 \cdot 3^1 + \alpha_2 \cdot 1 \cdot 3^1 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$
- Therefore, $\alpha_2 = 1$

Plugging these values back into our formula we get:

$$a_n = 3^n + n3^n$$

Homework Exercise

- Solve the recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

where $a_0 = a_1 = 1$

Conclusion

- In Chapter 7 we have examined:
  - What recurrence relations are
  - How they are used
  - The degree of a recurrence relation
  - Homogeneous recurrence relations
  - How to solve recurrence relations