MTH 2215

Applied Discrete Mathematics

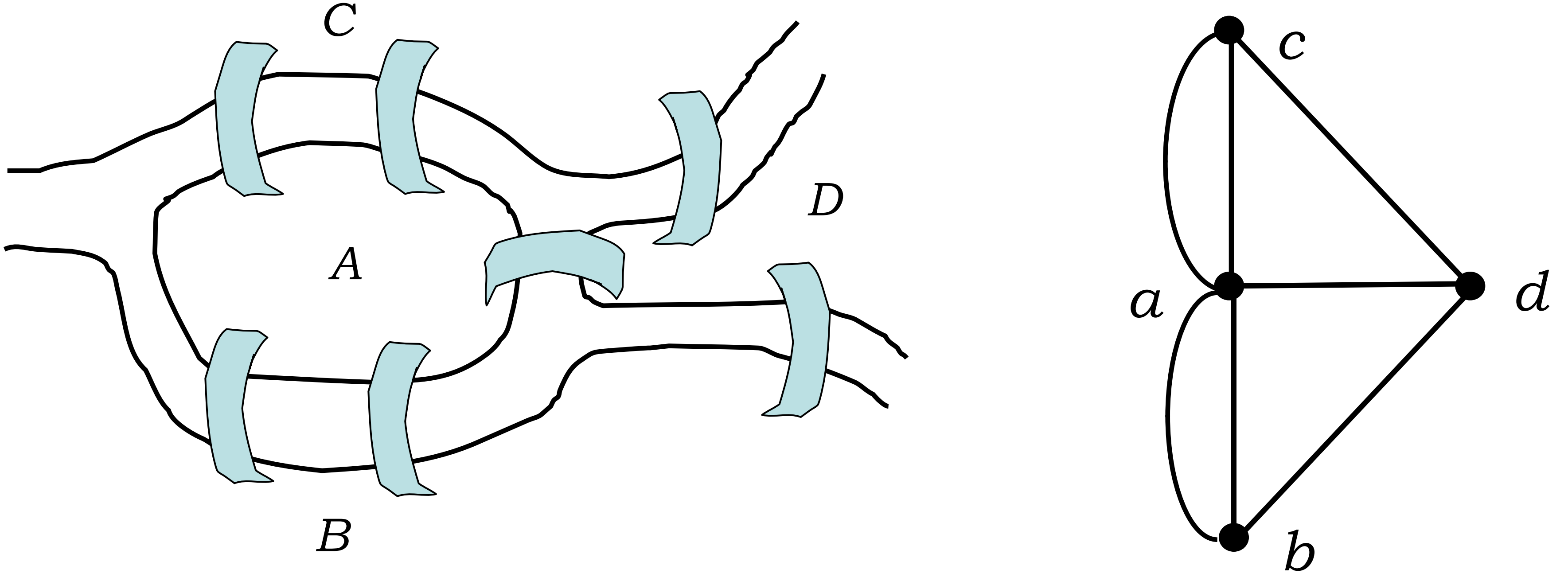
Chapter 9, Section 9.1

Introduction to Graphs

These class notes are based on material from our textbook, **Discrete Mathematics and Its Applications**, 6th ed., by Kenneth H. Rosen, published by McGraw Hill, Boston, MA, 2006. They are intended for classroom use only and are **not** a substitute for reading the textbook.

Problem

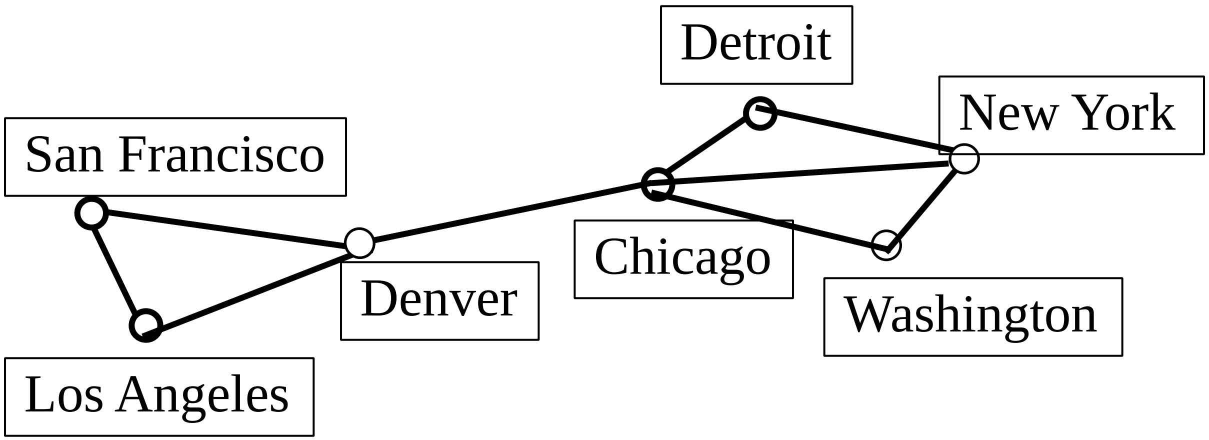
- The Seven bridges of Königsberg



Simple Graph

- A *graph* consists of
  - a nonempty set of *vertices* called V
  - a set of edges (unordered pairs of distinct elements of V) called E
- Notation: $G = (V,E)$
- A graph is *simple* if there is at most one edge linking two vertices.

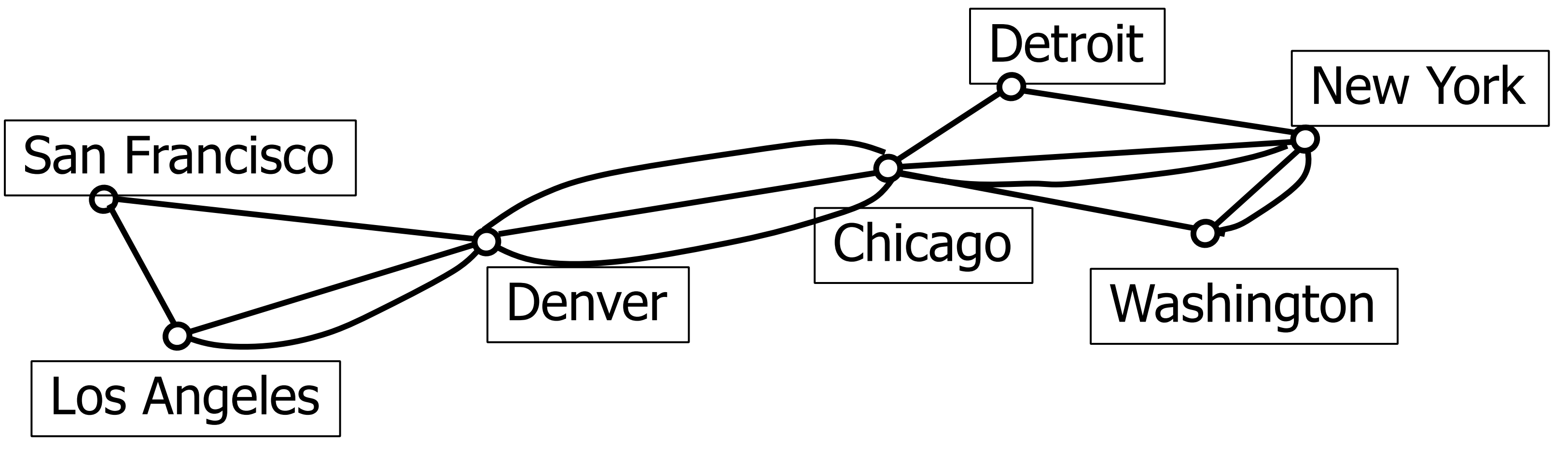
Simple Graph Example



- This simple graph represents a network.
- The network is made up of computers and telephone links between computers

Multigraph

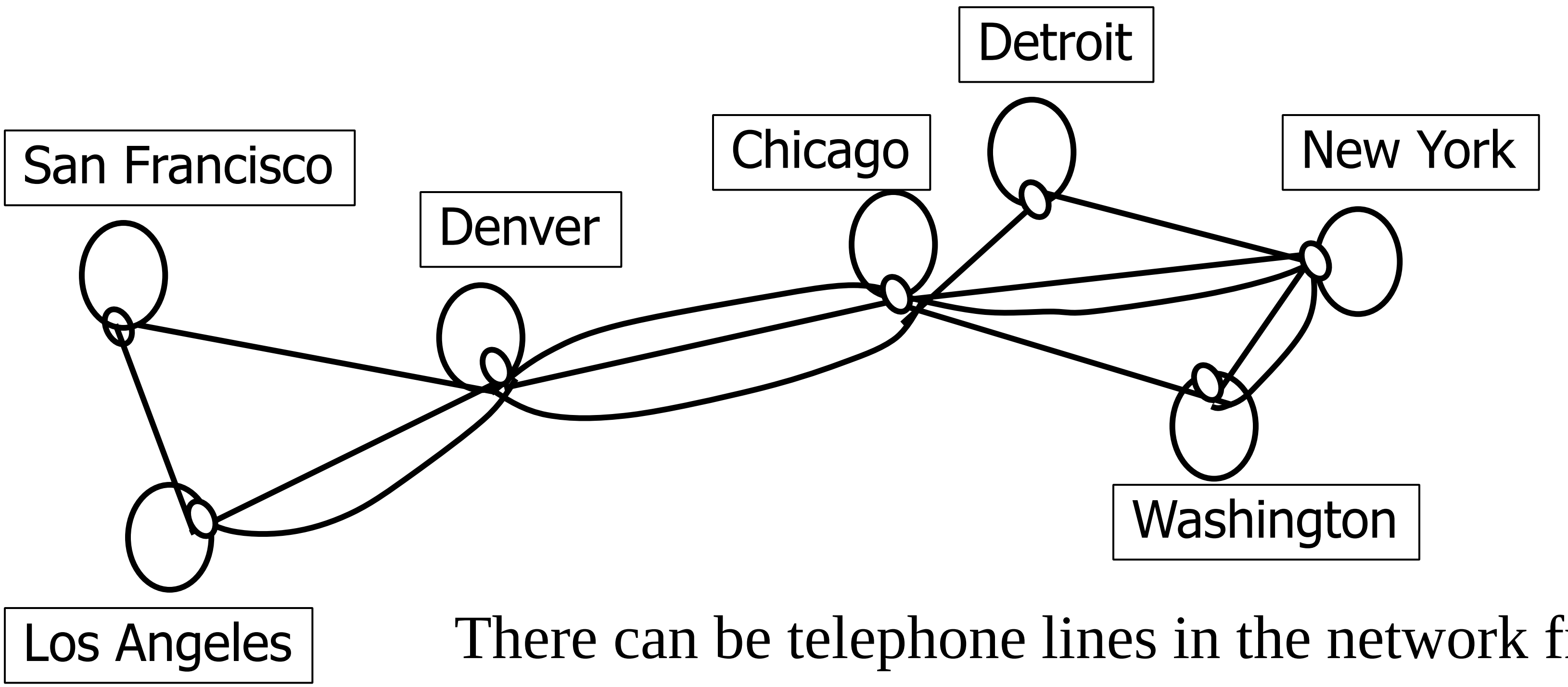
- A *multigraph* can have *multiple edges* (two or more edges connecting the same pair of vertices).



There can be multiple telephone lines between two computers in the network.

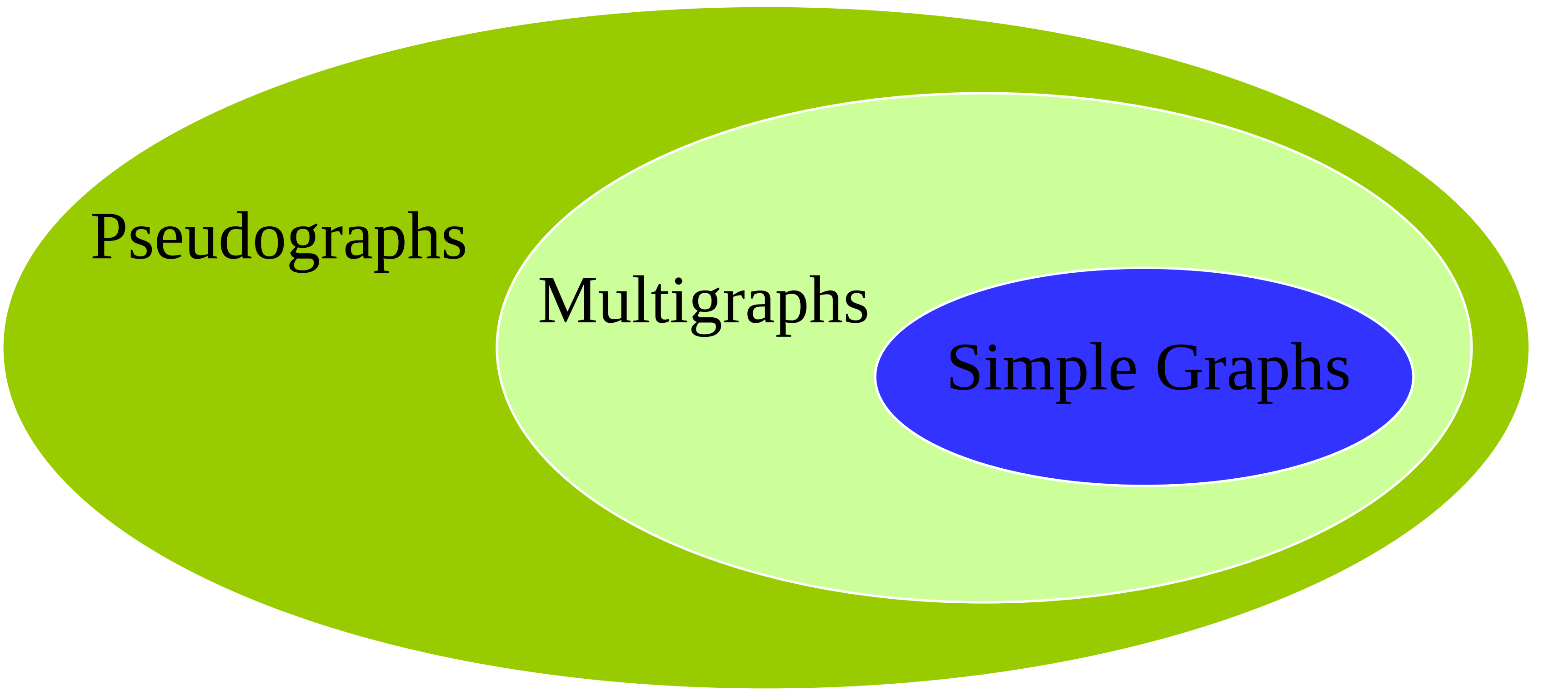
Pseudograph

- A *Pseudograph* can have multiple edges and *loops* (an edge connecting a vertex to itself).



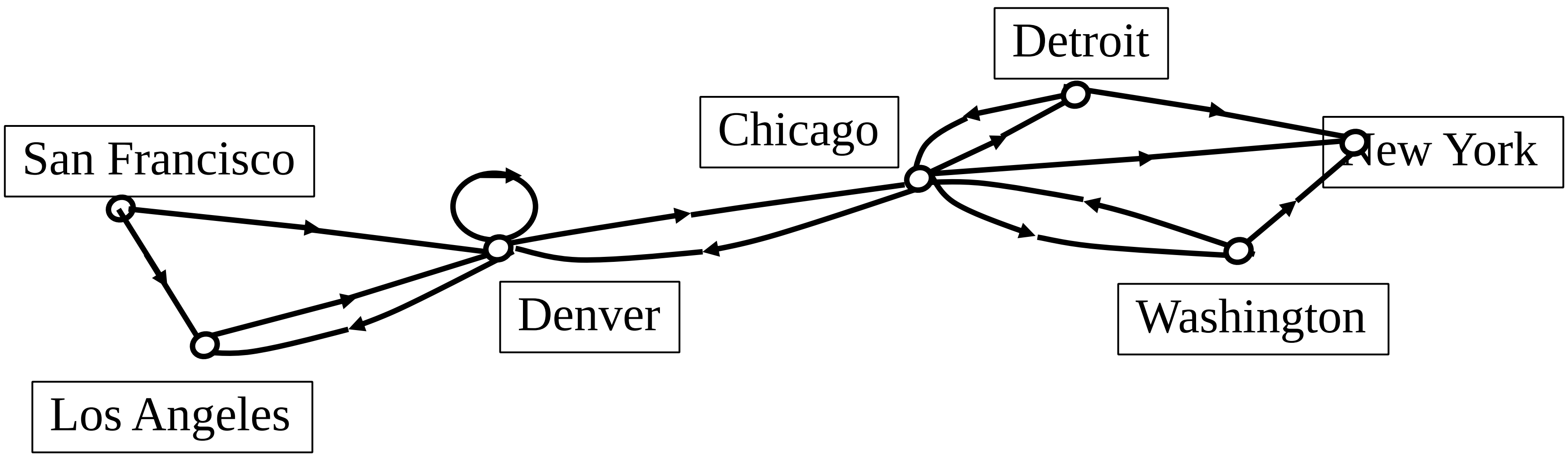
There can be telephone lines in the network from a computer to itself.

Types of Undirected Graphs



Directed Graph

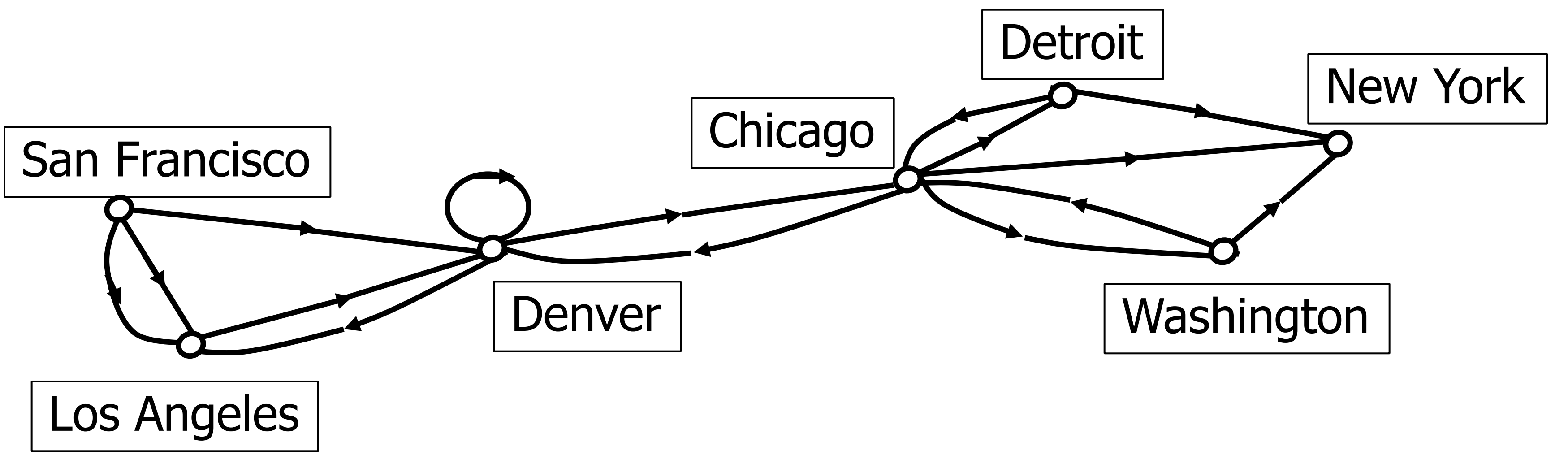
The edges are ordered pairs of (not necessarily distinct) vertices.



Some telephone lines in the network may operate in only one direction. Those that operate in two directions are represented by pairs of edges in opposite directions.

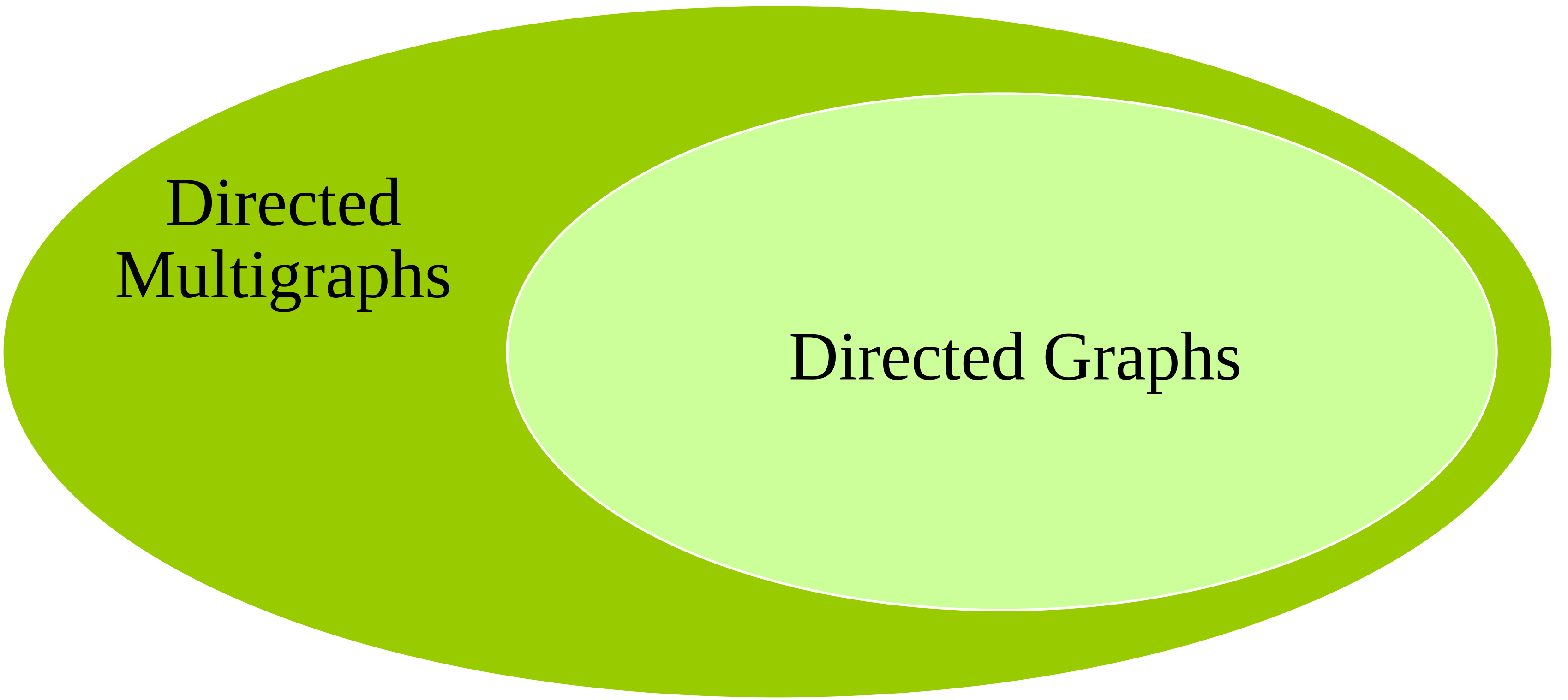
Directed Multigraph

A directed multigraph is a directed graph with multiple edges between the same two distinct vertices.



There may be several one-way lines in the same direction from one computer to another in the network.

Types of Directed Graphs



Graph Models

- Graphs can be used to model structures, sequences, and other relationships.
- Example: ecological niche overlay graph
  - Species are represented by vertices
  - If two species compete for food, they are connected by a vertex

Niche Overlay Graph

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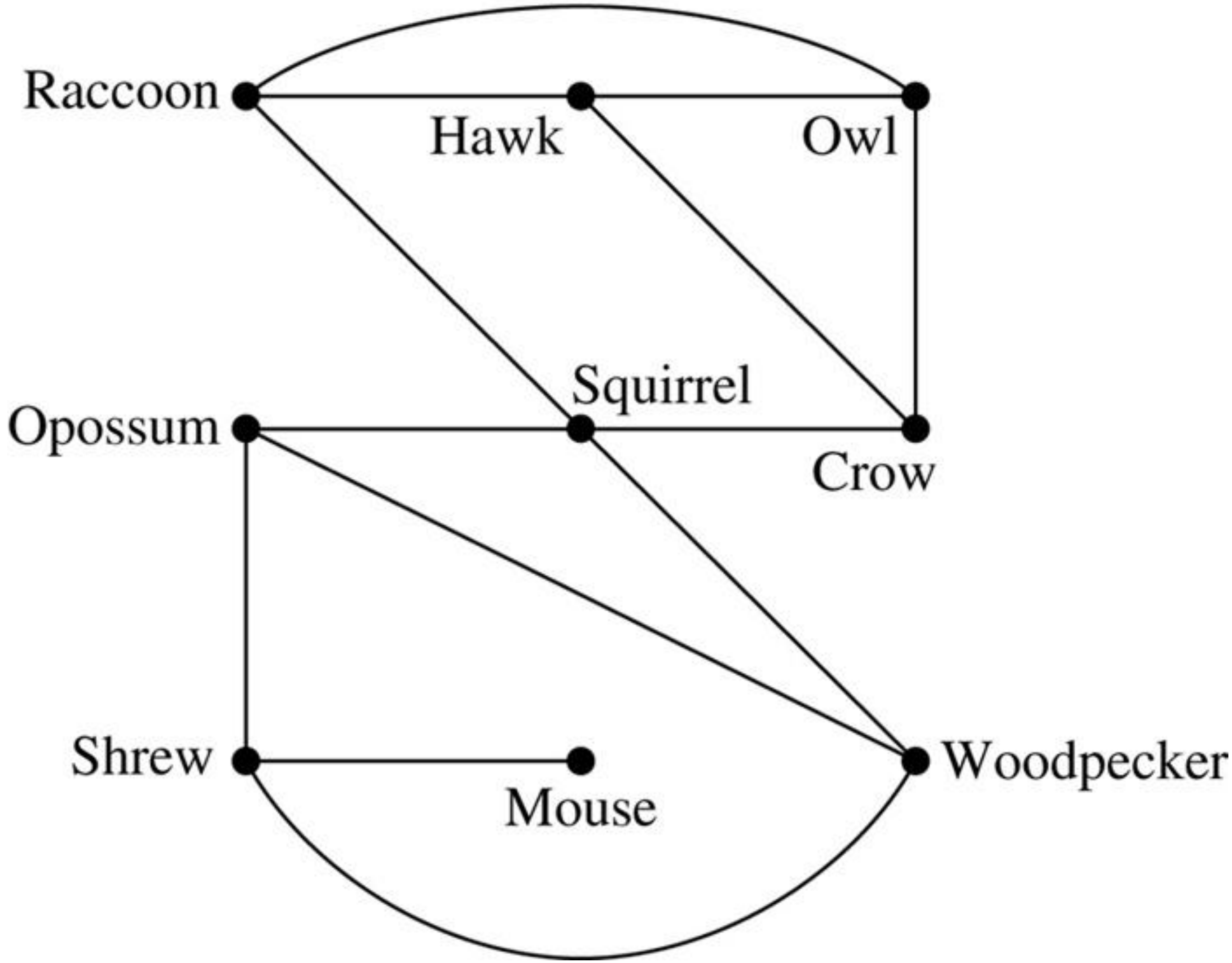

Acquaintanceship Graph

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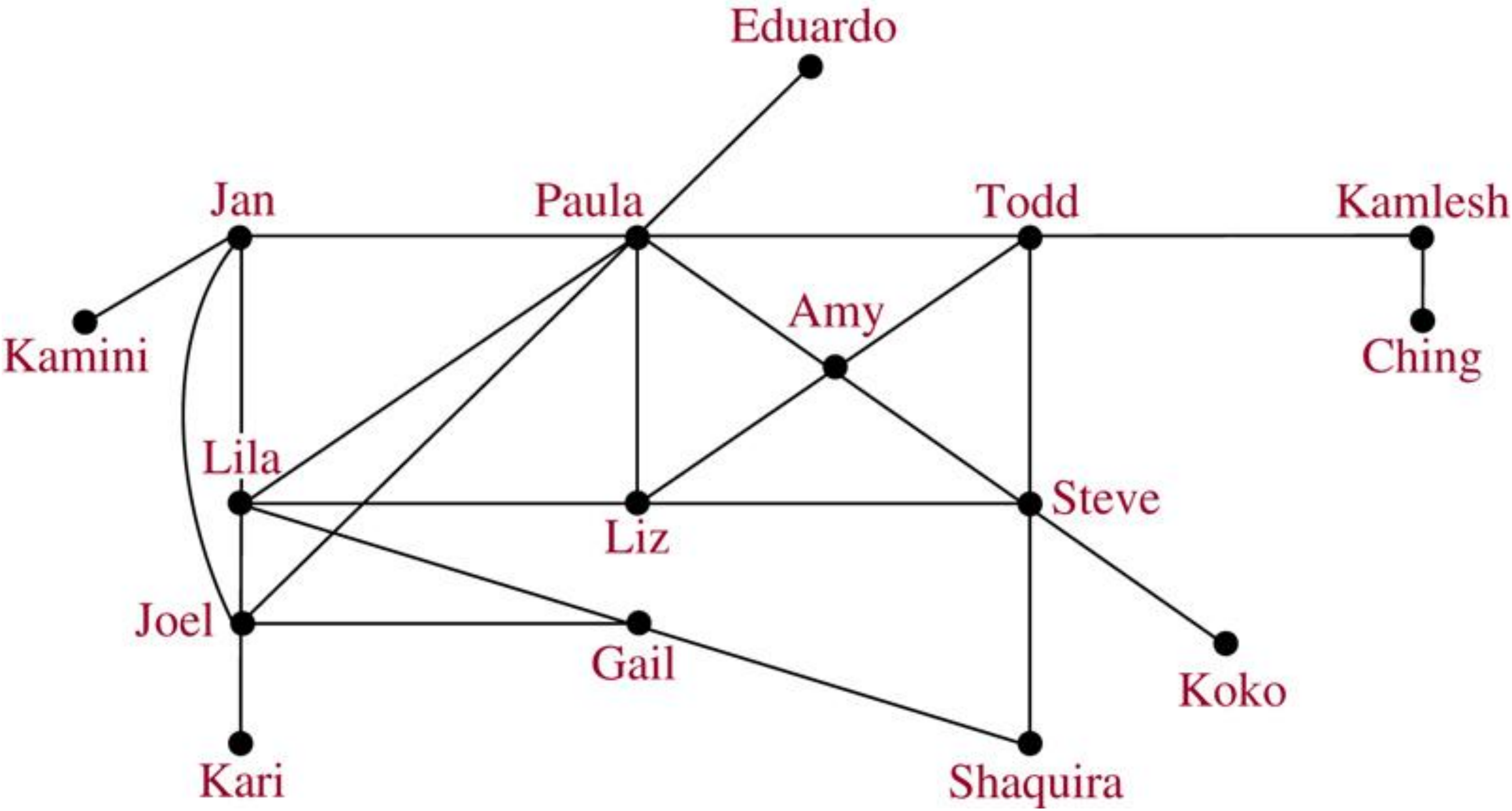

Influence Graph

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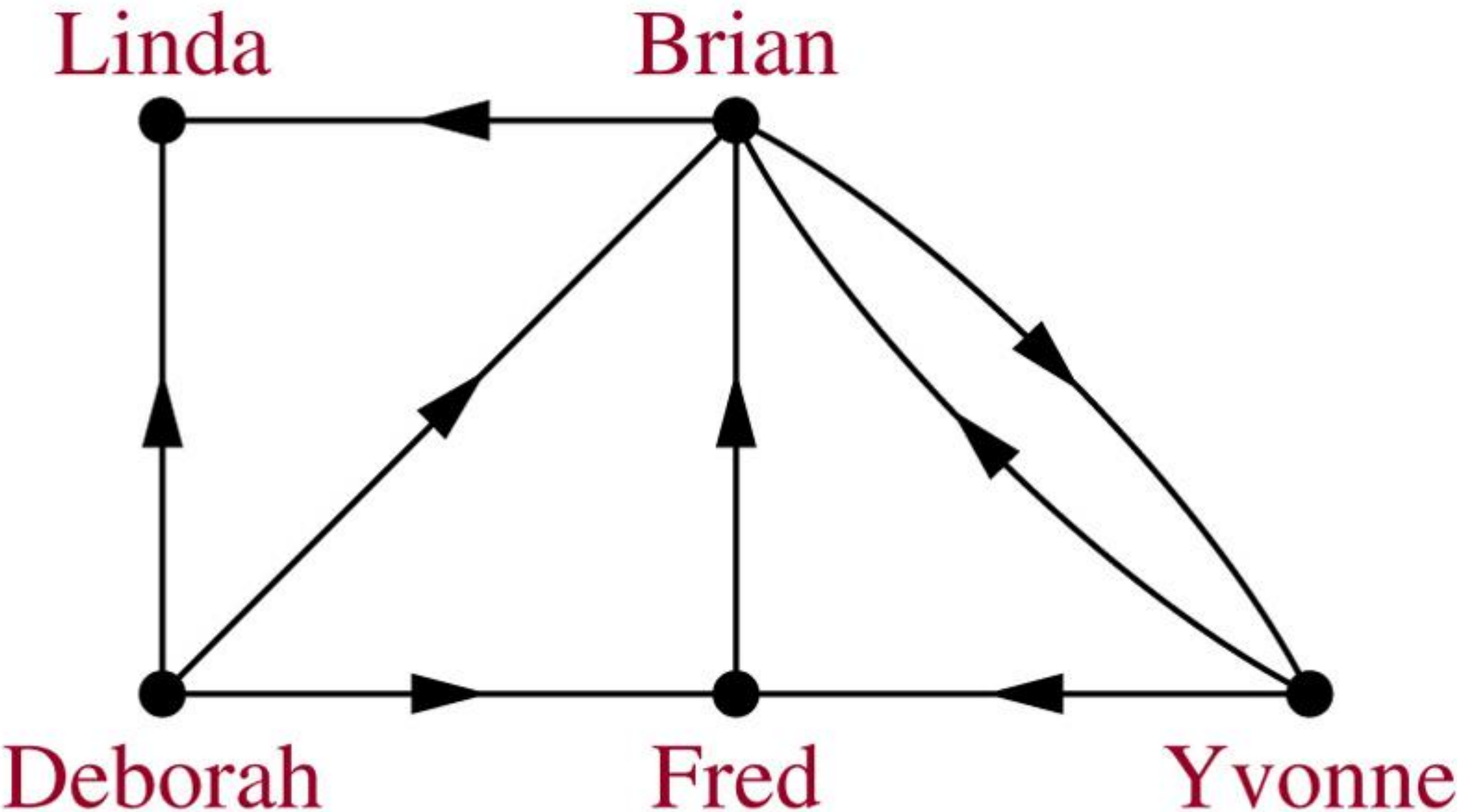
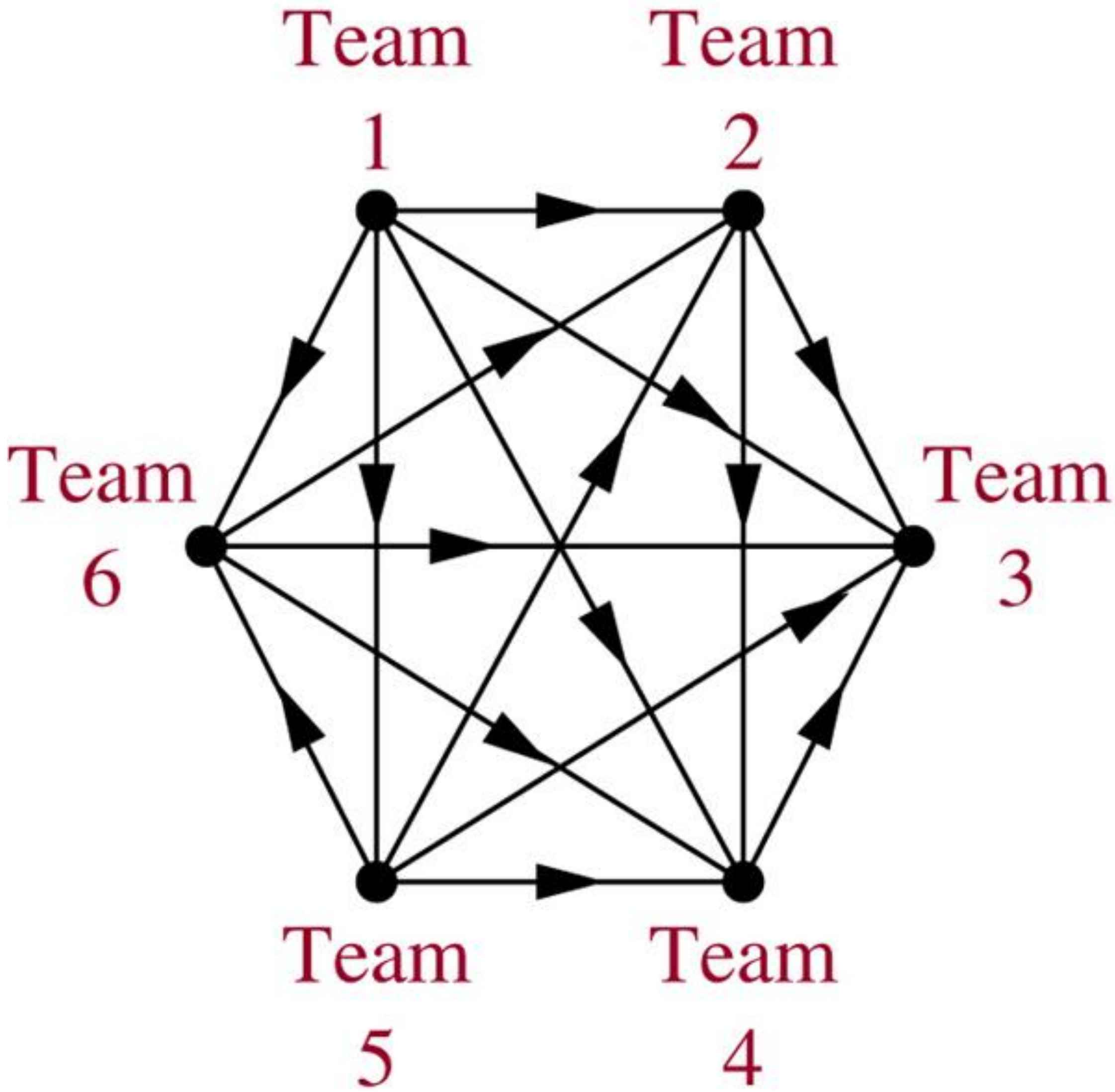


Round-Robin Tournament Graph



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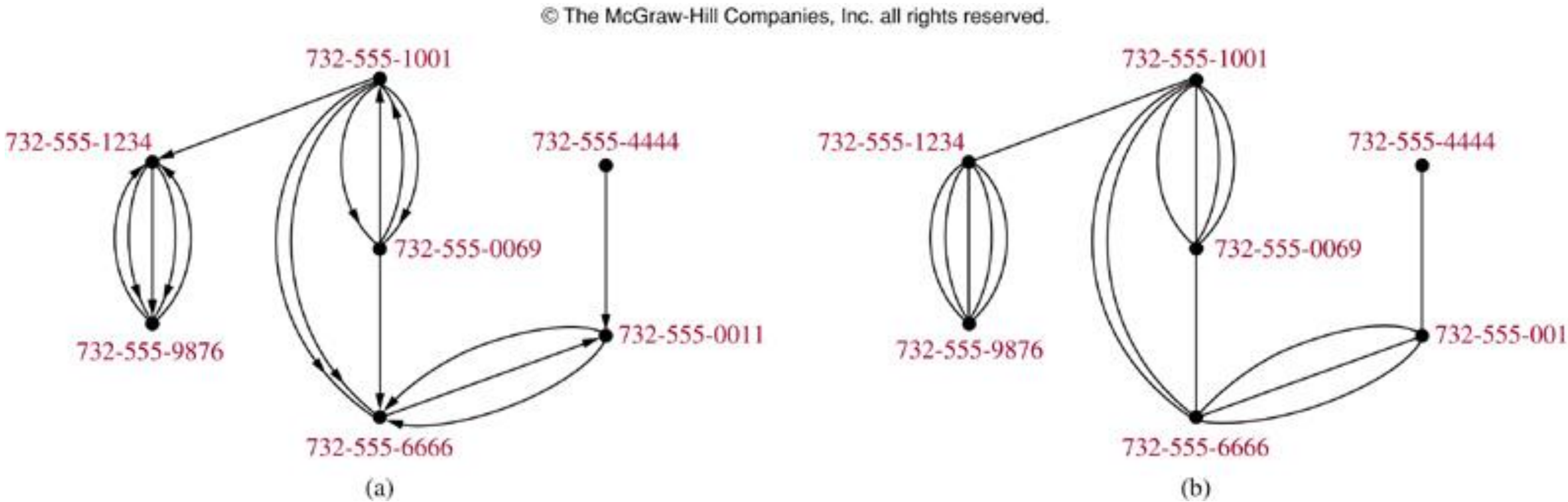
Call Graphs



Directed graph (a) represents calls *from* a telephone number *to* another.

Undirected graph (b) represents called *between* two numbers.

Hollywood Graph

- In the Hollywood graph:
  - Vertices represent actors
  - Edges represent the fact that the two actors have worked together on some movie
- As of October 2007, this graph had 893,283 vertices, and over 20 million edges.

Shortest-path Algorithms

- A decade or so ago a game called "Six Degrees of Kevin Bacon" was popular on college campuses.
- The idea, based on the idea that “it’s a small world”, was to try to find the fewest number of connections to link any other actor with Kevin Bacon.
- It was discovered that you could connect Kevin Bacon with just about any other actor in 6 links or so.

Bacon Numbers

- In the Hollywood Graph, the Bacon Number of an actor *x* is defined as the <u>length of the shortest path</u> connecting *x* and the actor Kevin Bacon.
- The average Kevin Bacon number is 2.957
- For more information, see the Oracle of Bacon website at the University of Virginia Computer Science Department.

Bacon Numbers

Kevin Bacon Number	**Number of People**
0	1 (Kevin Bacon himself)
1	2030
2	190213
3	557245
4	133450
5	9232
6	958
7	137
8	17

Shortest-path Algorithms

- Writing an efficient program for finding the shortest path in a graph is an important optimizing task in Computer Science.
- For futher information about shortest-path algorithms, take CSE 4833, or read the course textbook, *Introduction to Algorithms* by Cormen, Leiserson, Rivest, and Stein.

Summary

Type	**Edges**	**Loops**	**Multiple Edges**
Simple Graph	Undirected	NO	NO
Multigraph	Undirected	NO	YES
Pseudograph	Undirected	YES	YES
Directed Graph	Directed	YES	NO

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Chapter 9.2
Graph Terminology

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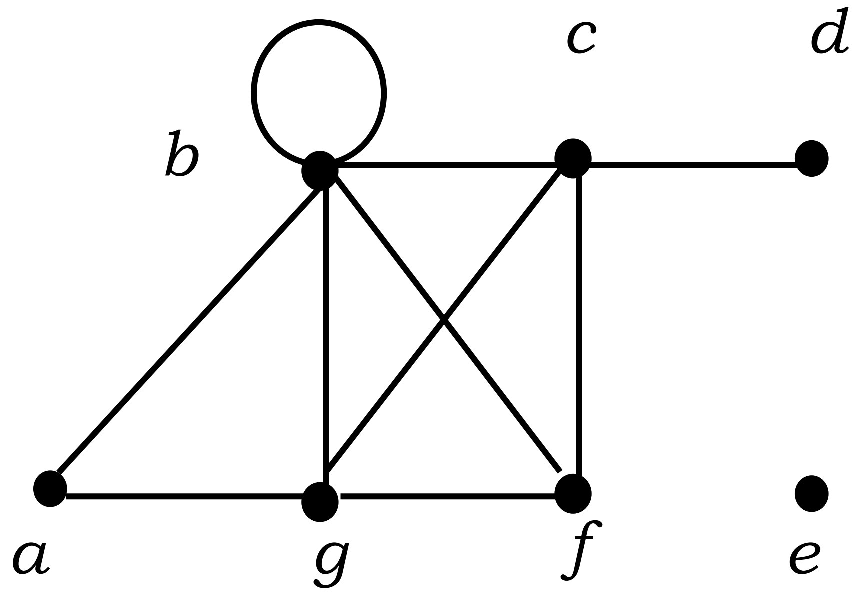
Adjacent Vertices in Undirected Graphs

- Two vertices, *u* and *v* in an undirected graph *G* are called *adjacent* (or neighbors) in *G*, if $\{u,v\}$ is an edge of *G*.
- An edge *e* connecting *u* and *v* is called *incident* with vertices *u* and *v*, or is said to <u>connect</u> *u* and *v*.
- The vertices *u* and *v* are called *endpoints* of edge $\{u,v\}$.

Degree of a Vertex

- The *degree of a vertex* in an undirected graph is the number of edges incident with it
  - except that a loop at a vertex contributes twice to the degree of that vertex
- The degree of a vertex v is denoted by deg(v).

Example



- Find the degrees of all the vertices:

deg(a) = 2, deg(b) = 6, deg(c) = 4, deg(d) = 1,

deg(e) = 0, deg(f) = 3, deg(g) = 4

Complete Graph

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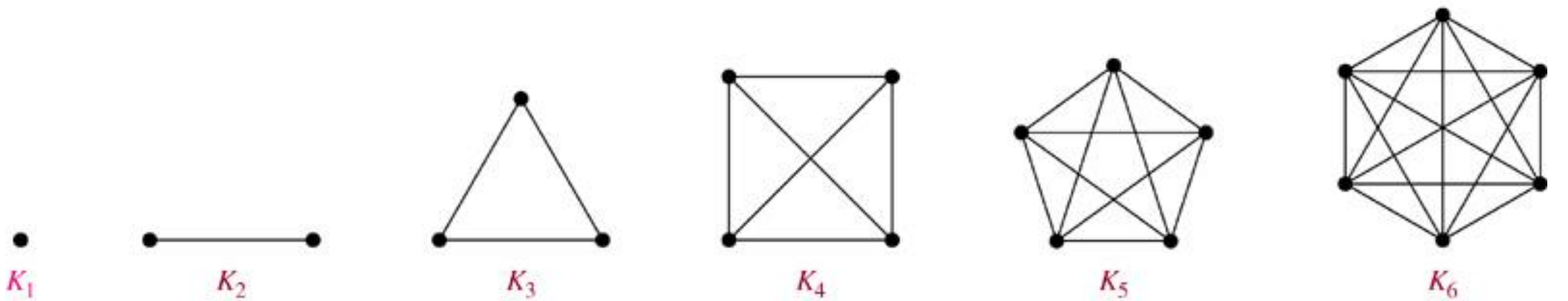
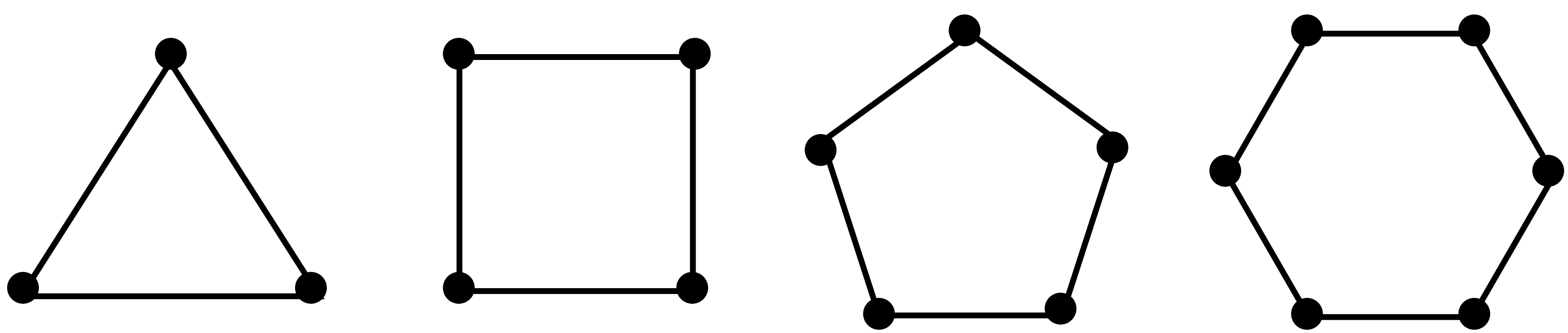


- The *complete graph* on *n* vertices (K_n) is the simple graph that contains exactly one edge between each pair of distinct vertices.
- The figures above represent the complete graphs, K_n, for n = 1, 2, 3, 4, 5, and 6.

Cycle

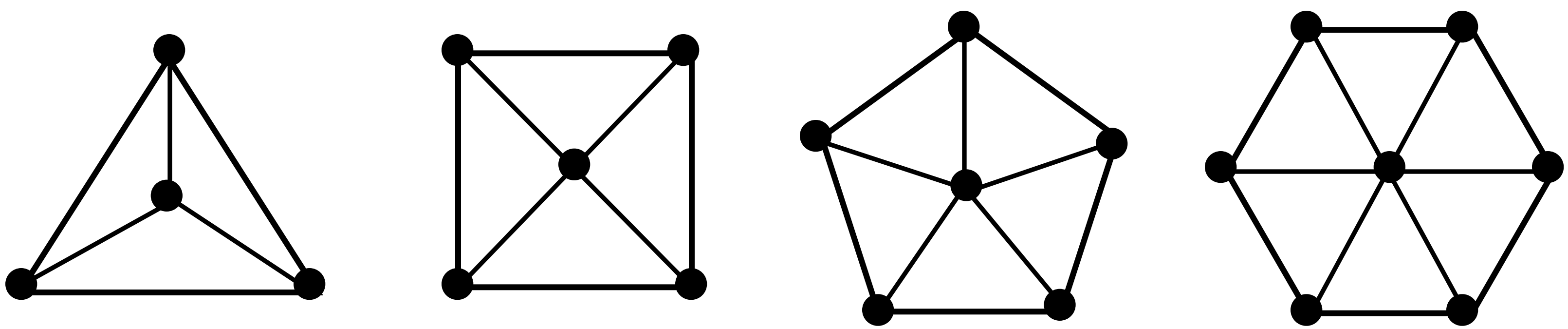
- The *cycle* C_n ($n \geq 3$), consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1,v_2\}$, $\{v_2,v_3\}$, …, $\{v_{n-1},v_n\}$, and $\{v_n,v_1\}$.



Cycles: C_3 C_4 C_5 C_6

Wheel

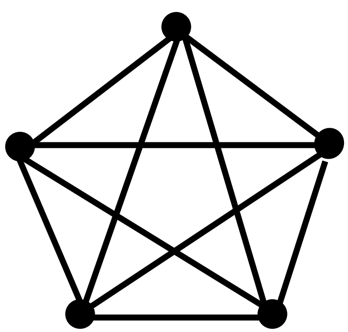
When a new vertex is added to a cycle C_n and this new vertex is connected to each of the n vertices in C_n, we obtain a *wheel* W_n.



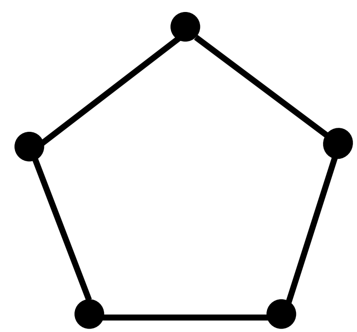
Wheels: W_3 W_4 W_5 W_6

Subgraph

- A *subgraph* of a graph $G = (V,E)$ is a graph $H = (W,F)$ where $W \subseteq V$ and $F \subseteq E$.



K_5

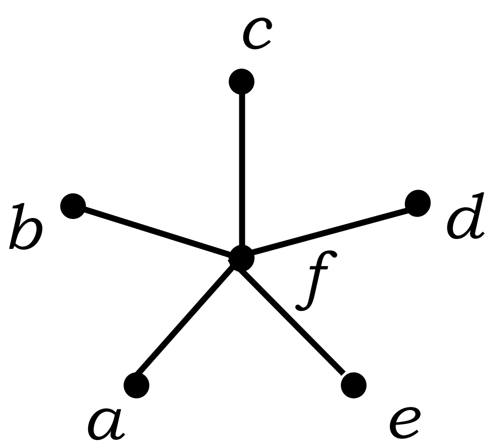


C_5

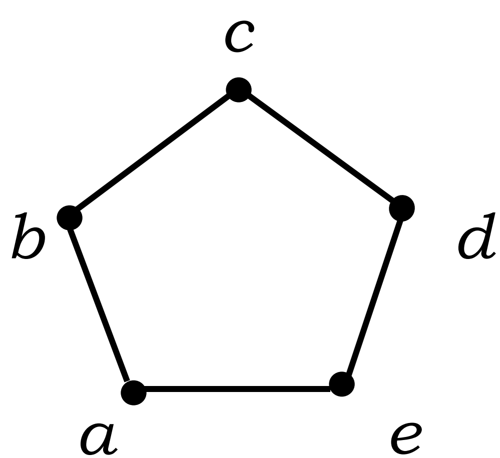
Is C_5 a subgraph of K_5?

Union

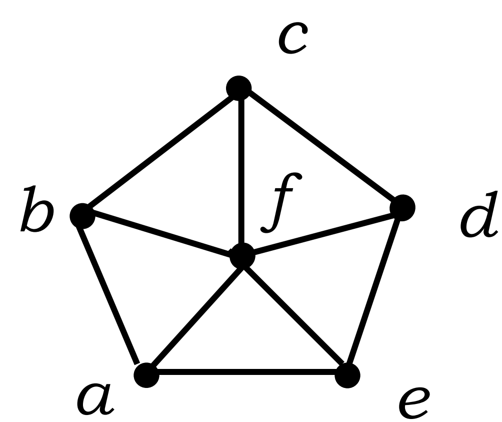
- The *union* of 2 simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The union is denoted by $G_1 \cup G_2$.



S_5



C_5



W_5

$S_5 \cup C_5 = W_5$

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Chapter 9.4
Connectivity

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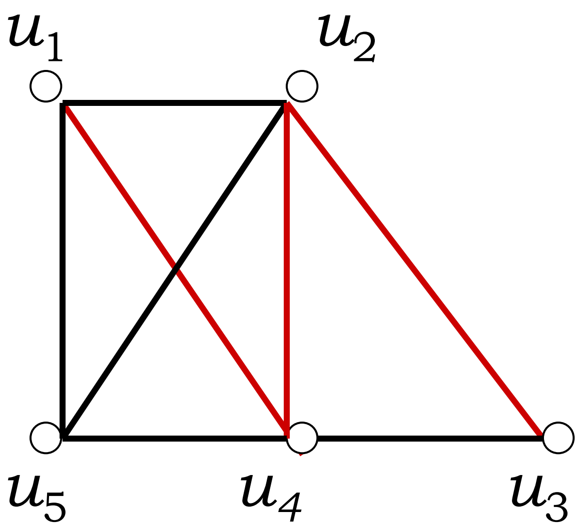
Paths in Undirected Graphs

- There is a *path* from vertex v_0 to vertex v_n if there is a sequence of edges from v_0 to v_n
  - This path is labeled as $v_0, v_1, v_2, \ldots, v_n$ and has a length of n.
- The path is a *circuit* if the path begins and ends with the same vertex.
- A path is *simple* if it does not contain the same edge more than once.

Paths in Undirected Graphs

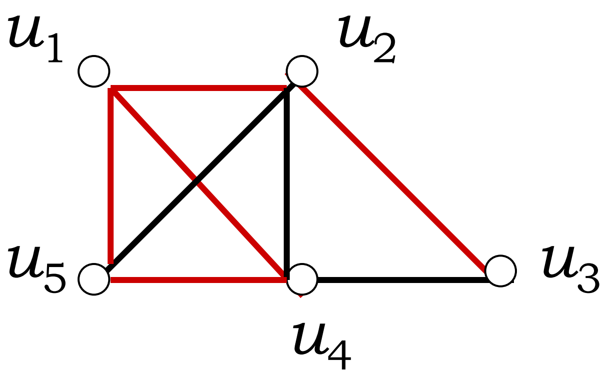
- A path or circuit is said to *pass through* the vertices v_0, v_1, v_2, …, v_n or *traverse* the edges e_1, e_2, …, e_n.

Example



- u_1, u_4, u_2, u_3
  - Is it simple?
  - *yes*
  - What is the length?
  - *3*
  - Does it have any circuits?
  - *no*

Example



- $u_1, u_5, u_4, u_1, u_2, u_3$
  - Is it simple?
  - *yes*
  - What is the length?
  - *5*
  - Does it have any circuits?
  - Yes; u_1, u_5, u_4, u_1

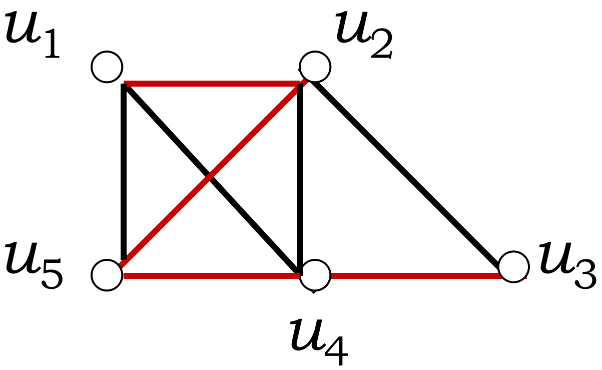
Example

- u_1, u_2, u_5, u_4, u_3

  – Is it simple?
  – *yes*

  – What is the length?
  – *4*

  – Does it have any circuits?
  – *no*

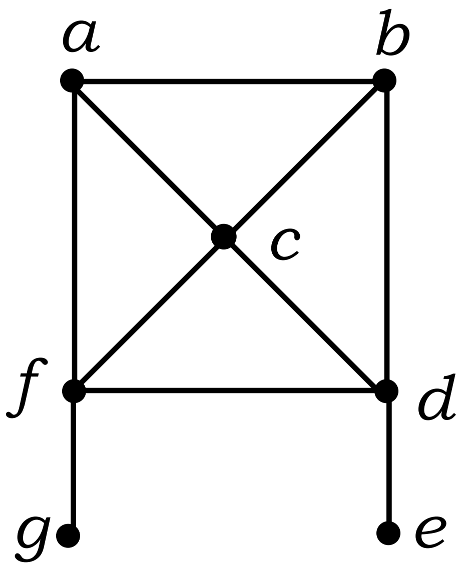


Connectedness

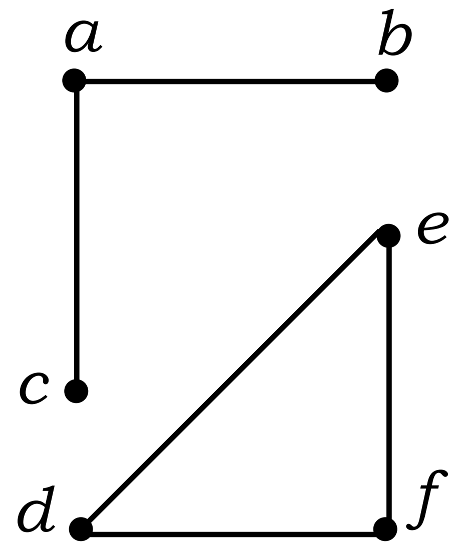
- An undirected graph is called *connected* if there is a path between every pair of distinct vertices of the graph.
- There is a simple path between every pair of distinct vertices of a connected undirected graph.

Example

Are the following graphs connected?



Yes



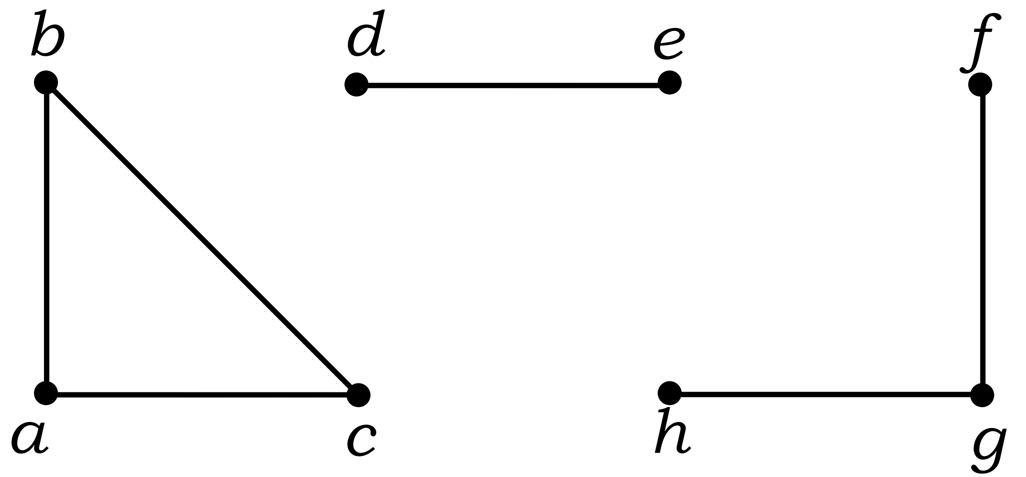
No

Connectedness (Cont.)

- A graph that is not connected is the union of two or more disjoint connected subgraphs (called the *connected components* of the graph).

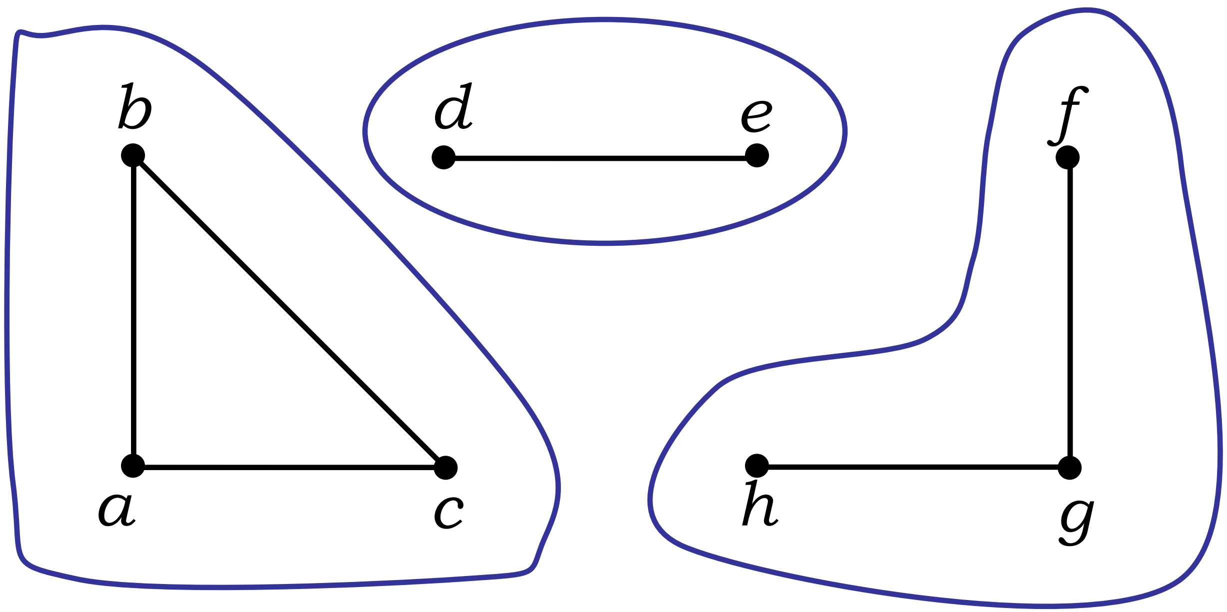
Example

- What are the connected components of the following graph?



Example

- What are the connected components of the following graph?



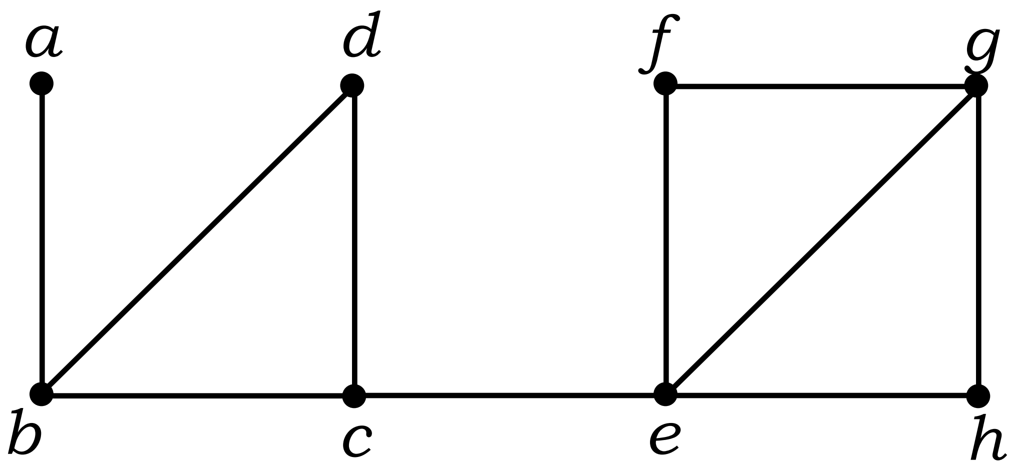
{a, b, c}, {d, e}, {f, g, h}

Cut edges and vertices

- If one can remove a vertex (and all incident edges) and produce a graph with more connected components, the vertex is called a *cut vertex*.
- If removal of an edge creates more connected components the edge is called a *cut edge* or *bridge*.

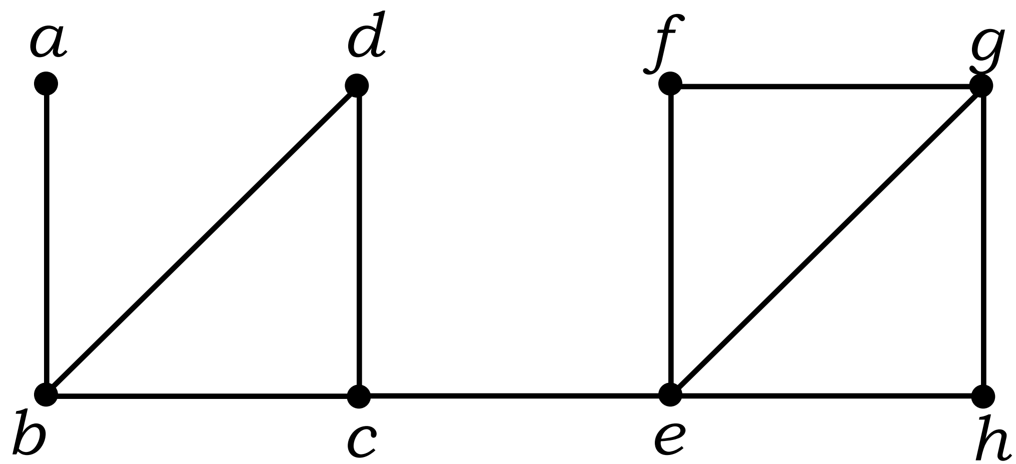
Example

- Find the cut vertices and cut edges in the following graph.



Example

- Find the cut vertices and cut edges in the following graph.



Cut vertices: c, e and b
Cut edge: (c, e)

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Chapter 9.5

Euler and Hamilton Paths

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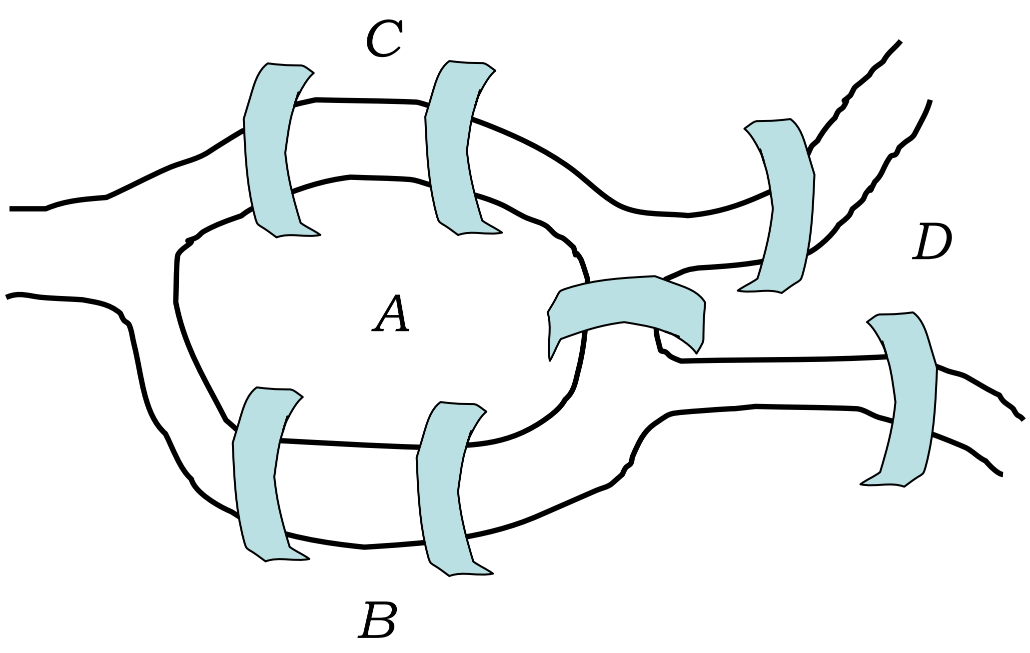
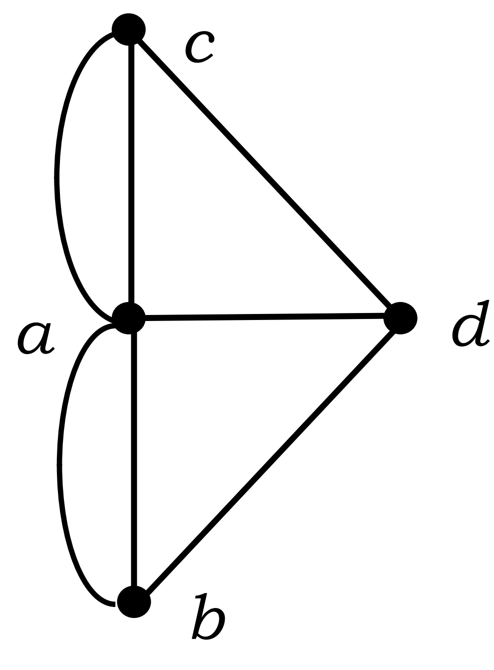
Euler Paths and Circuits

- The Seven bridges of Königsberg



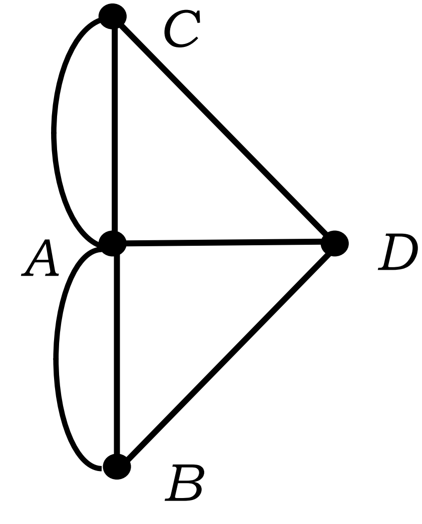


Euler Paths and Circuits

- An *Euler path* is a path using every edge of the graph *G* exactly once.
- An *Euler circuit* is an Euler path that returns to its start.

Does this graph have an Euler circuit?

No.

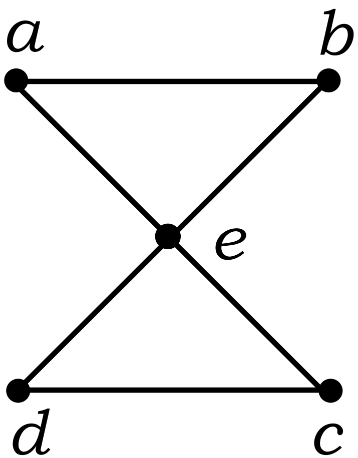


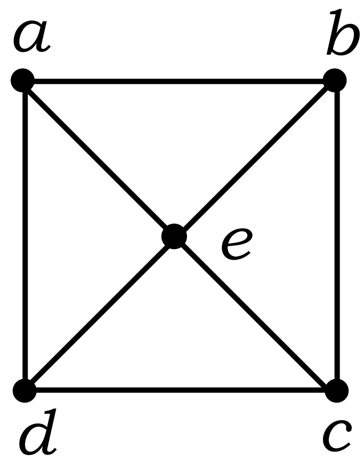
Necessary and Sufficient Conditions

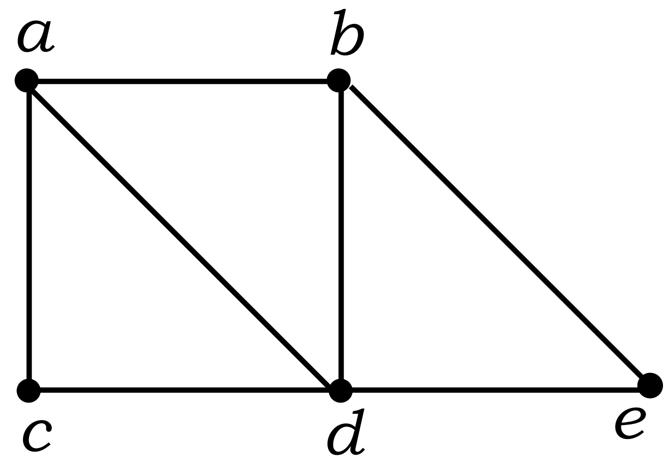
- How about multigraphs?
- A connected multigraph has a Euler circuit iff *each of its vertices has an even degree*.
- A connected multigraph has a Euler path but not an Euler circuit iff *it has exactly two vertices of odd degree*.

Example

- Which of the following graphs has an Euler *circuit*?







yes

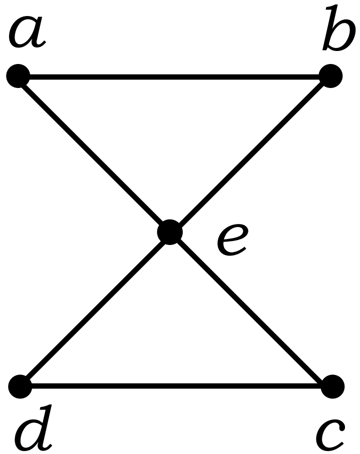
(a, e, c, d, e, b, a)

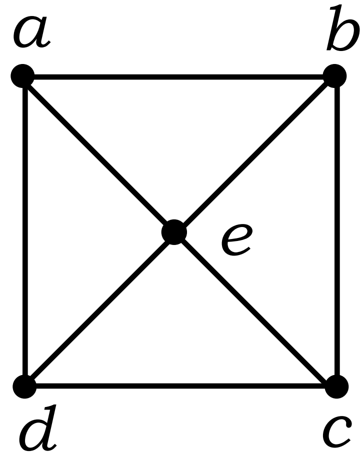
no

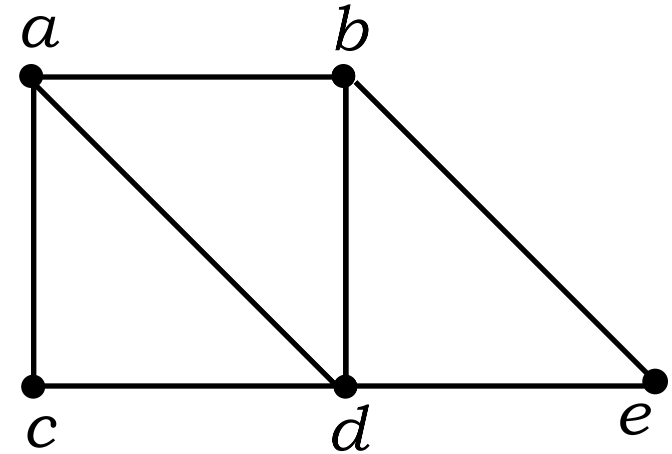
no

Example

- Which of the following graphs has an Euler *path*?







yes
(a, e, c, d, e, b, a)

no

yes
(a, c, d, e, b, d, a, b)

Hamilton Paths and Circuits

- A *Hamilton path* in a graph G is a path which visits every vertex in G exactly once.

- A *Hamilton circuit* is a Hamilton path that returns to its start.

Hamilton Circuits

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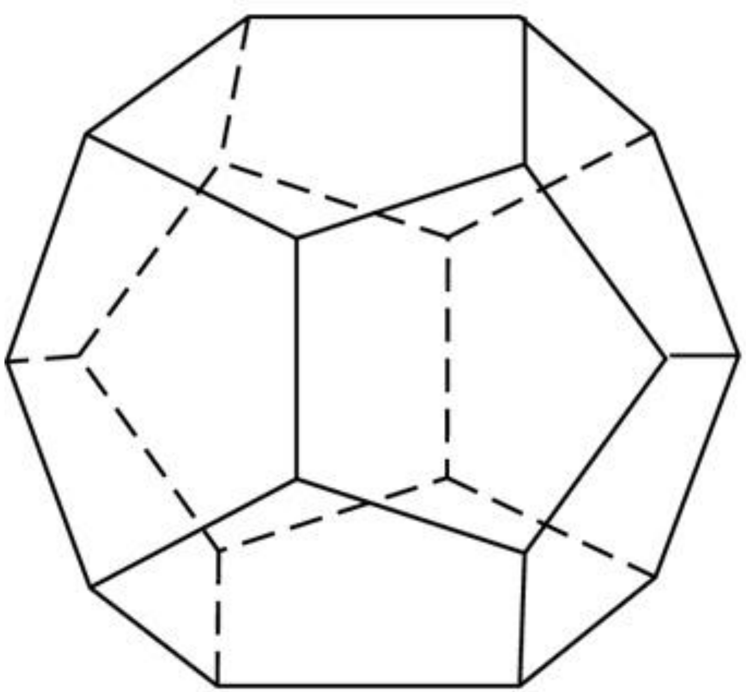
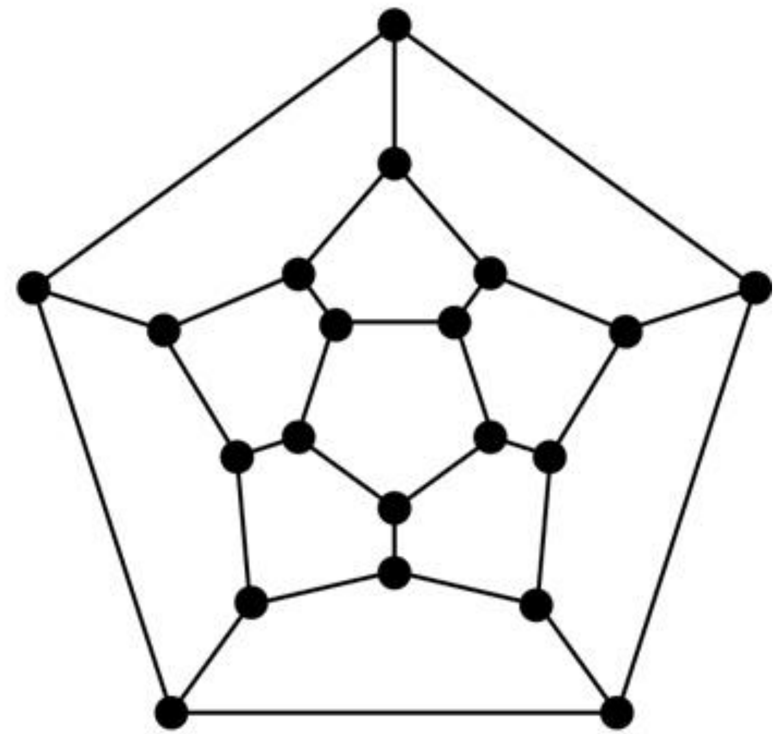


(a)



(b)

Is there a circuit in this graph that passes through each vertex exactly once?

Hamilton Circuits

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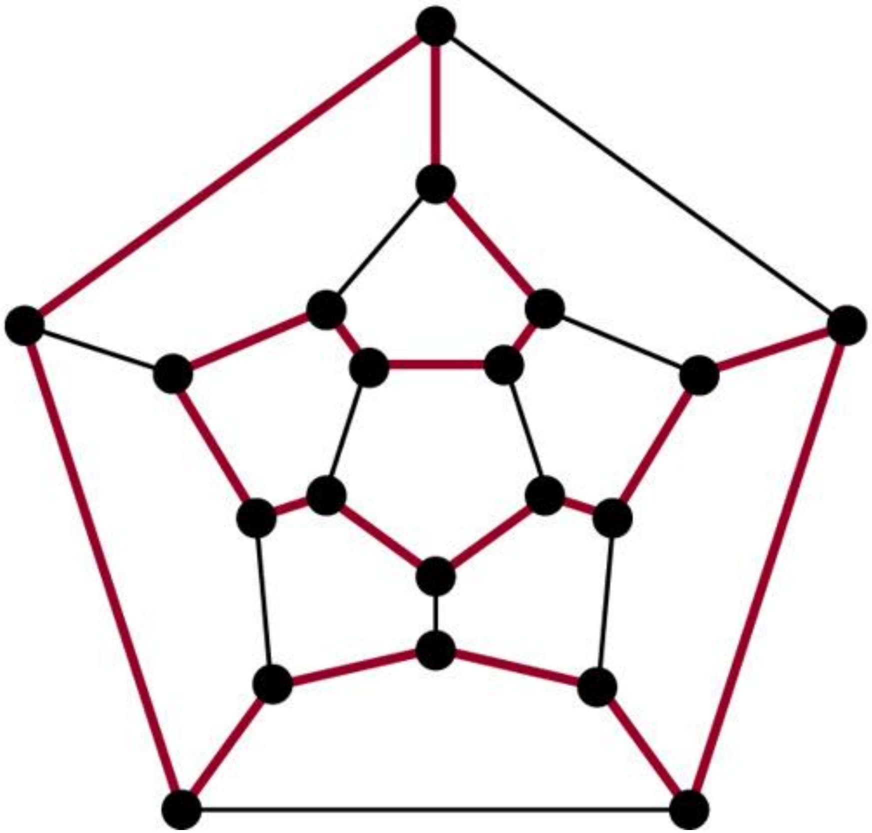

Yes; this is a circuit that passes through each vertex exactly once.

Finding Hamilton Circuits

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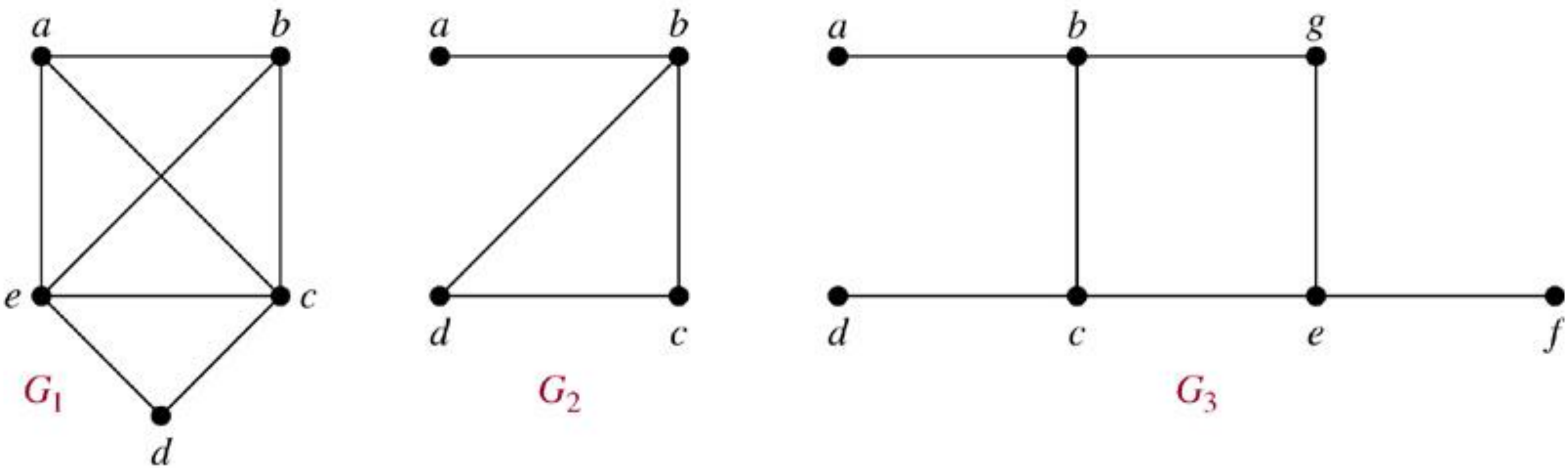

Which of these three figures has a Hamilton circuit?
Of, if no Hamilton circuit, a Hamilton path?

Finding Hamilton Circuits

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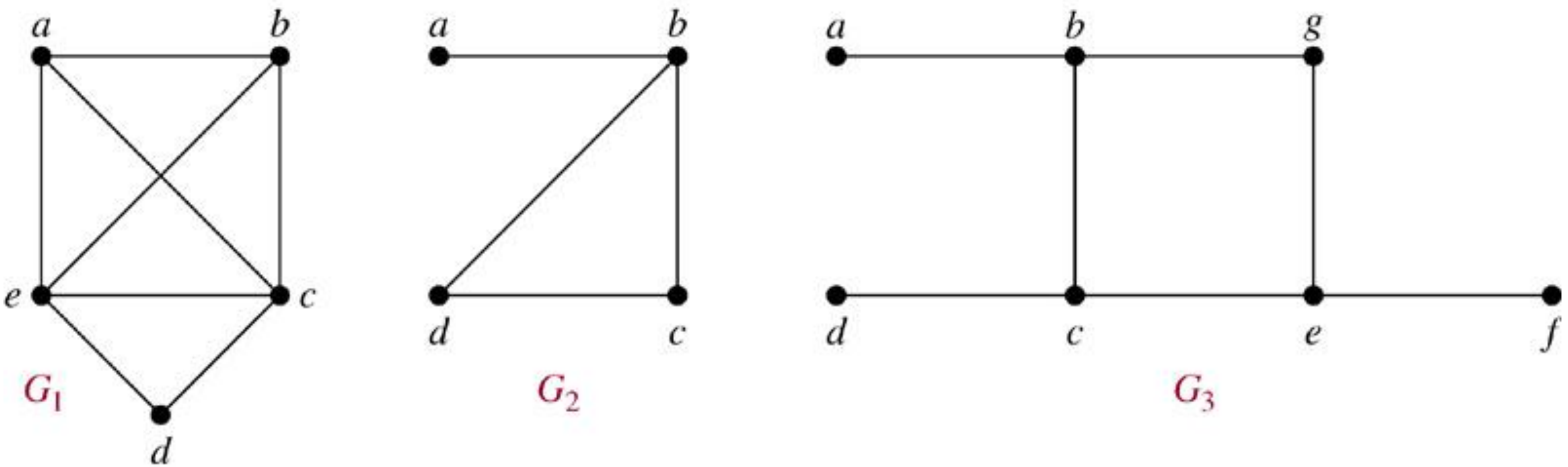

- G_1 has a Hamilton circuit: a, b, c, d, e, a
- G_2 does not have a Hamilton circuit, but does have a Hamilton path: a, b, c, d
- G_3 has neither.

Finding Hamilton Circuits

- Unlike the Euler circuit problem, finding Hamilton circuits is hard.
- There is no simple set of necessary and sufficient conditions, and no simple algorithm.

Properties to look for ...

- No vertex of degree 1
- If a node has degree 2, then both edges incident to it must be in any Hamilton circuit.
- No smaller circuits contained in any Hamilton circuit (the start/endpoint of any smaller circuit would have to be visited twice).

A Sufficient Condition

Let G be a connected simple graph with n vertices with $n \geq 3$.

G has a Hamilton circuit if the degree of each vertex is $\geq n/2$.

Travelling Salesman Problem

A Hamilton circuit or path may be used to solve practical problems that require visiting “vertices”, such as:

- road intersections
- pipeline crossings
- communication network nodes

A classic example is the Travelling Salesman Problem – finding a Hamilton circuit in a complete graph such that the total weight of its edges is minimal.

Summary

Property	**Euler**	**Hamilton**
Repeated visits to a given node allowed?	Yes	No
Repeated traversals of a given edge allowed?	No	No
Omitted nodes allowed?	No	No
Omitted edges allowed?	No	Yes