Chapter 8. Rotational Motion of Rigid body

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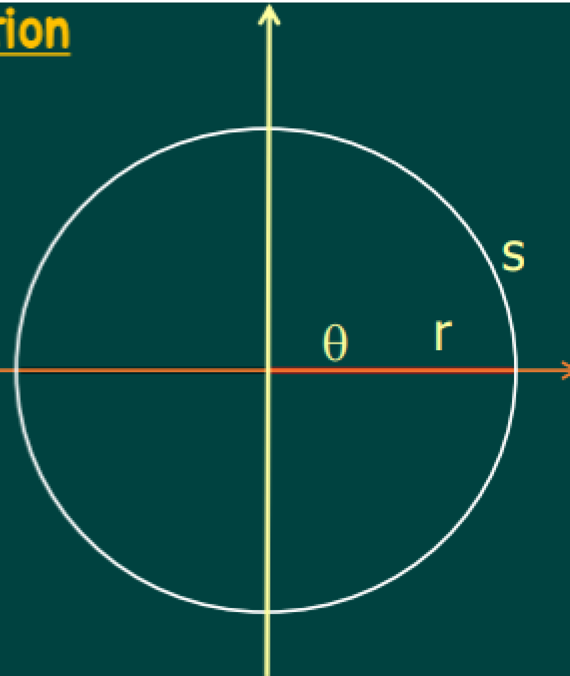
8.1. Angular position, velovity and acceleration

8.1.1. Angular Position and Radian



- What is the circumference C ?

$$C = (2\pi)r \qquad => \qquad 2\pi = \frac{C}{r}$$

- θ can be defined as the arc length s along a circle divided by the radius r :

$$\theta = \frac{s}{r}$$

- θ is a pure number, but commonly is given the artificial unit, radian ("rad") .

- Converting from degrees to radians:

$$\theta\ (rad) = \frac{\pi}{180°}\ \theta\ (\text{degrees})$$

- Converting from radians to degrees

$$\theta(\text{deg}r\,ees) = \frac{180°}{\pi}\theta(rad)$$

$$1\ rad = \frac{360°}{2\pi} = 57.3°$$

Whenever using rotational equations, you must use angles expressed in radians

Angular Displacement

- The angular displacement is defined as the angle the object rotates through during some time interval

$$\Delta\theta = \theta_f - \theta_i$$

- SI unit: radian (rad)
- This is the angle that the reference line of length r sweeps out

8.1.2. Average and Instantaneous Angular Speed

- The ***average*** angular speed, ω_{avg}, of a rotating rigid object is the ratio of the angular displacement to the time interval

$$\omega_{avg} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

- The ***instantaneous*** angular speed is defined as the limit of the average speed as the time interval approaches zero

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

- SI unit of angular speed: radian per second (rad/s)
- Angular speed positive if rotating in counterclockwise.
- Angular speed will be negative if rotating in clockwise.

8.1.3. Angular Acceleration

- The ***average angular acceleration***, β, of an object is defined as the ratio of the change in the angular speed to the time it takes for the object to undergo the change:

$$\beta_{avg} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

$t = t_i$: w_i

$t = t_f$: w_f

8.1.3. Angular Acceleration

- The **instantaneous** angular acceleration is defined as the limit of the average angular acceleration as the time goes to 0

$$\beta = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

- **SI units** of angular acceleration: **rad/s²**
- Positive angular acceleration is counterclockwise (RH rule – curl your fingers in the direction of motion).
  - if an object rotating counterclockwise is speeding up
  - if an object rotating clockwise is slowing down
- Negative angular acceleration is clockwise.
  - if an object rotating counterclockwise is slowing down
  - if an object rotating clockwise is speeding up

Rotational Kinematics

- A number of parallels exist between the equations for rotational motion and those for linear motion.

$$v_{avg} = \frac{x_f - x_i}{t_f - t_i} = \frac{\Delta x}{\Delta t} \qquad \omega_{avg} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta \theta}{\Delta t}$$

- Under **constant angular acceleration**, we can describe the motion of the rigid object using a set of kinematic equations
  - These are similar to the kinematic equations for linear motion
  - The rotational equations have the same mathematical form as the linear equations

Analogy with Linear Kinematics

- Start with angular acceleration: $\boldsymbol{\beta} = \frac{d\omega}{dt}$
- Integrate once: $\omega_f = \int \boldsymbol{\beta} dt = \omega_i + \boldsymbol{\beta} t$

$$\boxed{v_f = v_i + at}$$

- Integrate again: $\theta_f = \int (\omega_i + \boldsymbol{\beta} t) dt = \theta_i + \omega_i t + \frac{1}{2} \boldsymbol{\beta} t^2$

$$\boxed{x_f = x_i + v_i t + \frac{1}{2} a t^2}$$

- Just substitute symbols, and all of the old equations apply:

$$x \Rightarrow \theta$$
$$v \Rightarrow \omega$$
$$a \Rightarrow \boldsymbol{\beta}$$

Example: A Rotating Wheel

- A wheel rotates with a constant angular acceleration of 3.5 rad/s². If the angular speed of the wheel is 2.0 rad/s at $t = 0$

  (a) through what angle does the wheel rotate between

$t = 0$ and $t = 2.0$ s? Given your answer in radians and in revolutions.

  (b) What is the angular speed of the wheel at $t = 2.0$ s?

$$\omega_i = 2.0\, rad/s$$
$$\beta = 3.5\, rad/s^2$$
$$t = 2.0\, s$$

$$\theta_f - \theta_i = ?$$
$$\omega_f = ?$$

Relationship Between Angular and Linear Quantities

- Every point on the rotating object has the same angular motion
- Every point on the rotating object does **not** have the same linear motion

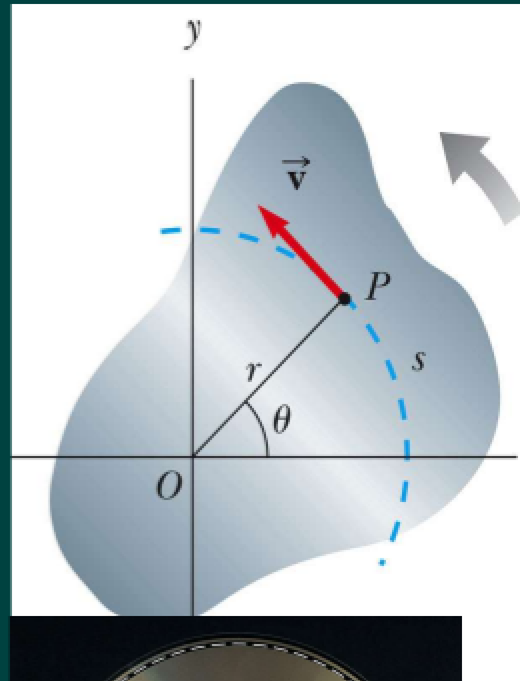
  Displacement: $s = \theta r$

  Speeds: $v = \omega r$

  Acelerations: $a = \beta r$

- The linear velocity is always tangent to the circular path
  - Called the tangential velocity
- The magnitude of velocity is defined by the tangential speed

$$\Delta\theta = \frac{\Delta s}{r} \qquad \frac{\Delta\theta}{\Delta t} = \frac{\Delta s}{r\Delta t} = \frac{1}{r}\frac{\Delta s}{\Delta t}$$



Acceleration Comparison

- The tangential acceleration is the derivative of the tangential velocity

$$\Delta v = r\Delta\omega$$

$$\frac{\Delta v}{\Delta t} = r\frac{\Delta\omega}{\Delta t} = r\boldsymbol{\beta}$$

$$a_t = r\boldsymbol{\beta}$$

Speed and Acceleration Note

- All points on the rigid object will have the same angular speed, but not the same tangential speed
- All points on the rigid object will have the same angular acceleration, but not the same tangential acceleration
- The tangential quantities depend on r, and r is not the same for all points on the object

$$\omega = \frac{v}{r} \quad or \quad v = r\omega$$

$$a_t = r\boldsymbol{\beta}$$

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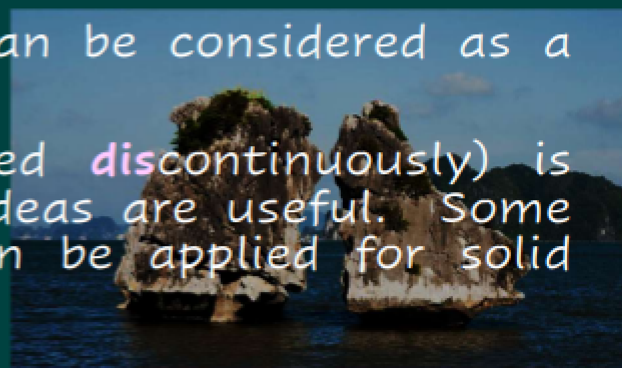
8.2. Rigid body and motion of rigid body

A rigid body/object: is a solid body in which deformation is zero or negligibly small.

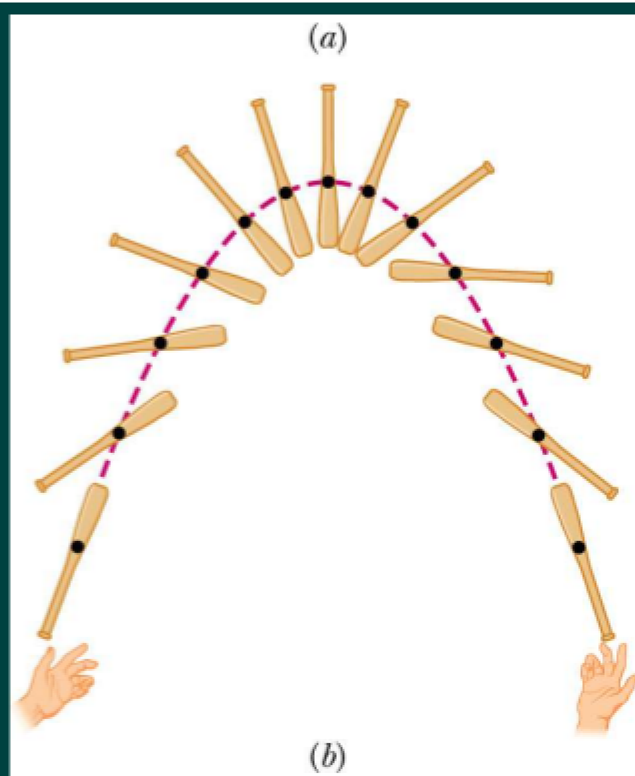
- The relative locations of all particles making up the object remain constant
- All real objects are deformable to some extent, but the rigid object model is very useful in many situations where the deformation is negligible. This simplification allows analysis of the motion of an extended object

- **A rigid body** can be considered as **a system of point particles** in which the distance between any two particles is always a constant.
- On the other hand, **a rigid body** can be considered as a **continuous distribution of mass**.
- The first idea (mass is distributed **dis**continuously) is opposite to the second. But both ideas are useful. Some results studied in this chapter can be applied for solid bodies which are not rigid.



8.2.1. The Center of Mass

- How should we define the position of the moving body (the **baseball bat**)?
- What is height h for $PE_g = mgh$?
- Take the average position of mass. Call "Center of Mass" (COM or CM)



- There is a special point in a system or object, called the ***center of mass***, that moves as if all of the mass of the system is concentrated at that point
- The CM of an object or a system is the point, where the object or the system can be **balanced** in the uniform gravitational field.

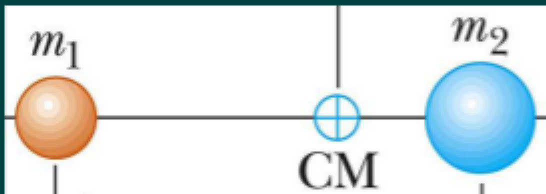
8.2.1. The Center of Mass (cont's)

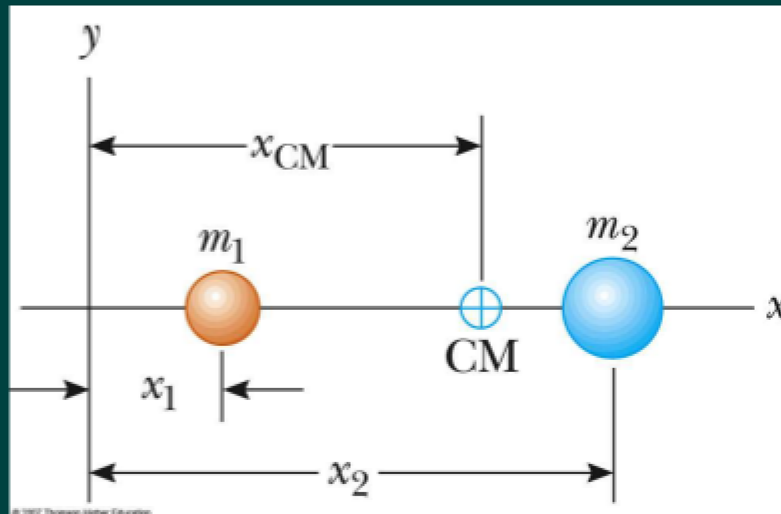
- The center of mass of any symmetric object lies on an axis of symmetry and on any plane of symmetry
  - If the object has uniform density
- The CM may reside inside the body, or outside the body

Where is the Center of Mass ?

- The center of mass of particles

Two bodies in 1 dimension





$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Where is the Center of Mass ?

- Assume $m_1 = 1$ kg, $m_2 = 3$ kg, and $x_1 = 1$ m, $x_2 = 5$ m, where is the center of mass of these two objects?

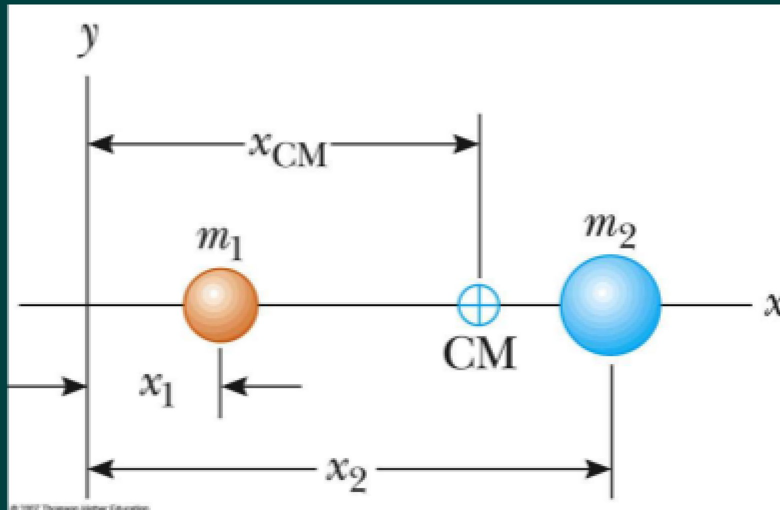
A) $x_{CM} = 1$ m

B) $x_{CM} = 2$ m

C) $x_{CM} = 3$ m

D) $x_{CM} = 4$ m

E) $x_{CM} = 5$ m



$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Center of Mass for many particles in 3D?

Center of Mass for a System of Particles

- Two bodies and one dimension

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

- General case: n bodies and three dimension

$$\vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i$$

  - where $M = m_1 + m_2 + m_3 + \ldots$

$$\boxed{x_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i}$$

$$\boxed{y_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i}$$

$$\boxed{z_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i}$$

Sample Problem :

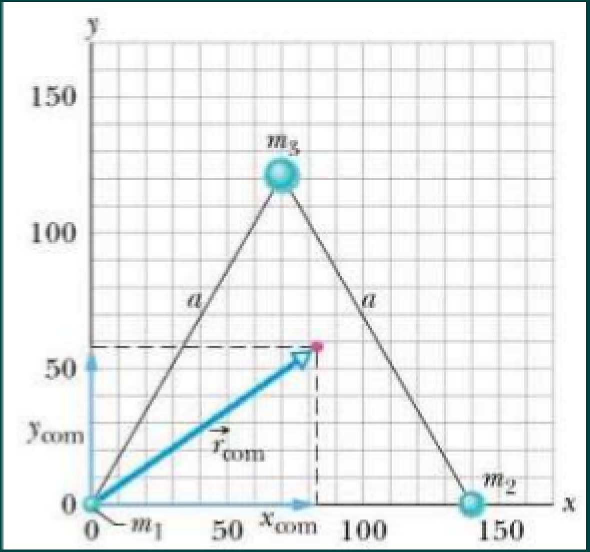
Three particles of masses m_1 = 1.2 kg, m_2 = 2.5 kg, and m_3 = 3.4 kg form an equilateral triangle of edge length a = 140 cm. Where is the center of mass of this system?

(Hint: m_1 is at (0,0), m_2 is at (140 cm,0), and m3 is at (70 cm, 120 cm), as shown in the figure below.)

$$\boxed{x_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}}$$

$$\boxed{y_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}}$$

$x_{CM} =$ 82.8 cm and $y_{CM} = 57.5$ cm



Translation and rotation

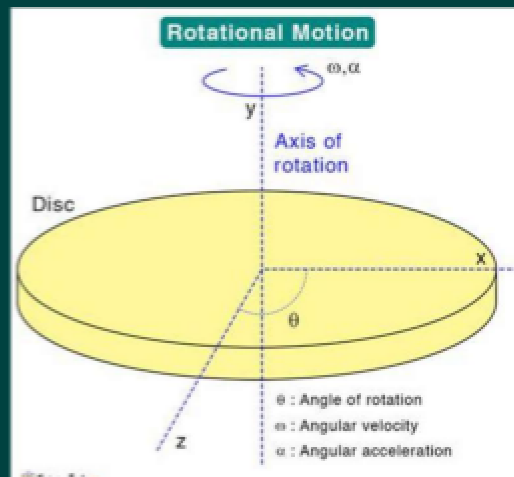
Translational motion is the motion such that, at any instant, all the particles of the body move parallel to each other.

- A translational motion can be rectilinear or curvilinear.The body is shifted from point to point without rotation.
- Momentarily, all the particles (of the body) move in the same direction.
- All the particles have the same velocity and acceleration.
- Therefore, in studying a **pure** translational motion, we only need to study the motion of the CM of the body.

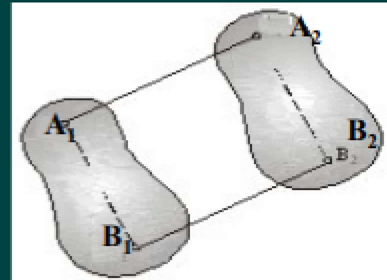
Translation and rotation (cont.)

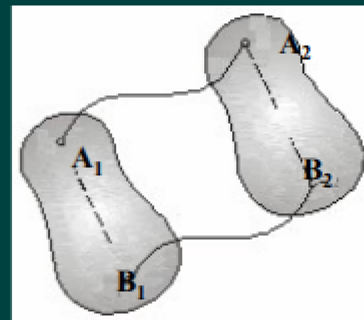
In a pure rotational motion of a rigid body, the body rotates about an axis which is fixed in an inertial reference frame.

- The orbit of any particle (in the body) is a circle which is perpendicular to the axis and centered at the axis.
- All particles rotate the same angle in the same time interval, and have the same angular velocity w and angular acceleration β.



Motion of a rigid body:





Translational motion:

- Every point in the body experiences the same displacement that points undergo the exact same motions and the distance between the points is also maintained. It means that all the particles forming the body move along parallel paths which may be either rectilinear (straight line) or curvilinear.
- Every point in the body experiences the same velocity and acceleration at any instant of time.

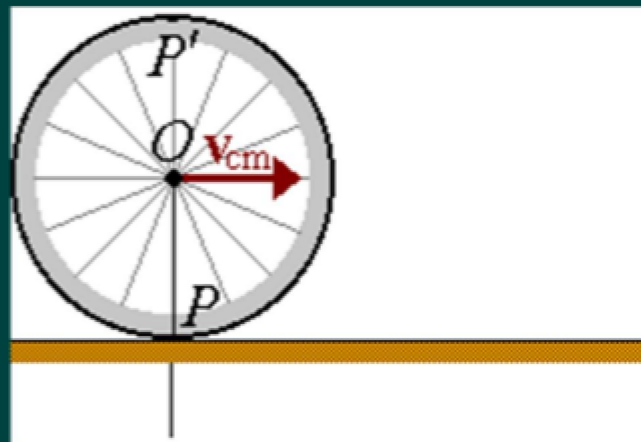
Rotational motion / a fix axe:

- Particles forming the rigid body move in parallel planes along circles centered (***O***) on the same fixed axis (*Δ*) with different radiuses (***r***).
- Have the same angular velocity, and angular acceleration

Motion of a rigid body:

Rotating without slipping:

- .....



8.2.2. Translational Motion of a System of Particles or a rigid object

- Assume the total mass, M, of the system remains **constant**
- We can describe the translational motion of the system in terms of the velocity and acceleration of **the center of mass** of the system.
- We can also describe the momentum of the system and Newton's Second Law for the system.

Velocity and Momentum of a System of Particles

- The velocity of the center of mass of a system of particles is

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{1}{M}\sum_i m_i\vec{v}_i$$

- The momentum can be expressed as

$$M\vec{v}_{CM} = \sum_i m_i\vec{v}_i = \sum_i \vec{p}_i = \vec{p}_{tot}$$

- The total linear momentum of the system equals the total mass multiplied by the velocity of the center of mass.

Acceleration and Force of the Center of Mass

- The acceleration of the center of mass can be found by differentiating the velocity with respect to time

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \vec{a}_i$$

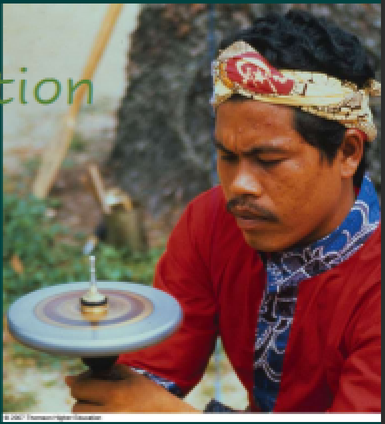
- The acceleration can be related to a force: $\boldsymbol{M\vec{a}_{CM} = \sum_i \vec{F}_i}$
- If we sum over all the internal forces, they cancel in pairs and the net force on the system is caused only by the external forces,
- So, the net external force equals the total mass of the system multiplied by the acceleration of the center of mass:

$$\sum \vec{F}_{ext} = M\vec{a}_{CM}$$

Newton's Second Law for a System of Particles/a rigid object

- The center of mass of a system of particles/a rigid object of combined mass *M* moves like an equivalent particle of mass *M* would move under the influence of the net external force on the system.

Chapter 8. Rotational Motion of Rigid body



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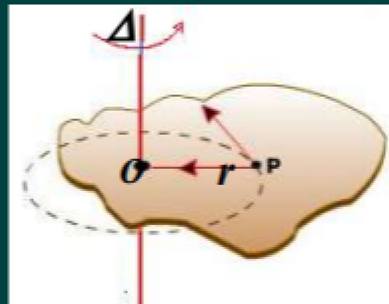
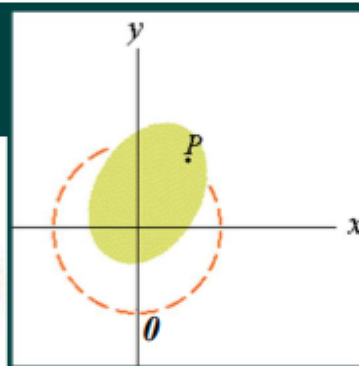
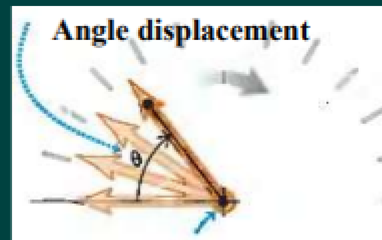
Kinematic characterizations





- Particles forming the rigid body move in parallel planes along circles centered (**0**) on the same fixed axis (Δ) with different radius (***r***).
- Displacement: of point *P* on body is seen as line *0P* that is fixed in the body and rotates with it forming the angles (θ) and describing the **rotational position** of the body as coordinate for rotation.

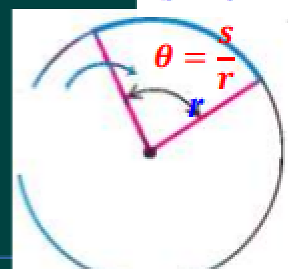


Rotation axis passes origin and points in

- The angle subtended at the center of a circle by an arc with a length *s* equal to the radius of the circle (*r*). The value of θ (in radians) is defined as $\theta = \frac{s}{r}$

An angle is ratio of the arc length *s* to the radius *r*

$$s = \theta r$$

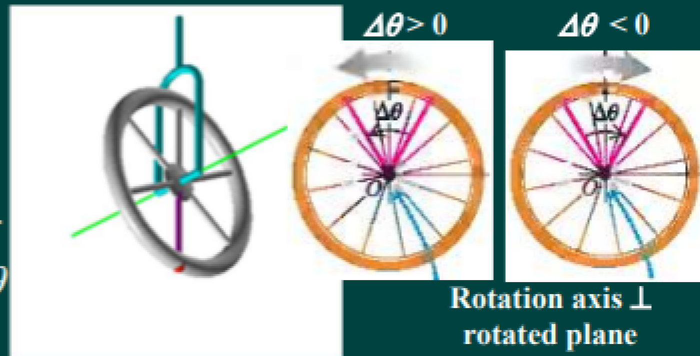


8.3. Rotation of Rigid body

Kinematic characterizations (cont.)

☞ Average angular velocity of the body as the ratio of the angular displacement in the time interval

$$\omega = \frac{\Delta\theta}{\Delta t}$$

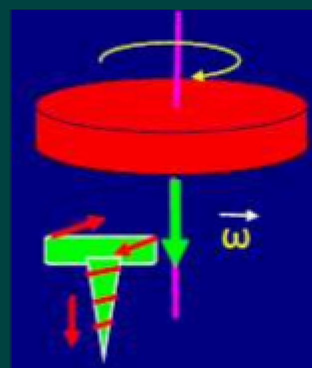


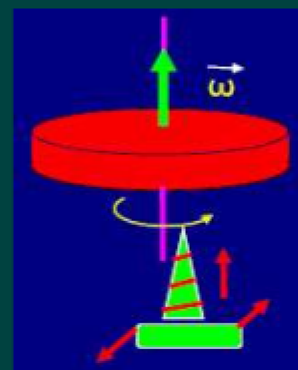
☞ The instantaneous angular velocity is the derivative of θ with respect to t: $\omega = \frac{d\theta}{dt}$

☞ When rigid body rotates all points have the same angular velocity .

☞ **Angular velocity vector $\vec{\omega}$** :

- Orientation is lying on the rotation axis;
- Direction is determined by the right-hand rule as vector product.



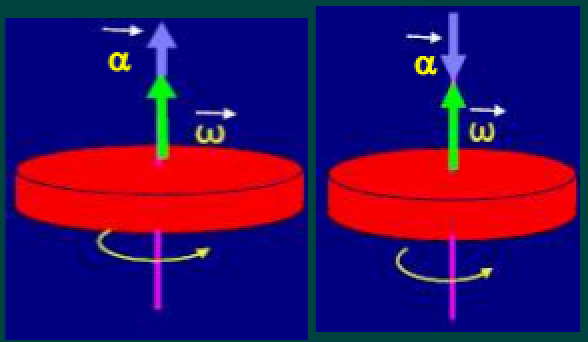


8.3. Rotation of Rigid body

Kinematic characterizations (cont.)

☞ **Acceleration vector: $\vec{\alpha}$**

- Orientation is lying on the rotation axis,
- Same direction of $\vec{\omega}$ if ω is increasing, inversely if ω is decreasing.
- Magnitude: $\beta = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$



☞ For points located on rotation axis both velocity and acceleration are zero.

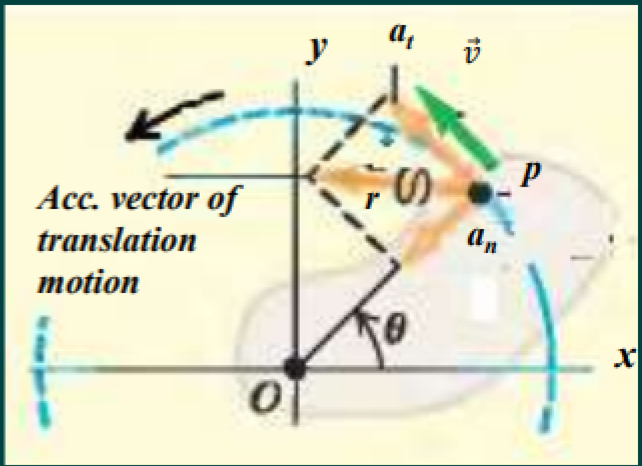
☞ Relationship between: $\vec{v}, \vec{a}$ and $\vec{\omega}, \vec{\alpha}$

- ♦ $\vec{v}$ and $\vec{\omega}$:

$$v = \omega r \text{ or } \vec{v} = \vec{\omega} \times \vec{r}$$

- ♦ $\vec{a}_t$ and $\vec{\alpha}$: $a_t = \alpha . r$ or $\vec{a}_t = \vec{\alpha} \times \vec{r}$

- ♦ $\vec{a}_n$ and $\vec{\omega}$: $a_n = \frac{v^2}{r} = \omega^2 r$



8.3.1. Torque/moment of a force acting on a rigid object

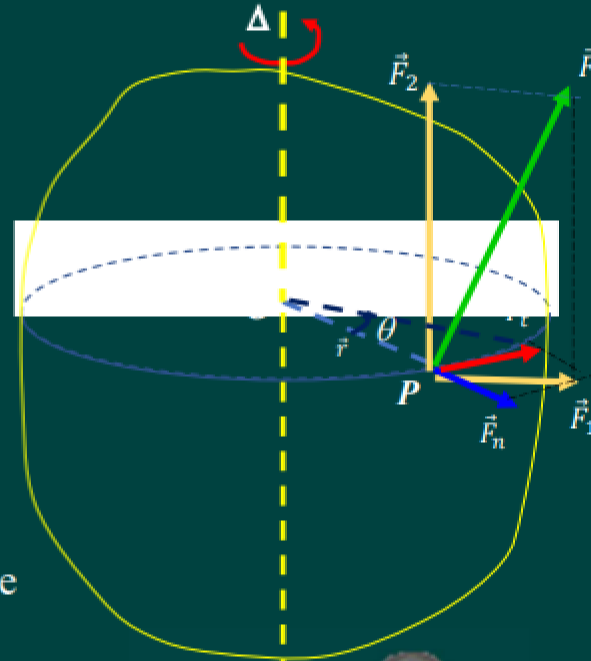
☞ what aspect of a force determines the cause or change in rotational motion?

☞ Consider force $\vec{F}$ acting on a rigid body at P of which the position is defined by vector $\overrightarrow{0P} = \vec{r}$

(0 is lying on the rotation axis Δ). It's components are as follows,

$$\vec{F}\begin{cases} \vec{F}_2 // \Delta\text{: Impossible to cause the rotation} \\ \vec{F}_1 \perp \Delta \begin{cases} \vec{F}_n\text{: radical force} \Rightarrow \text{impossible to cause the rotation} \\ \vec{F}_t\text{: causes the rotation} \end{cases} \end{cases}$$

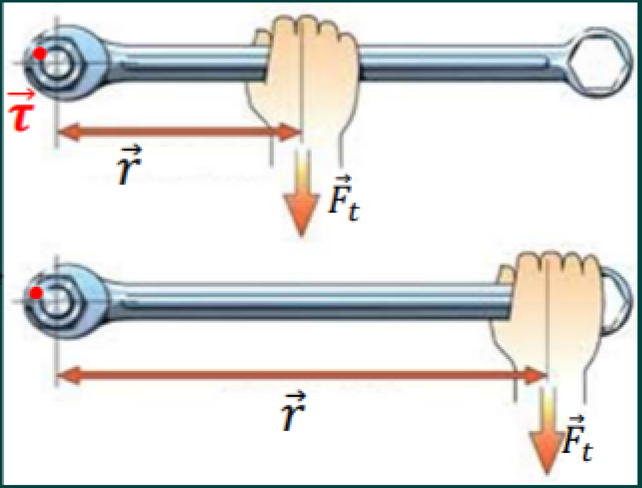
♦ Conclusion: **Only the tangent force can cause the rotation about fixed axis of a rigid body.**



8.3.1. Torque (cont.)

☞ **Definition: for a force perpendicular to the rotating axis**

$$\vec{\tau} = \vec{r} \times \vec{F}_t \qquad (r\text{: Lever arm})$$



♦ Magnitude: $\tau = rF_t \sin(\vec{r}, \vec{F}_t)$

☞ when : $\vec{\tau} = \text{const}$

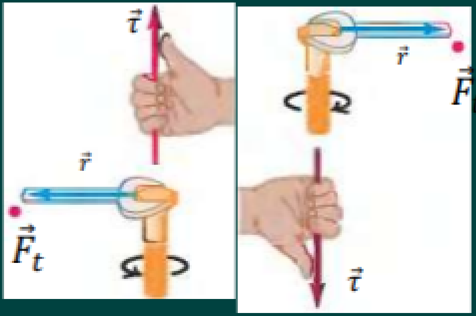
♦ shorter $r \Rightarrow$ greater F;

♦ longer $r \Rightarrow$ smaller F.

☞ **Vector $\vec{\tau}$ has**

♦ Orientation coincides the rotation axis;

♦ Direction is determined by using the right –hand-side rule.



8.3.2. Torque and two conditions for equilibrium (reading)

- An object in mechanical equilibrium must satisfy the following two conditions

  (1) The net external force must be zero:

$$F_{net} = \Sigma \vec{F}_{i.external} = 0$$

  (2) The net external torque must be zero: $\Sigma \tau_i = 0$

Example 3: Balancing seesaw

$$\tau_{pivot} + \tau_{gravity} + \tau_{man} + \tau_{woman} = 0$$

The first two torques are zero.
Let x represent the man's distance from the pivot.
The woman is at a distance $L/2$ from the pivot.

$$0 + 0 - Mgx + mg(L/2) = 0$$

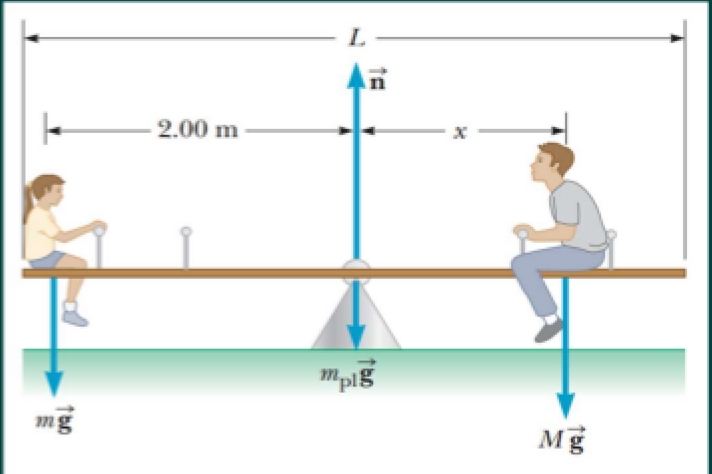


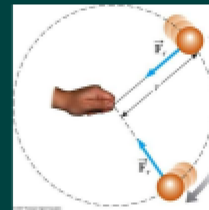
Figure 8.8 (Example 8.3) The system consists of two people and a seesaw. Because the sum of the forces and the sum of the torques acting on the system are both zero, the system is said to be in equilibrium.

8.3.3. Moment of Inertia

a) Moment of Inertia of Point Mass

- For a single particle, the definition of moment of inertia is:

$$I = mr^2$$



  - m is the mass of the single particle
  - r is the rotational radius
- SI units of moment of inertia are kg·m²
- **Moment of inertia** and **mass** of an object are different quantities.
- It depends on both the quantity of matter and its distribution (through the r^2 term).

- **For a composite particle**, the definition of moment of inertia is

$$I = \sum m_i {r_i}^2 = m_1 {r_1}^2 + m_2 {r_2}^2 + m_3 {r_3}^2 + m_4 {r_4}^2 + \ldots$$

  - m_i and r_i are the mass and rotational radius of the i^{th} single particle

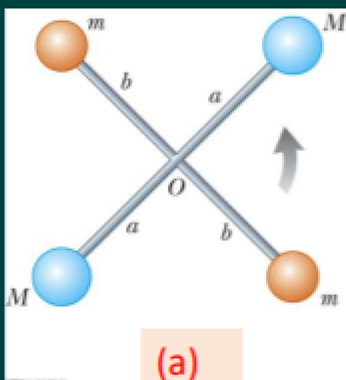
Example 4. The Baton Twirler

- Consider an unusual baton made up of four sphere fastened to the ends of very light rods. Each rod is 1.0m long ($a = b = 1.0$ m). M = 0.3 kg and m = 0.2 kg.

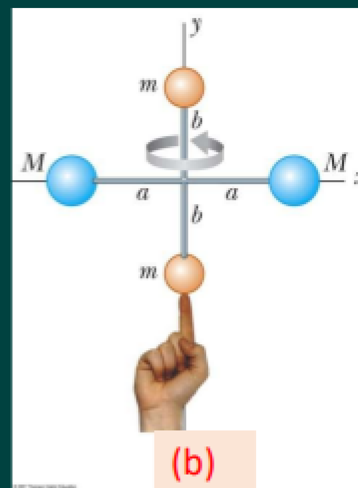
(a) Find I about an axis perpendicular to the page and passing through the point where the rods cross.

(b) The majorette tries spinning her strange baton about the axis y, calculate I of the baton about this axis.

(a) $I = \sum m_i r_i^2 = mb^2 + Ma^2 + mb^2 + Ma^2 = 2Ma^2 + 2mb^2$



(a)



(b)

b) Moment of Inertia of Extended Objects

- Divided the extended objects into many small volume elements, each of mass Δm_i
- We can rewrite the expression for I in terms of Δm

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \, \Delta m_i = \int r^2 dm$$

- Consider a small volume such that $dm = \rho \, dV$, with ρ is mass density; Then

$$I = \int \rho r^2 dV$$

- If ρ is constant, the integral can be evaluated with known geometry, otherwise its variation with position must be known

(Mass) Densities

- You know the density (volume density) as mass/unit volume
  - $\rho = M/V = dm/dV \qquad => dm = \rho dV$
- We can define other densities such as surface density (mass/unit area)
  - $\sigma = M/A = dm/dA \qquad => dm = \sigma dA$
- Or linear density (mass/unit length)
  - $\lambda = M/L = dm/dx \qquad => dm = \lambda dx$

Moment of Inertia of a Uniform Rigid Rod

- The shaded area has a mass

$$dm = \lambda\, dx$$

- Then the moment of inertia is

: Rotating axis y (CM)

By y-axis:

$$I_y = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L}\, dx$$

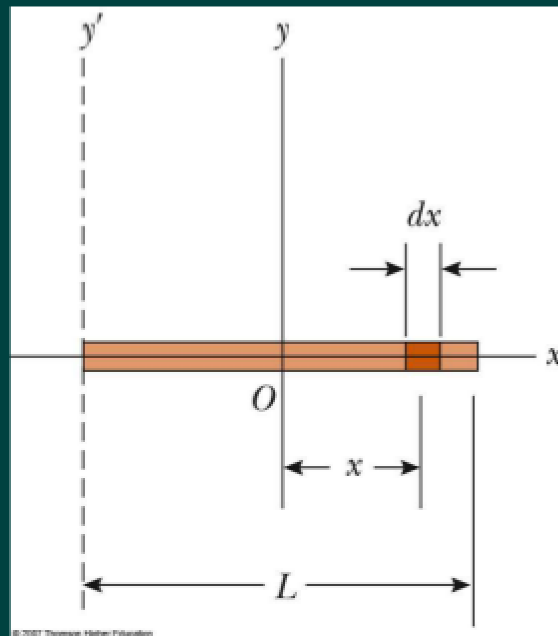
$$= \frac{M}{L}\int_{-L/2}^{L/2} x^2\, dx$$

$$\boldsymbol{I = \frac{1}{12} ML^2}$$

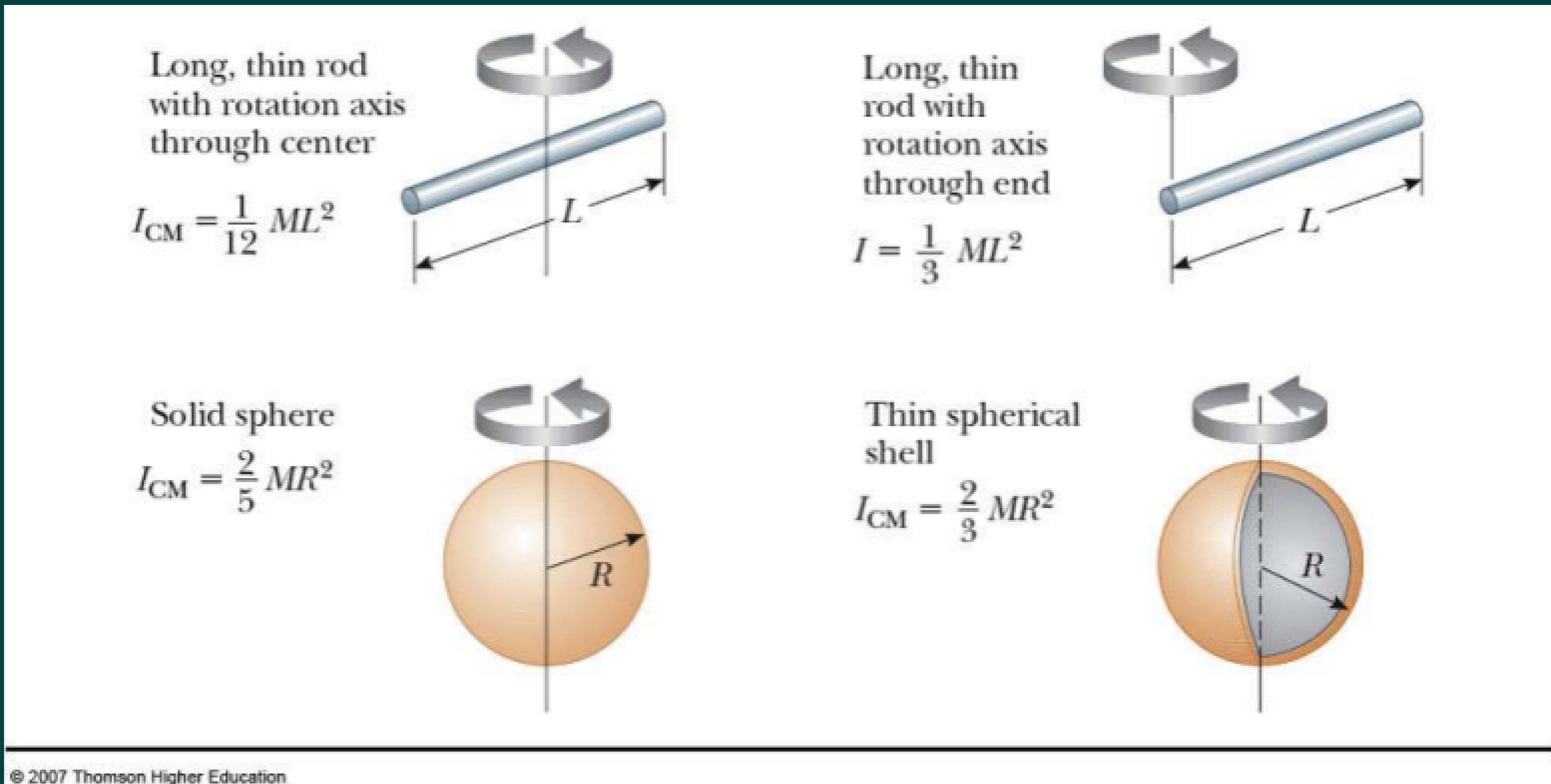
By y'-axis:

$$I_{y'} = \int r^2 dm = \int_0^L x^2 \frac{M}{L}\, dx$$

$$\boldsymbol{I_{y'} = \frac{1}{3} ML^2}$$



Moment of Inertia for some other common shapes

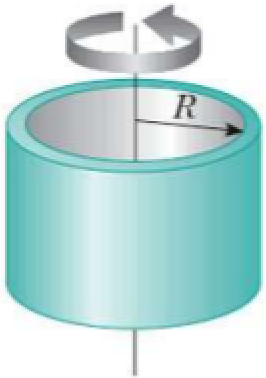


Moment of Inertia for some other common shapes

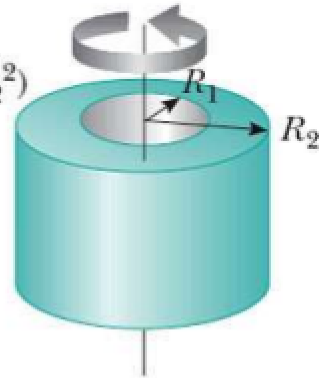
TABLE 10.2

Moments of Inertia of Homogeneous Rigid Objects with Different Geometries

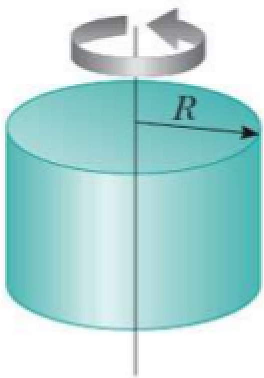
Hoop or thin cylindrical shell
$I_{CM} = MR^2$



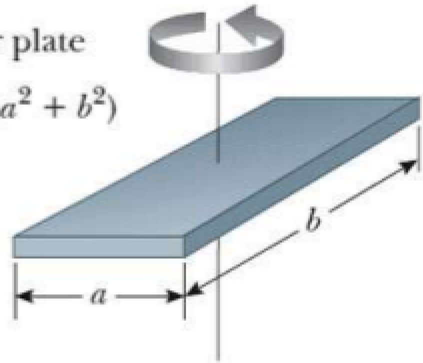
Hollow cylinder
$I_{CM} = \frac{1}{2}M(R_1^2 + R_2^2)$



Solid cylinder or disk
$I_{CM} = \frac{1}{2}MR^2$

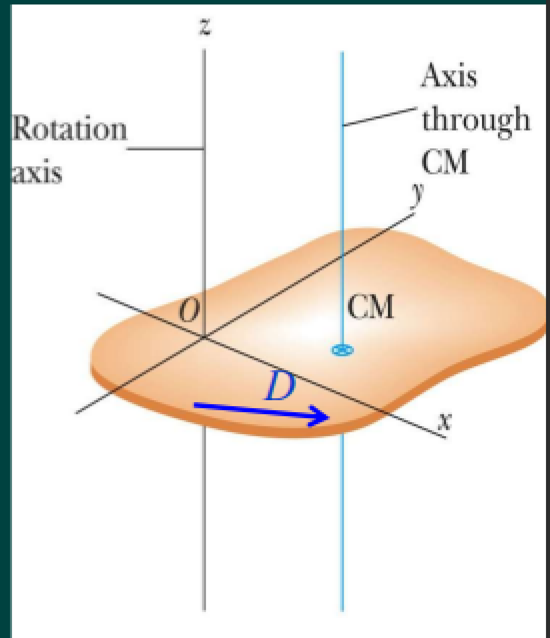


Rectangular plate
$I_{CM} = \frac{1}{12}M(a^2 + b^2)$



c) Parallel-Axis Theorem

- In the previous examples, the axis of rotation calculations
- The theorem states coincides with the axis of symmetry of the object
- For an arbitrary axis, the parallel-axis theorem often simplifies



$$\boxed{I = I_{CM} + MD^2}$$

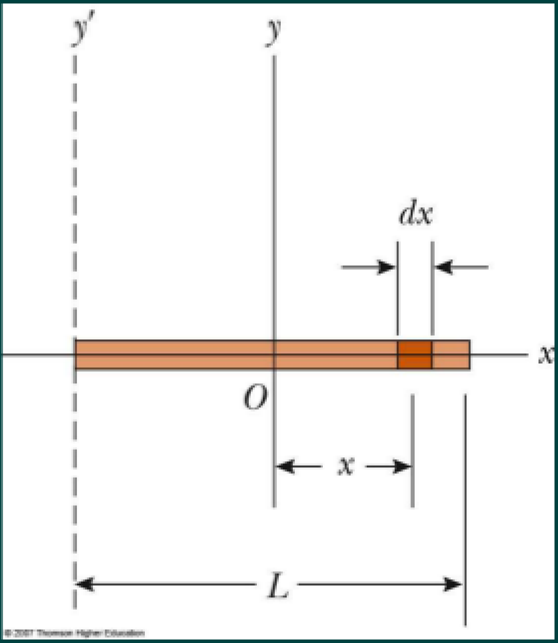
Where: I is about any axis parallel to the axis through the center of mass of the object

I_{CM} is about the axis through the center of mass

D is the distance from the center of mass axis to the arbitrary axis

c) Parallel-Axis Theorem (cont.)

Moment of Inertia of a Uniform Rigid Rod (cont.)



- The moment of inertia about y is

$$I_y = I_{CM} = \frac{1}{12}ML^2$$

- The moment of inertia about y′ is

$$I_{y'} = I_{CM} + MD^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$

Review

- **Translational motion:** The net external force equals the total mass of the system multiplied by the acceleration of the center of mass:

$$\sum \vec{F}_{ext} = M\vec{a}_{CM}$$

- **Torque Definition**

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \tau = r.F\sin\theta = d.F$$

$$d = r.\sin\theta$$

- **Moment of Inertia**

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$

Chapter 8. Rotational Motion of Rigid body

8.3.4. Newton's Second Law for a Rotating Object

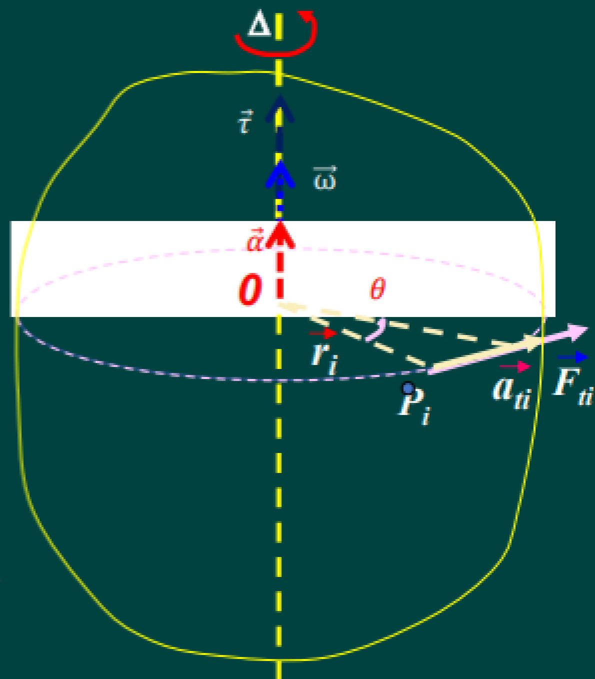
- When a rigid object is subject to a net torque (≠0), it undergoes an angular acceleration

$$\boxed{\Sigma\tau_i = I\alpha}$$

- The angular acceleration α is directly proportional to the net torque $\Sigma\tau_i$
- => the angular acceleration is inversely proportional to the moment of inertia of the object: $\alpha = \frac{\Sigma\tau_i}{I}$
- The relationship is analogous to $\sum \vec{F}_i = m\vec{a}$

8.3.4. Newton's Second Law for a Rotating Object

Dynamic equation



☞ Consider a rigid body rotates about fixed axis Δ when acted by tangent force $\vec{F}_t$ Let's see $\vec{F}_{ti}$ acting on P_i, one of particles forming the rigid body. As a result, P_i moves with acceleration a_{ti} along a circle having center at *0* on Δ .

- The dynamic equation for P_i: $m_i\vec{a}_{ti} = \vec{F}_{ti}$
- Multiple both sides by $\vec{r}_i$: $m\vec{r}_i \times \vec{a}_{ti} = \vec{r}_i \times \vec{F}_{ti}$
- $RHS = \vec{r} \times \vec{F} = \vec{\tau}_i$
- $LHS = m_i\vec{r}_i \times \vec{a}_{ti} = m_i[\vec{r}_i \times (\vec{\alpha} \times \vec{r}_i)] = m_i[(\vec{r}_i\vec{r}_i)\vec{\alpha} - (\vec{r}_i.\vec{\alpha})\vec{r}_i] = m_i\vec{r}_i^2\vec{\alpha}$

  hence $m_i.r_i^2.\vec{\alpha} = \vec{\tau}_i$
- Finally, for all particles forming the rigid body, $\boxed{\left(\sum_i m_i.r_i^2\right)\vec{\alpha} = \sum_i \vec{\tau}_i}$

☞ Dynamic equation for a rigid body rotated about a fixed axis, $\boxed{I\vec{\alpha} = \vec{\tau}}$

- Where $I = \sum_i m_i.r_i^2$ is so-called *Moment of Inertia*.

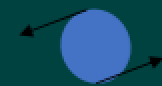
Strategy to use the Newton 2nd Law

- Draw or sketch system. Adopt coordinates, indicate rotation axes, list the known and unknown quantities, …
- Draw free body diagrams of key parts. Show forces at their points of application. Find torques about a (common) axis
- May need to apply Second Law twice, once to each part

Translation: $\vec{F}_{net} = \sum \vec{F}_i = m\vec{a}$

Rotation: $\vec{\tau}_{net} = \sum \vec{\tau}_i = I\vec{\alpha}$

Note: can have $F_{net} = 0$ but $\tau_{net} \neq 0$



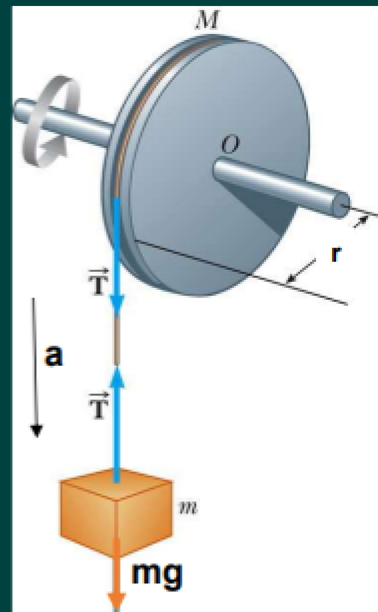
- Make sure there are enough (N) equations; there may be constraint equations (extra conditions connecting unknowns)
- Simplify and solve the set of (simultaneous) equations.
- Find unknown quantities and check answers

Example 7: The Falling Object

A solid, frictionless cylindrical reel of mass M = 2.5 kg and radius R = 0.2 m is used to draw water from a well. A bucket of mass m = 1.2 k is attached to a cord that is wrapped around the cylinder.

(a) Find the tension T in the cord and acceleration a of the object.

(b) If the object starts from rest at the top of the well and falls for 3.0 s before hitting the water, how far does it fall ?



Solution:

- Draw free body diagrams of each object
- Only the cylinder is rotating, so apply

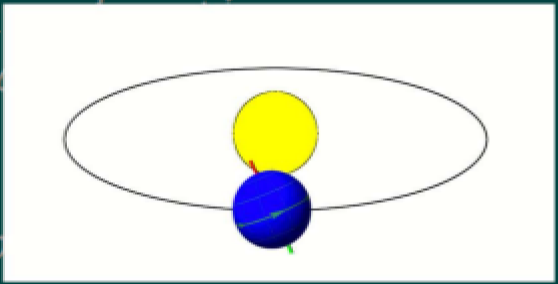
  $\Sigma\, \tau = I\alpha$

- The bucket is falling, but not rotating, so apply

  $\Sigma\, F = ma$

- Remember that $a = \alpha r$ and solve the resulting equations

Chapter 8. Rotational Motion of Rigid body



8.1. Angular position, velovity and a
- 8.1.1. Angular Position and Ra
- 8.1.2. Angular Velocity
- 8.1.3. Angular Acceleration

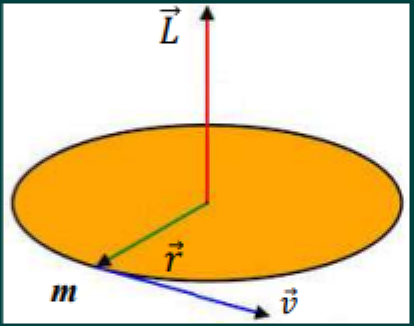
8.2. Rigid body and motion of rigid b
- 8.2.1. The Center of Mass
- 8.2.2. Translational Motion of a System of Particles/rigid body

8.3. Rotation of Rigid body
- 8.3.1. Torque (moment of a force)
- 8.3.2. Torque and two condition for equilibrium
- 8.3.3. Moment of inertia
- 8.3.4. Newton’s Second Law for a Rotating Object
- **8.3.5. Angular momentum**
- **8.3.6. Energy of a rotating object**

8.3.5. Angular Momentum

Definition:

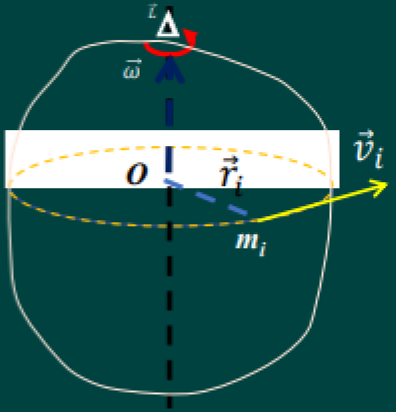
- For a particle mass m at position, vector $\vec{r}$ moves in a plane with constant velocity $\vec{v}$ relative to the origin O of an inertial frame,



$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

- Magnitude: $L = rmv\sin(\vec{r}, \vec{v}) = rmv$
- Direction: Right-hand rule.

For a system of particles or a rigid body



$$L = \sum_i L_i = \sum_i m_i r_i (r_i \omega) = \left[\sum_i (m_i r_i^2)\right] \omega = I\omega$$

☞ Theorems of angular momentum

♦ **1st Law:** *The rate of change of angular momentum equals the net torque of all external forces acting on,* $\dfrac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$

♦ **2nd Law :** *The change in angular momentum equals the impulse of of the net torque acting on in a duration Δt,* $\Delta\vec{L} = \vec{L}_2 - \vec{L}_1 = \vec{\tau}\Delta t$

Comparing Linear momentum $\vec{p}$ and angular momentum $\vec{L}$

	Linear momentum $\vec{p}$	Angular momentum $\vec{L}$
Definition	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{\omega}$
1st Law	$\frac{d\vec{p}}{dt} = \vec{F}_{net}$	$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$
2nd Law	$\Delta\vec{p} = \vec{p}_2 - \vec{p}_1 = \int_{t_1}^{t_2} F_{net} dt$ Change of momentum = Impulse of (net) force.	$\Delta\vec{L} = \vec{L}_2 - \vec{L}_1 = \int_{t_1}^{t_2} (\Sigma\vec{\tau}_i) dt$ Change of angular momentum = Impulse of (total) torque.
Conservation of momentum		

8.3.5. Angular Momentum

- Definition: $\vec{L} = I.\vec{\omega}$
- 1st Law of angular momentum:

  Starting from the Newton's Law of rotational motion

$$\Sigma\vec{\tau}_i = I\vec{\alpha} = I.\frac{d\vec{\omega}}{dt} = \frac{d(I.\vec{\omega})}{dt} = \frac{d\vec{L}}{dt}$$

- 2nd law of angular momentum:

$$d\vec{L} = (\Sigma\vec{\tau}_i)dt$$

I = const

$$\int_{\vec{L}_1}^{\vec{L}_2} d\vec{L} = \int_{t_1}^{t_2} (\Sigma\vec{\tau}_i)dt$$

$$\Delta\vec{L} = \vec{L}_2 - \vec{L}_1 = \int_{t_1}^{t_2} (\Sigma\vec{\tau}_i)dt$$

Change of angular momentum = Impulse of (total) torque.

- Conservation of angular momentum:

  if $\Sigma\vec{\tau}_i = 0$ then $\vec{L}$ is unchanged.

8.3.5. Angular Momentum

- **Conservation of angular momentum**

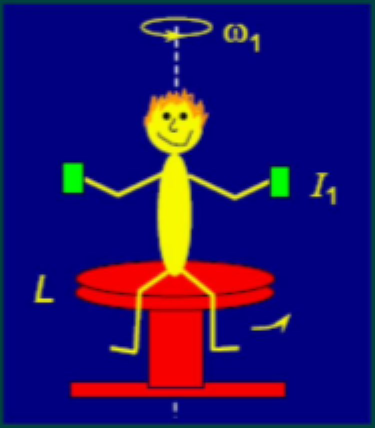
$$\text{If } \sum \vec{\tau_i} = 0 \text{ then } \frac{d\vec{L}}{dt} = 0 \text{ and } \vec{L} = const$$

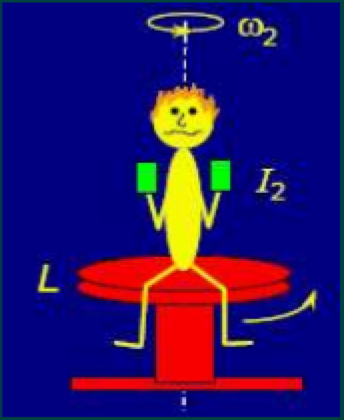
When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved).

Eg: A person is sitting on a turntable holding his arms extended horizontally with a same dumbbell in each hand.

♦ Initially his arms are out-stretched, ⇒ angular velocity ω_1 ⇒ angular momentum, $\vec{L}_1 = I_1\vec{\omega}_1$

♦ When the arms are folded ⇒ angular velocity ω_2 ⇒ angular momentum, $\vec{L}_2 = I_2\vec{\omega}_2$



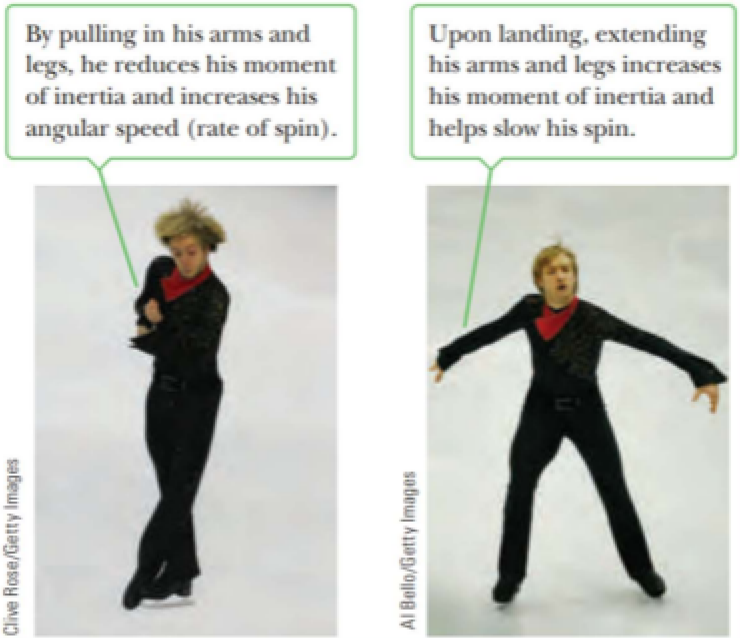


♦ If friction in the turntable is neglected, no external torques act about the vertical (z) axis., hence the angular momentum about this axis is constant.

$$I_1\vec{\omega}_1 = I_2\vec{\omega}_2 \Rightarrow \vec{\omega}_2 = \frac{I_1}{I_2}\vec{\omega}_1 \ \text{since } I_1 > I_2 \Rightarrow \vec{\omega}_2 > \vec{\omega}_1$$

Application of conservation of angular momentum

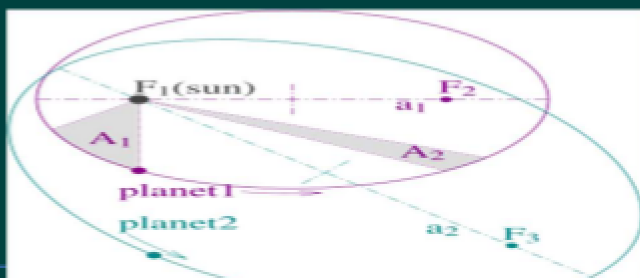
Figure 8.30 Evgeni Plushenko varies his moment of inertia to change his angular speed.



Tightly curling her body, a diver decreases her moment of inertia, increasing her angular speed.

A skater spinning in the finale of his act

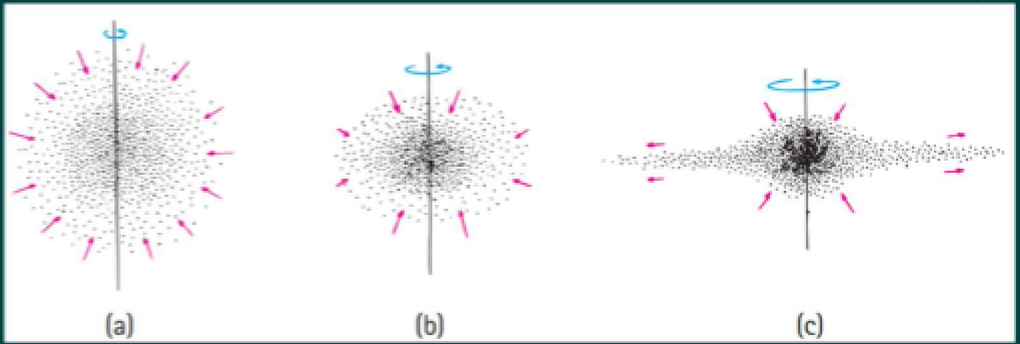
a diver wishes to make several somersaults



Third Kepler Law

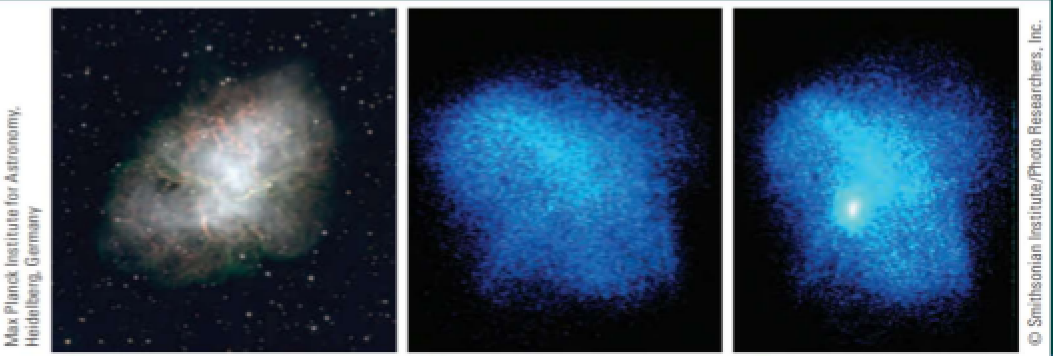
The Solar System and Its Formation

- **The Nebular theory:** Theory that the Sun and planets formed together from a cloud of gas and dust—a nebula.
- Gravitation between materials in the cloud pulled it inward. When pulled inward, spin increased in accord with the conservation of angular momentum.
- The spinning cloud conformed to the shape of a spinning disk.



Rotating Neutron Stars

Figure 8.31 (a) The Crab Nebula in the constellation Taurus. This nebula is the remnant of a supernova seen on Earth in A.D. 1054. It is located some 6 300 light-years away and is approximately 6 light-years in diameter, still expanding outward. A pulsar deep inside the nebula flashes 30 times every second. (b) Pulsar off. (c) Pulsar on.



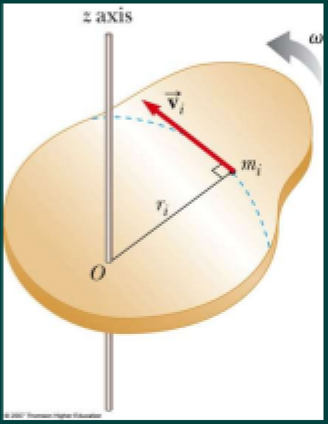
Chapter 8. Rotational Motion of Rigid body

8.3.6. Rotational Kinetic Energy

a) Definition:

- An object rotating around (fixed) z axis with an angular speed, ω, has **rotational kinetic energy**
- The total rotational kinetic energy of the rigid object is the sum of the energies of all its particles

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$



$$K_R = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2} I\omega^2$$

$$K_R = \frac{1}{2} I\omega^2$$

where I: the moment of inertia

- Unit of **rotational KE** is Joule (J)

b) Work-Energy Theorem for pure Translational motion (recalled)

- The work-energy theorem tells us

$$W_{net} = \Delta KE = KE_f - KE_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

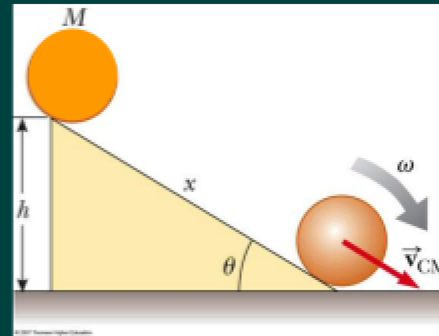
Kinetic energy is for point mass only, ignoring rotation.

- Work done by a force: $\boldsymbol{W_{net} = \int dW = \int \vec{F} \cdot d\vec{s}}$

- Power of a force $\boldsymbol{P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{s}}{dt} = \vec{F} \cdot \vec{v}}$

Total Energy of a System

A ball is rolling down a ramp



- Described by three types of energy

- Gravitational potential energy

$$PE = Mgh$$

- Translational kinetic energy

$$KE_t = \frac{1}{2}Mv_{CM}^2$$

- Rotational kinetic energy

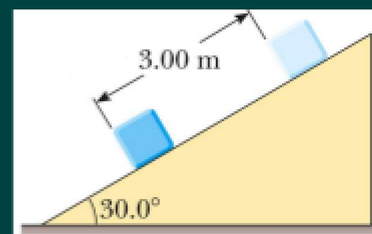
$$KE_r = \frac{1}{2}I\omega^2$$

$$\boxed{KE = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I\omega^2}$$

- Total energy of a system

$$\boxed{E = KE + PE = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I\omega^2 + Mgh}$$

Comparing with an object sliding down



- Total energy $E = KE + PE = \frac{1}{2}mv_{CM}^2 + mgh$

Work done by a torue during a pure rotation

- Apply force F to mass at point P, distance r, causing rotation-only about axis
- The work done by F applied to the object at P as it rotates through an infinitesimal distance ds

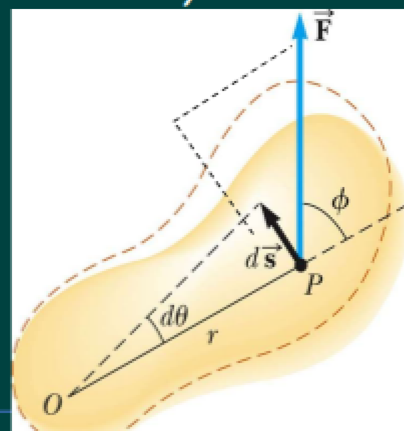
$$dW = \vec{F} \cdot d\vec{s} = F\cos(90° - \phi)ds$$
$$= F\sin\phi\, ds = Fr\sin\phi\, d\theta$$

- Only transverse component of F does work – the same component that contributes to torque:

$$dW = \tau d\theta$$

- As object rotates from θ_i to θ_f, work done by the torque

$$W = \int_{\theta_i}^{\theta_f} dW = \int_{\theta_i}^{\theta_f} \tau d\theta$$



Work-Kinetic Theorem pure rotation

- As object rotates from θ_i to θ_f, work done by the torque

$$W = \int_{\theta_i}^{\theta_f} dW = \int_{\theta_i}^{\theta_f} \tau d\theta = \int_{\theta_i}^{\theta_f} I\alpha d\theta = \int_{\theta_i}^{\theta_f} I\frac{d\omega}{dt}d\theta = \int_{\theta_i}^{\theta_f} I\omega d\omega$$

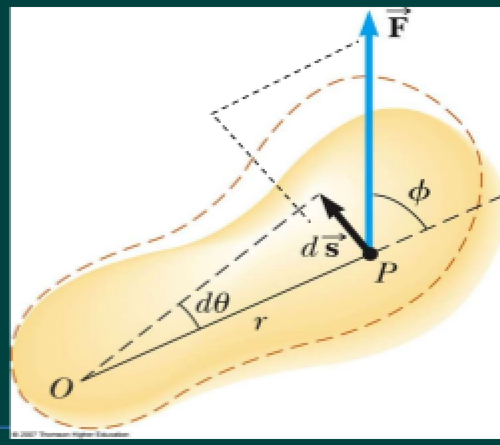
I is constant for rigid object

$$W = \int_{\theta_i}^{\theta_f} I\omega d\omega = I\int_{\theta_i}^{\theta_f} \omega d\omega = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = \Delta KE_r$$

ΔKE_R

- Power

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$$



Example 8. Work done and power of force

An motor attached to a grindstone exerts a constant torque of 10 N-m. The moment of inertia of the grindstone is I = 2 kg.m². The system starts from rest.

Find the kinetic energy after 8 s

$$K_f = \frac{1}{2} I \omega_f^2 = 1600J \Leftarrow \omega_f = \omega_i + \alpha t = 40 \text{ rad/s} \Leftarrow \alpha = \frac{\tau}{I} = 5 \text{ rad/s}^2$$

Find the work done by the motor during this time

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta = \tau(\theta_f - \theta_i) = 10 \times 160 = 1600J$$

$$(\theta_f - \theta_i) = \omega_i t + \frac{1}{2} \alpha t^2 = 160 \text{ rad}$$

$$W = K_f - K_i = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2 = 1600 \text{ J}$$

Find the average power delivered by the motor

$$P_{avg} = \frac{dW}{dt} = \frac{1600}{8} = 200 \text{ watts}$$

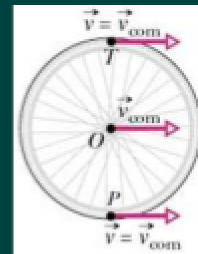
Find the instantaneous power at t = 8 s

$$P = \tau \omega = 10 \times 40 = 400 \text{ watts}$$

Review: Work-Energy Theorem

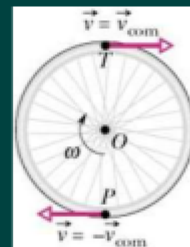
- For pure translation motion

$$W_{net} = \Delta K_{cm} = K_{cm,f} - K_{cm,i} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$



- For pure rotation

$$W_{net} = \Delta K_{rot} = K_{rot,f} - K_{rot,i} = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$



- Rolling: pure rotation + pure translation

$$W_{net} = \Delta K_{total} = (K_{rot,f} + K_{cm,f}) - (K_{rot,i} + K_{cm,i})$$
$$= \left(\frac{1}{2}I\omega_f^2 + \frac{1}{2}mv_f^2\right) - \left(\frac{1}{2}I\omega_i^2 + \frac{1}{2}mv_i^2\right)$$