One-Sample Tests of Hypothesis

Chapter 10

McGraw-Hill/Irwin

Learning Objectives

LO1 Define a hypothesis.

LO2 Explain the five-step hypothesis-testing procedure.

LO3 Describe Type I and Type II errors.

LO4 Define the term test statistic and explain how it is used.

LO5 Distinguish between a one-tailed and two-tailed hypothesis

LO6 Conduct a test of hypothesis about a population mean.

LO7 Compute and interpret a p-value.

LO8 Conduct a test of hypothesis about a population proportion.

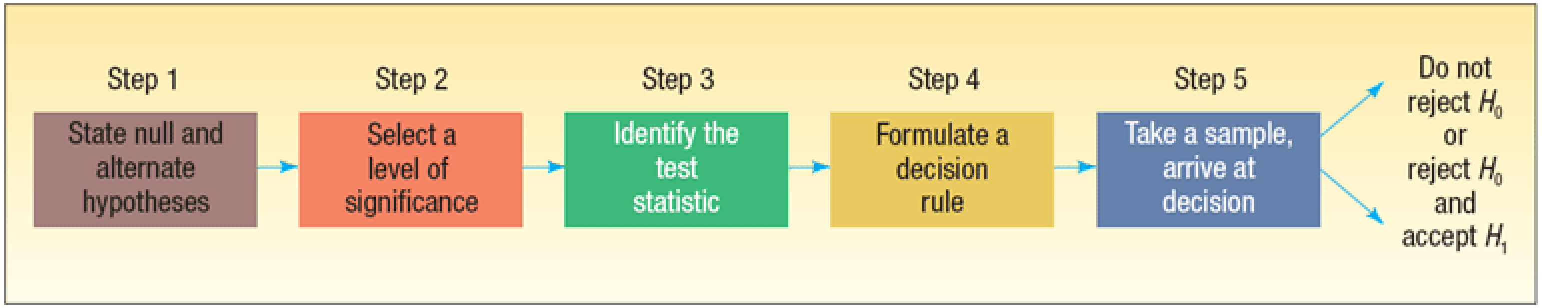
LO9 Compute the probability of a Type II error.

LO1 Define a hypothesis.
LO2 Explain the five-step hypothesis-testing procedure.

Hypothesis and Hypothesis Testing

HYPOTHESIS A statement about the value of a population parameter developed for the purpose of testing.

HYPOTHESIS TESTING A procedure based on sample evidence and probability theory to determine whether the hypothesis is a reasonable statement.



TEST STATISTIC A value, determined from sample information, used to determine whether to reject the null hypothesis.

CRITICAL VALUE The dividing point between the region where the null hypothesis is rejected and the region where it is not rejected.

Step 1: State the Null Hypothesis (H_0) and the Alternate Hypothesis (H_1)

NULL HYPOTHESIS A statement about the value of a population parameter developed for the purpose of testing numerical evidence.

- The alternate hypothesis describes what you will conclude if you reject the null hypothesis.

ALTERNATE HYPOTHESIS A statement that is accepted if the sample data provide sufficient evidence that the null hypothesis is false.

Step 2: Select a Level of Significance

LEVEL OF SIGNIFICANCE The probability of rejecting the null hypothesis when it is true.

TYPE I ERROR Rejecting the null hypothesis, H_0, when it is true.

TYPE II ERROR Accepting the null hypothesis when it is false.

- The following table summarizes the decisions the researcher could make and the possible consequences.

Null Hypothesis	Researcher: Does Not Reject H_0	Researcher: Rejects H_0
H_0 is true	Correct decision	Type I error
H_0 is false	Type II error	Correct decision

Step 3: Select the Test Statistic

TEST STATISTIC A value, determined from sample information, used to determine whether to reject the null hypothesis.

- In hypothesis testing for the mean (μ) when is known, the test statistic z is computed by:

TESTING A MEAN, σ KNOWN

$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \qquad \textbf{[10–1]}$$

Step 4: Formulate the Decision Rule

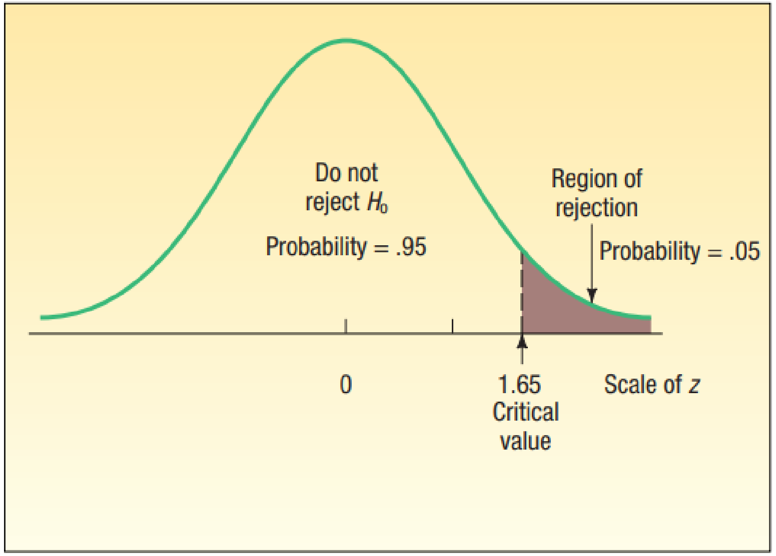


CHART 10–1 Sampling Distribution of the Statistic z, a Right-Tailed Test, .05 Level of Significance

CRITICAL VALUE The dividing point between the region where the null hypothesis is rejected and the region where it is not rejected.

Step 5: Make a Decision

- The fifth and final step in hypothesis testing is computing the test statistic, comparing it to the critical value, and making a decision to reject or not to reject the null hypothesis

SUMMARY OF THE STEPS IN HYPOTHESIS TESTING

1. Establish the null hypothesis (H_0) and the alternate hypothesis (H_1).
2. Select the level of significance, that is, α.
3. Select an appropriate test statistic.
4. Formulate a decision rule based on steps 1, 2, and 3 above.
5. Make a decision regarding the null hypothesis based on the sample information. Interpret the results of the test.

Important Things to Remember about H_0 and H_1

- H_0: null hypothesis and H_1: alternate hypothesis
- H_0 and H_1 are mutually exclusive and collectively exhaustive
- H_0 is always presumed to be true
- H_1 has the burden of proof
- A random sample (n) is used to "*reject* H_0"
- If we conclude 'do not reject H_0', this does not necessarily mean that the null hypothesis is true, it only suggests that there is not sufficient evidence to reject H_0; rejecting the null hypothesis then, suggests that the alternative hypothesis may be true.
- Equality is always part of H_0 (e.g. "=" , "≥" , "≤").
- "≠" "<" and ">" always part of H_1
- In actual practice, the status quo is set up as H_0
- If the claim is "boastful" the claim is set up as H_1 (we apply the Missouri rule – "show me"). Remember, H_1 has the burden of proof
- In problem solving, look for **key words** and convert them into symbols. Some key words include: "***improved, better than, as effective as, different from, has changed***, etc."

Keywords	Inequality Symbol	Part of:
Larger (or more) than	>	H_1
Smaller (or less)	<	H_1
No more than	≤	H_0
At least	≥	H_0
Has increased	>	H_1
Is there difference?	≠	H_1
Has not changed	=	H_0
Has "improved", "is better than". "is more effective"	See left text	H_1

Hypothesis Setups for Testing a Mean (μ) or a Proportion (π)

MEAN

H_0: μ = value H_1: $\mu \neq$ value Reject H_0 if: $\lvert Z \rvert > Z_{\alpha/2}$ $\lvert t \rvert > t_{\alpha/2,\ n-1}$	H_0: $\mu \geq$ value H_1: $\mu <$ value Reject H_0 if: $Z < -Z_{\alpha}$ $t < -t_{\alpha,\ n-1}$	H_0: $\mu \leq$ value H_1: $\mu >$ value Reject H_0 if: $Z > Z_{\alpha}$ $t > t_{\alpha,\ n-1}$

$$\boxed{t = \frac{\bar{x} - \mu}{s/\sqrt{n}}} \qquad \boxed{z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}}$$

PROPORTION

H_0: π = value H_1: $\pi \neq$ value Reject H_0 if: $\lvert Z \rvert > Z_{\alpha/2}$	H_0: $\pi \geq$ value H_1: $\pi <$ value Reject H_0 if: $Z < -Z_{\alpha}$	H_0: $\pi \leq$ value H_1: $\pi >$ value Reject H_0 if: $Z > Z_{\alpha}$

$$\boxed{z = \frac{p - \pi}{\sqrt{\dfrac{\pi(1-\pi)}{n}}}}$$

Testing for a Population Mean with a Known Population Standard Deviation- Example

EXAMPLE

Jamestown Steel Company manufactures and assembles desks and other office equipment . The weekly production of the Model A325 desk at the Fredonia Plant follows the normal probability distribution with a mean of **200** and a standard deviation of **16**. Recently, new production methods have been introduced and new employees hired. The vice president of manufacturing would like to investigate whether there has been a **change** in the weekly production of the Model A325 desk. Is the mean number of desks produced at the Fredonia Plant different from 200 at the 0.01 significance level. The mean number of desks produced last year is 203.5

Step 1: State the null hypothesis and the alternate hypothesis.

H_0: $\mu = 200$

H_1: $\mu \neq 200$

(note: keyword in the problem "has changed")

Step 2: Select the level of significance.

$\alpha = 0.01$ as stated in the problem

Step 3: Select the test statistic.

Use Z-distribution since σ is known

Step 4: Formulate the decision rule.

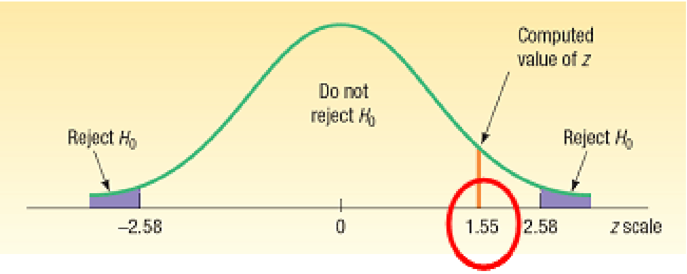
Reject H_0 if $|Z| > Z_{\alpha/2}$

$$|Z| > Z_{\alpha/2}$$

$$\left|\frac{\bar{X} - \mu}{\sigma / \sqrt{n}}\right| > Z_{\alpha/2}$$

$$\left|\frac{203.5 - 200}{16 / \sqrt{50}}\right| > Z_{.01/2}$$

$$1.55 \text{ is not} > 2.58$$



Step 5: Make a decision and interpret the result.

Because 1.55 does not fall in the rejection region, H_0 is not rejected. We conclude that the population mean is not different from 200. So we would report to the vice president of manufacturing that the sample evidence does not show that the production rate at the plant has changed from 200 per week.

Testing for a Population Mean with a Known Population Standard Deviation- Another Example

Suppose in the previous problem the vice president wants to know whether there has been an ***increase*** in the number of units assembled. To put it another way, can we conclude, because of the improved production methods, that the mean number of desks assembled in the last 50 weeks was **more than 200**?

Recall: $\sigma=16$, $n=200$, $\alpha=.01$

Step 1: State the null hypothesis and the alternate hypothesis.

H_0: $\mu \le 200$

H_1: $\mu > 200$

(note: keyword in the problem "an ***increase"***)

Step 2: Select the level of significance.

$\alpha = 0.01$ as stated in the problem

Step 3: Select the test statistic.

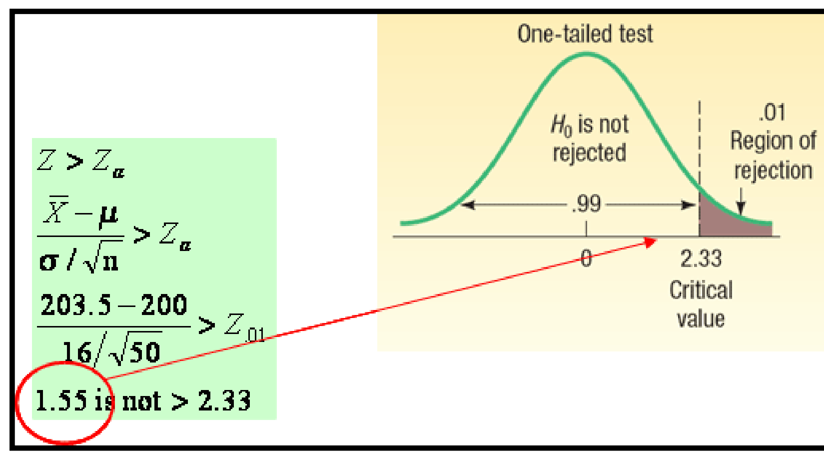
Use Z-distribution since σ is known

Step 4: Formulate the decision rule.

Reject H_0 if $Z > Z_\alpha$

Step 5: Make a decision and interpret the result.

Because 1.55 does not fall in the rejection region, H_0 is not rejected. We conclude that the average number of desks assembled in the last 50 weeks is not more than 200



LO6

Testing for the Population Mean: Population Standard Deviation Unknown

When the population standard deviation **(σ)** is unknown, the sample standard deviation **(s)** is used in its place the *t*-distribution is used as test statistic, which is computed using the formula:

TESTING A MEAN, σ UNKNOWN

$$t = \frac{\overline{X} - \mu}{s/\sqrt{n}}$$

with $n - 1$ degrees of freedom, where:

$\overline{X}$ is the sample mean.
μ is the hypothesized population mean.
s is the sample standard deviation.
n is the number of observations in the sample.

EXAMPLE

The McFarland Insurance Company Claims Department reports the **mean** cost to process a claim is at least **$60**. An industry comparison showed this amount to be larger than most other insurance companies, so the company instituted cost-cutting measures. To evaluate the effect of the cost-cutting measures, the Supervisor of the Claims Department selected a random sample of **26** claims processed last month. The sample information is reported below.

At the **.01** significance level is it reasonable a claim is ***now less than $60***?

$45	$49	$62	$40	$43	$61
48	53	67	63	78	64
48	54	51	56	63	69
58	51	58	59	56	57
38	76				

LO6

Testing for the Population Mean: Population Standard Deviation Unknown - Example

Step 1: State the null hypothesis and the alternate hypothesis.

H0: $\mu \geq \$60$

H1: $\mu < \$60$

Step 2: Select the level of significance.

$\alpha = 0.01$ as stated in the problem

Step 3: Select the test statistic.

Use t-distribution since σ is unknown

Step 4: Formulate the decision rule.

Reject H_0 if $t < -t_{\alpha,n-1}$

Step 5: Make a decision and interpret the result.

Because -1.818 does not fall in the rejection region, H_0 is not rejected at the .01 significance level. We have not demonstrated that the cost-cutting measures reduced the mean cost per claim to less than \$60. The difference of \$3.58 (\$56.42 - \$60) between the sample mean and the population mean could be due to sampling error.

TABLE 10–1 A Portion of the *t* Distribution Table

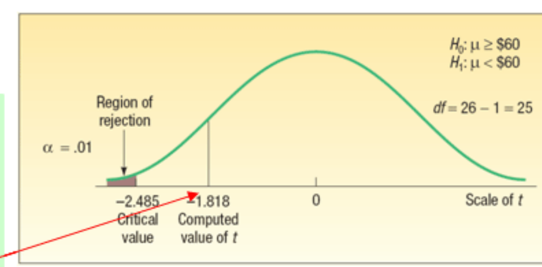
	Confidence Intervals					
	80%	90%	95%	98%	99%	99.9%
	Level of Significance for One-Tailed Test, α					
df	0.100	0.050	0.025	0.010	0.005	0.0005
	Level of Significance for Two-Tailed Test, α					
	0.20	0.10	0.05	0.02	0.01	0.001
⋮	⋮	⋮	⋮	⋮	⋮	⋮
21	1.323	1.721	2.080	2.518	2.831	3.819
22	1.321	1.717	2.074	2.508	2.819	3.792
23	1.319	1.714	2.069	2.500	2.807	3.768
24	1.318	1.711	2.064	2.492	2.797	3.745
25	1.316	1.708	2.060	2.485	2.787	3.725
26	1.315	1.706	2.056	2.479	2.779	3.707
27	1.314	1.703	2.052	2.473	2.771	3.690
28	1.313	1.701	2.048	2.467	2.763	3.674
29	1.311	1.699	2.045	2.462	2.756	3.659
30	1.310	1.697	2.042	2.457	2.750	3.646

$$t < -t_{\alpha,n-1}$$

$$\frac{\bar{X}-\mu}{s/\sqrt{n}} < -t_{\alpha,n-1}$$

$$\frac{\$56.42-\$60}{\$10.04/\sqrt{26}} < -t_{.01,26-1}$$

$$-1.818 \text{ is not } < -2.485$$



Type of Errors and *p*-value in Hypothesis Testing

- **Type I Error -**
  - Defined as the probability of rejecting the null hypothesis when it is actually true.
  - This is denoted by the Greek letter "α"
  - Also known as the *significance level* of a test
- **Type II Error:**
  - Defined as the probability of "accepting" the null hypothesis when it is actually false.
  - This is denoted by the Greek letter "β"
- ***p*-VALUE** is the probability of observing a sample value as extreme as, or more extreme than, the value observed, given that the null hypothesis is true.
- In testing a hypothesis, we can also compare the *p*-value to the significance level (α).
- Decision rule using the p-value:
  Reject H_0 if *p*-value < significance level

EXAMPLE *p*-Value

Recall the last problem where the hypothesis and decision rules were set up as:

H_0: $\mu \le 200$

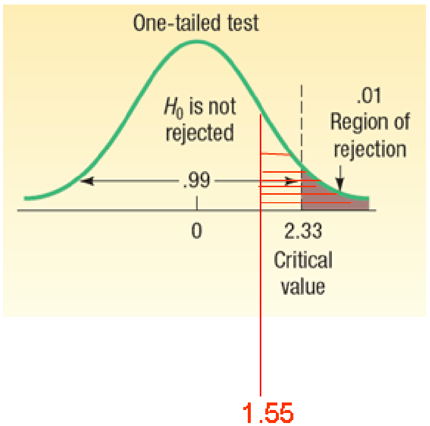
H_1: $\mu > 200$

Reject H_0 if $Z > Z_\alpha$

where $Z = 1.55$ and $Z_\alpha = 2.33$

Reject H_0 if p-value < α

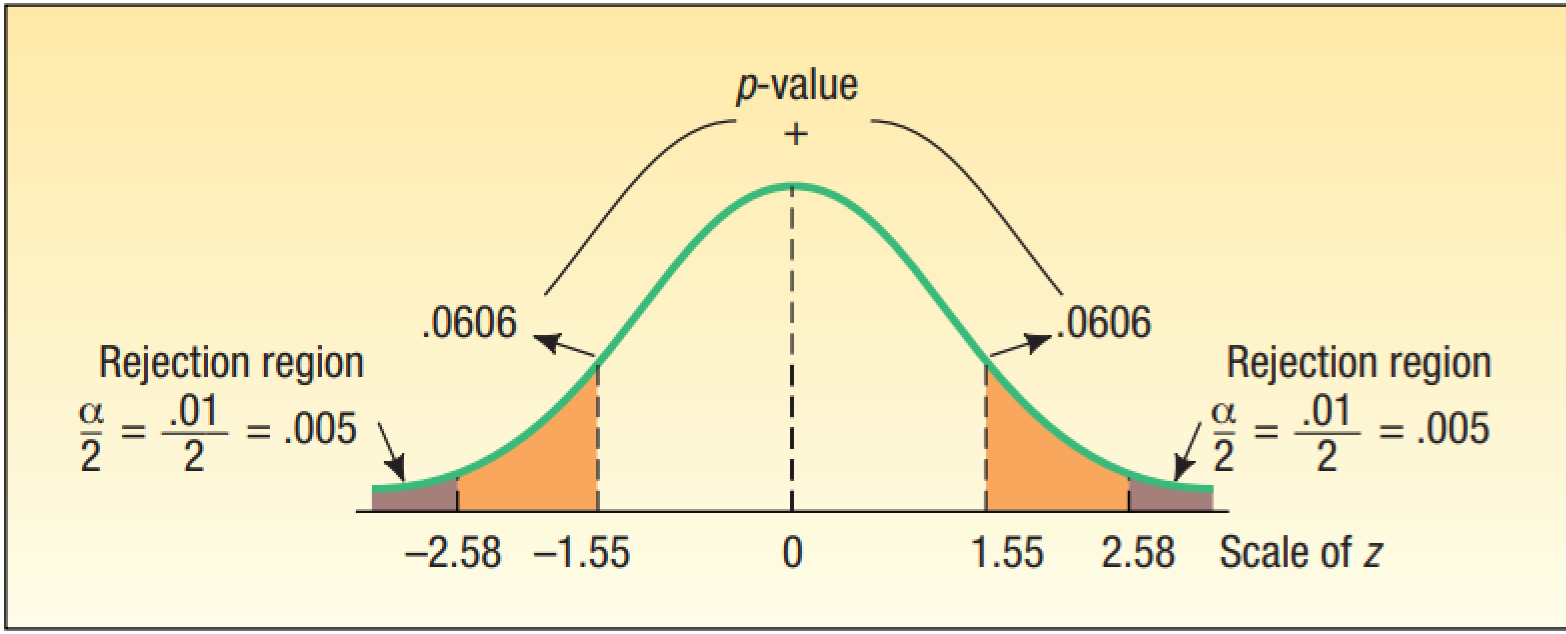
0.0606 is not < 0.01



Conclude: Fail to reject H_0

P(Z>1.55)=.5000 -.4394
P-value = .0606

p-value



INTERPRETING THE WEIGHT OF EVIDENCE AGAINST H_0 If the p-value is less than

(a) .10, we have *some* evidence that H_0 is not true.
(b) .05, we have *strong* evidence that H_0 is not true.
(c) .01, we have *very strong* evidence that H_0 is not true.
(d) .001, we have *extremely strong* evidence that H_0 is not true.

LO8 Conduct a test of hypothesis about a population proportion.

Tests Concerning Proportion using the z-Distribution

- A **Proportion** is the fraction or percentage that indicates the part of the population or sample having a particular trait of interest.
- The sample proportion is denoted by *p* and is found by ***x/n***
- It is assumed that the binomial assumptions discussed in Chapter 6 are met:

  (1) the sample data collected are the result of counts;

  (2) the outcome of an experiment is classified into one of two mutually exclusive categories—a "success" or a "failure";

  (3) the probability of a success is the same for each trial; and (4) the trials are independent
- Both $n\pi$ and $n(1-\pi)$ are at least 5.
- When the above conditions are met, the normal distribution can be used as an approximation to the binomial distribution
- The test statistic is computed as follows:

TEST OF HYPOTHESIS, ONE PROPORTION

$$z = \frac{p - \pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} \quad [10\text{–}3]$$

where:

π is the population proportion.

p is the sample proportion.

n is the sample size.

LO8

Test Statistic for Testing a Single Population Proportion - Example

EXAMPLE

Suppose prior elections in a certain state indicated it is necessary for a candidate for governor to receive **at least** 80 percent of the vote in the northern section of the state to be elected. The incumbent governor is interested in assessing his chances of returning to office and plans to conduct a survey of 2,000 registered voters in the northern section of the state. Using the hypothesis-testing procedure, assess the governor's chances of reelection.

Step 1: State the null hypothesis and the alternate hypothesis.

H_0: $\pi \geq .80$

H_1: $\pi < .80$

(note: keyword in the problem "*at least*")

Step 2: Select the level of significance.

α = 0.05 as stated in the problem

Step 3: Select the test statistic.

Use Z-distribution since the assumptions are met and $n\pi$ and $n(1-\pi) \geq 5$

Step 4: Formulate the decision rule.

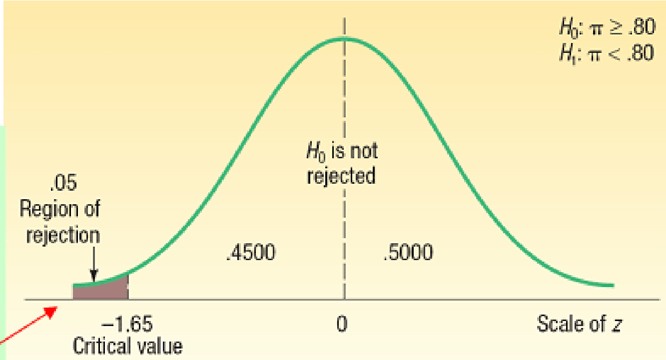
Reject H_0 if $Z < -Z_\alpha$

$$Z < -Z_\alpha$$

$$\frac{p-\pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} < -z_\alpha$$

$$\frac{\frac{1{,}550}{2{,}000} - .80}{\sqrt{\frac{.80(1-.80)}{2{,}000}}} < -1.65$$

$$-2.80 < -1.65$$



Step 5: Make a decision and interpret the result.

The computed value of z (-2.80) is in the rejection region, so the null hypothesis is rejected at the .05 level. The evidence at this point does not support the claim that the incumbent governor will return to the governor's mansion for another four years.

LO9 Compute the probability of a Type II error.

Type II Error

- Recall **Type I Error**, the level of significance, denoted by the Greek letter "α", is defined as the probability of rejecting the null hypothesis when it is actually true.

- **Type II Error**, denoted by the Greek letter "β",is defined as the probability of "accepting" the null hypothesis when it is actually false.

EXAMPLE

Western Wire Products purchases steel bars to make cotter pins. Past experience indicates that the mean tensile strength of all incoming shipments is 10,000 psi and that the standard deviation, σ, is 400 psi.

To make a decision about incoming shipments of steel bars, Western Wire Products set up this rule for the quality-control inspector to follow: "Take a sample of 100 steel bars. At the .05 significance level, if the sample mean $(\bar{X})$ strength falls between 9,922 psi and 10,078 psi, accept the lot. Otherwise, the lot is to be rejected." Refer to Chart 10–9, Region A. It shows the region where each lot is rejected and where it is not rejected. The mean of this distribution is designated μ_0. The tails of the curve represent the probability of making a Type I error, that is, rejecting the incoming lot of steel bars when in fact it is a good lot, with a mean of 10,000 psi.

Suppose the unknown population mean of an incoming lot, designated 1, is really 9,900 psi. What is the probability that the quality-control inspector will fail to reject the shipment (a Type II error)?

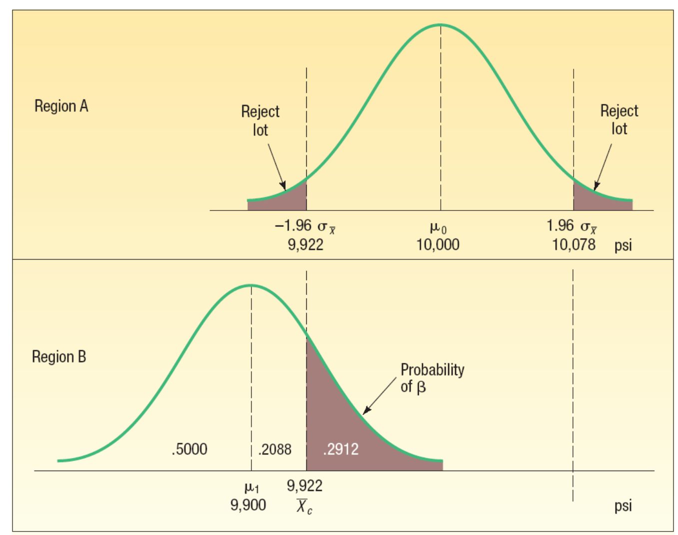
The number of standard units (z value) between the mean of the incoming lot (9,900), designated by μ_1, and $\bar{X}_c$, representing the critical value for 9,922, is computed by:

TYPE II ERROR $$z = \frac{\bar{X}_c - \mu_1}{\sigma/\sqrt{n}} \quad \text{[10–4]}$$

With $n = 100$ and $\sigma = 400$, the value of z is 0.55:

$$z = \frac{\bar{X}_c - \mu_1}{\sigma/\sqrt{n}} = \frac{9{,}922 - 9{,}900}{400/\sqrt{100}} = \frac{22}{40} = 0.55$$

The area under the curve between 9,900 and 9,922 (a z value of 0.55) is .2088. The area under the curve beyond 9,922 pounds is .5000 − .2088, or .2912; this is the probability of making a Type II error—that is, accepting an incoming lot of steel bars when the population mean is 9,900 psi.



Type I and Type II Errors Illustrated

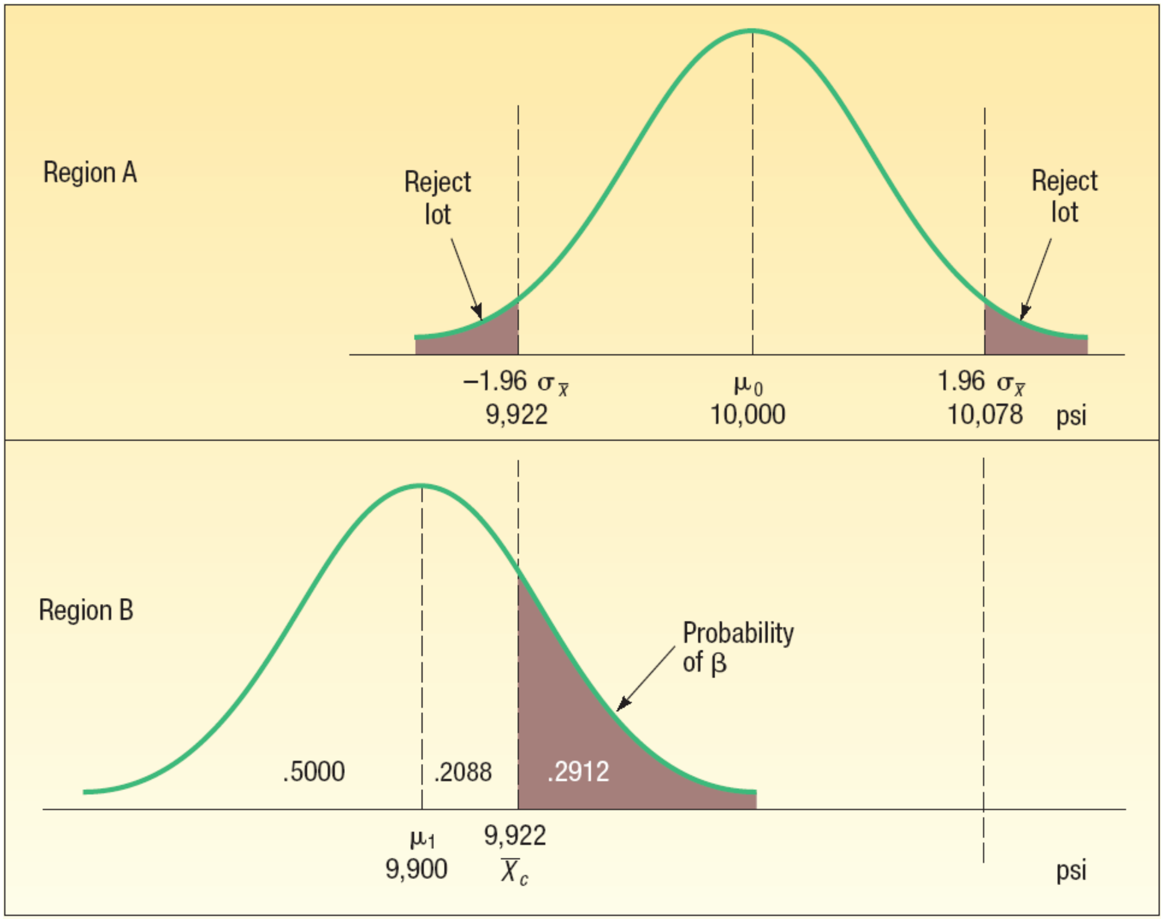
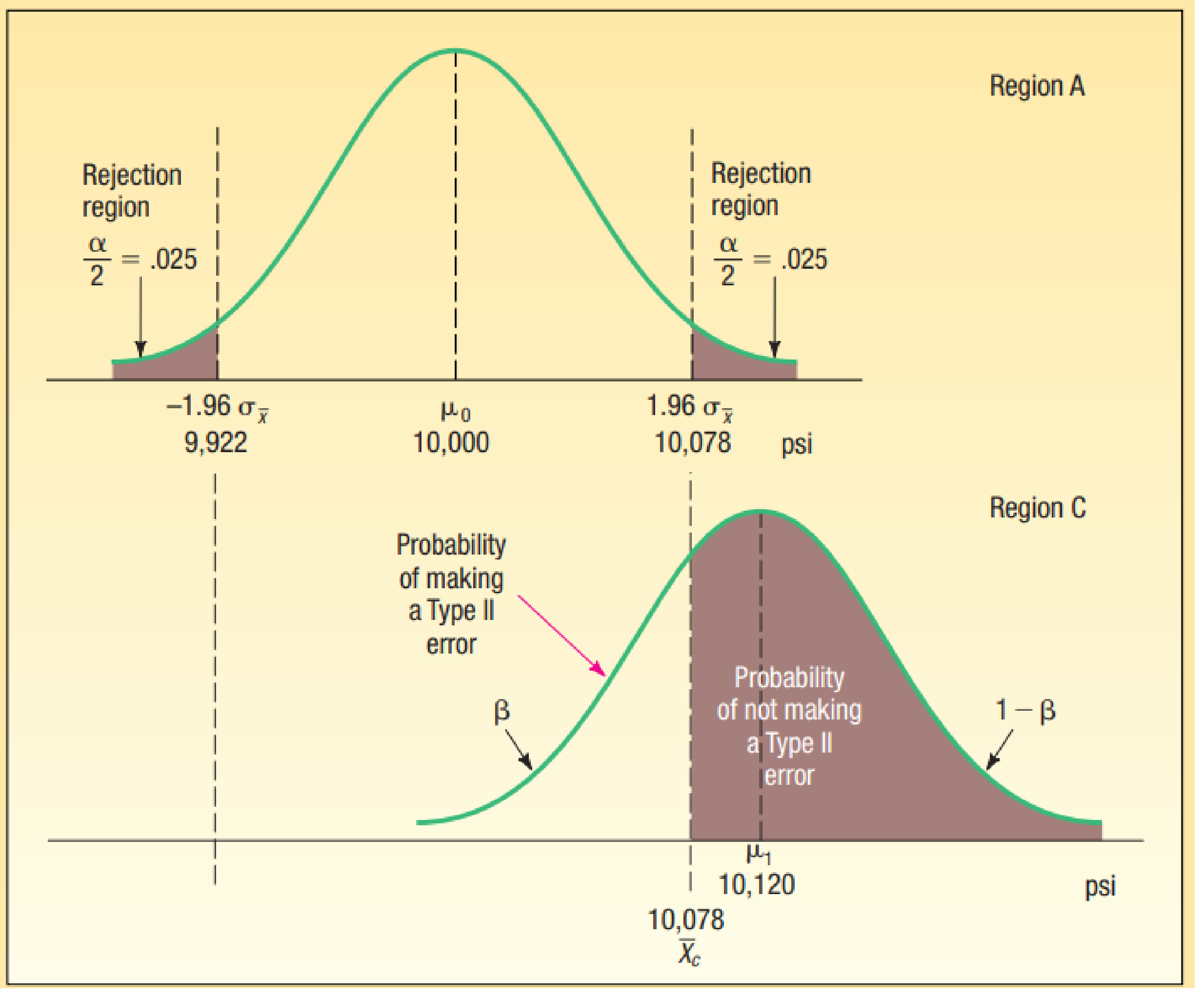


CHART 10–9 Charts Showing Type I and Type II Errors


Region A
Rejection region
$\frac{\alpha}{2} = .025$
Rejection region
$\frac{\alpha}{2} = .025$
$-1.96\ \sigma_{\bar{x}}$
9,922
μ_0
10,000
$1.96\ \sigma_{\bar{x}}$
10,078
psi
Region C
Probability of making a Type II error
β
Probability of not making a Type II error
$1 - \beta$
μ_1
10,120
psi
10,078
$\bar{X}_c$

LO9

Type II Errors For Varying Mean Levels

TABLE 10–4 Probabilities of a Type II Error for $\mu_0 = 10{,}000$ Pounds and Selected Alternative Means, .05 Level of Significance

Selected Alternative Mean (pounds)	Probability of Type II Error (β)	Probability of Not Making a Type II Error ($1 - \beta$)
9,820	.0054	.9946
9,880	.1469	.8531
9,900	.2912	.7088
9,940	.6736	.3264
9,980	.9265	.0735
10,000	—*	—
10,020	.9265	.0735
10,060	.6736	.3264
10,100	.2912	.7088
10,120	.1469	.8531
10,180	.0054	.9946

*It is not possible to make a Type II error when $\mu = \mu_0$.