Discrete Probability Distributions

Chapter 6

McGraw-Hill/Irwin

Learning Objectives

LO1 Identify the characteristics of a *probability distribution.*

LO2 Distinguish between *discrete* and *continuous random variable*.

LO3 Compute the *mean* of a probability distribution.

LO4 Compute the *variance and standard deviation* of a probability distribution.

LO5 Describe and compute probabilities for a *binomial distribution*.

What is a Probability Distribution?

PROBABILITY DISTRIBUTION A listing of all the outcomes of an experiment and the probability associated with each outcome.

CHARACTERISTICS OF A PROBABILITY DISTRIBUTION

1. The probability of a particular outcome is between 0 and 1 inclusive.
2. The outcomes are mutually exclusive events.
3. The list is exhaustive. So the sum of the probabilities of the various events is equal to 1.

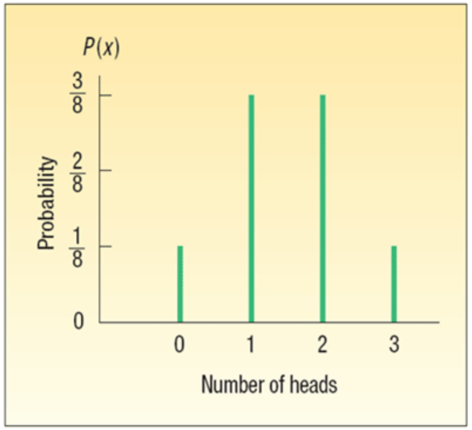
Experiment:

Toss a coin three times. Observe the number of heads. The possible results are: Zero heads, One head, Two heads, and Three heads.

What is the probability distribution for the number of heads?

Possible Result	Coin Toss First	Coin Toss Second	Coin Toss Third	Number of Heads
1	T	T	T	0
2	T	T	H	1
3	T	H	T	1
4	T	H	H	2
5	H	T	T	1
6	H	T	H	2
7	H	H	T	2
8	H	H	H	3

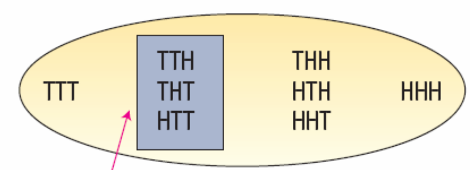
Number of Heads, x	Probability of Outcome, $P(x)$
0	$\frac{1}{8} = .125$
1	$\frac{3}{8} = .375$
2	$\frac{3}{8} = .375$
3	$\frac{1}{8} = .125$
Total	$\frac{8}{8} = 1.000$



Random Variables

RANDOM VARIABLE A quantity resulting from an experiment that, by chance, can assume different values.

Possible *outcomes* for three coin tosses



The *event* {one head} occurs and the *random variable* $x = 1$.

DISCRETE RANDOM VARIABLE A random variable that can assume only certain clearly separated values. It is usually the result of counting something.

EXAMPLES

1. The number of students in a class.
2. The number of children in a family.
3. The number of cars entering a carwash in a hour.
4. Number of home mortgages approved by Coastal Federal Bank last week.

CONTINUOUS RANDOM VARIABLE can assume an infinite number of values within a given range. It is usually the result of some type of measurement

EXAMPLES

1. The length of each song on the latest Tim McGraw album.
2. The weight of each student in this class.
3. The temperature outside as you are reading this book.
4. The amount of money earned by each of the more than 750 players currently on Major League Baseball team rosters.

Random Variable vs Probability distribution

A set of possible values from a random variable into a probability distribution, the result is a probability distribution

- Random Variable
  - Reporting the particular outcome of an experiment
- Probability distribution
  - Reporting all the possible outcomes as well as the corresponding probability

LO3 and LO4 Compute the *mean, standard deviation and variance* of a probability distribution.

The Mean and Variance of a Discrete Probability Distribution

MEAN

- The mean is a typical value used to represent the central location of a probability distribution.
- The mean of a probability distribution is also referred to as its **expected value.**

MEAN OF A PROBABILITY DISTRIBUTION $$\mu = \Sigma[xP(x)]$$ **[6–1]**

VARIANCE AND STANDARD DEVIATION

- Measures the amount of spread in a distribution
- The computational steps are:
  1. Subtract the mean from each value, and square this difference.
  2. Multiply each squared difference by its probability.
  3. Sum the resulting products to arrive at the variance.

  The standard deviation is found by taking the positive square root of the variance.

VARIANCE OF A PROBABILITY DISTRIBUTION $$\sigma^2 = \Sigma[(x - \mu)^2 P(x)]$$ **[6–2]**

Mean, Variance, and Standard Deviation of a Probability Distribution - Example

John Ragsdale sells new cars for Pelican Ford. John usually sells the largest number of cars on Saturday. He has developed the following probability distribution for the number of cars he expects to sell on a particular Saturday.

Number of Cars Sold, x	Probability, $P(x)$
0	.10
1	.20
2	.30
3	.30
4	.10
Total	1.00

MEAN

$$\mu = \Sigma[xP(x)]$$
$$= 0(.10) + 1(.20) + 2(.30) + 3(.30) + 4(.10)$$
$$= 2.1$$

Number of Cars Sold, x	Probability, $P(x)$	$x \cdot P(x)$
0	.10	0.00
1	.20	0.20
2	.30	0.60
3	.30	0.90
4	.10	0.40
Total	1.00	μ = 2.10

VARIANCE

$$\sigma^2 = \Sigma[(x - \mu)^2 P(x)]$$

Number of Cars Sold, x	Probability, $P(x)$	$(x - \mu)$	$(x - \mu)^2$	$(x - \mu)^2 P(x)$
0	.10	0 − 2.1	4.41	0.441
1	.20	1 − 2.1	1.21	0.242
2	.30	2 − 2.1	0.01	0.003
3	.30	3 − 2.1	0.81	0.243
4	.10	4 − 2.1	3.61	0.361
				σ^2 = 1.290

STANDARD DEVIATION

$$\sigma = \sqrt{\sigma^2} = \sqrt{1.290} = 1.136$$

LO5 Describe and compute probabilities for a *binomial distribution*.

Binomial Probability Distribution

- **A Widely occurring discrete probability distribution**
- **Characteristics of a Binomial Probability Distribution**

1. There are only two **possible outcomes** on a particular trial of an experiment, a success or a failure (mutually exclusive categories)
2. The random variable is the **result of counts.**
3. The probability of success and failure **stay the same for each trial.**
4. Each trial is ***independent*** of any other trial

How is a Binomial Probability Computed?

- To construct a particular binomial probability:
  - The number of trials
  - The probability of success on each trial

BINOMIAL PROBABILITY FORMULA $P(x) = {}_nC_x\, \pi^x(1-\pi)^{n-x}$ **[6–3]**

where:

C denotes a combination.
n is the number of trials.
x is the random variable defined as the number of successes.
π is the probability of a success on each trial.

MEAN OF A BINOMIAL DISTRIBUTION $\mu = n\pi$ **[6–4]**

VARIANCE OF A BINOMIAL DISTRIBUTION $\sigma^2 = n\pi(1-\pi)$ **[6–5]**

Binomial Probability Distribution

EXAMPLE

There are **five** flights daily from Pittsburgh via US Airways into the Bradford, Pennsylvania, Regional Airport. Suppose the probability that any flight arrives late is **.20**.

What is the probability that **none** of the flights are late today?

$$
\begin{aligned}
P(x=0) &= {}_nC_x \pi^x (1-\pi)^{n-x} \\
&= {}_5C_0 (.20)^0 (1-.20)^{5-0} \\
&= (1)(1)(.3277) \\
&= 0.3277
\end{aligned}
$$

What is the average number of late flights? What is the variance of the number of late flights?

$$
\begin{aligned}
\mu &= n\pi \\
&= (5)(0.20) = 1.0
\end{aligned}
$$

$$
\begin{aligned}
\sigma^2 &= n\pi(1-\pi) \\
&= (5)(0.20)(1-0.20) \\
&= (5)(0.20)(0.80) \\
&= 0.80
\end{aligned}
$$

Binomial Probability Tables

- Table 6–2 shows part of Appendix B.9 for $n=6$ and various values of π

TABLE 6–2 Binomial Probabilities for $n = 6$ and Selected Values of π

	$n = 6$ Probability										
$x\backslash\pi$	**.05**	.1	.2	.3	.4	.5	.6	.7	.8	.9	.95
0	**.735**	.531	.262	.118	.047	.016	.004	.001	.000	.000	.000
1	**.232**	.354	.393	.303	.187	.094	.037	.010	.002	.000	.000
2	**.031**	.098	.246	.324	.311	.234	.138	.060	.015	.001	.000
3	**.002**	.015	.082	.185	.276	.313	.276	.185	.082	.015	.002
4	**.000**	.001	.015	.060	.138	.234	.311	.324	.246	.098	.031
5	**.000**	.000	.002	.010	.037	.094	.187	.303	.393	.354	.232
6	**.000**	.000	.000	.001	.004	.016	.047	.118	.262	.531	.735

Binomial Probability Example

Example

Five percent of the worm gears produced by an automatic, high-speed Carter-Bell milling machine are defective. What is the probability that out of six gears selected at random none will be defective? Exactly one? Exactly two? Exactly three? Exactly four? Exactly five? Exactly six out of six?

- There are only two possible outcomes (a particular gear is either defective or acceptable)
- There is a fixed number of trials (6)
- There is a constant probability of success (.05)
- The trials are independent.
- $n = 6$ and $\pi = .05$

$$\mu = n\pi = (6)(.05) = 0.30$$

$$\sigma^2 = n\pi(1 - \pi) = 6(.05)(.95) = 0.285$$

Number of Defective Gears, x	Probability of Occurrence, $P(x)$	Number of Defective Gears, x	Probability of Occurrence, $P(x)$
0	.735	4	.000
1	.232	5	.000
2	.031	6	.000
3	.002		

Binomial Distribution table

- If n remains the same but π increases from .05 to .95, the shape of the distribution changes

TABLE 6–3 Probability of 0, 1, 2, . . . Successes for a π of .05, .10, .20, .50, and .70 and an n of 10

$x \backslash \pi$	.05	.1	.2	.3	.4	.5	.6	.7	.8	.9	.95
0	.599	.349	.107	.028	.006	.001	.000	.000	.000	.000	.000
1	.315	.387	.268	.121	.040	.010	.002	.000	.000	.000	.000
2	.075	.194	.302	.233	.121	.044	.011	.001	.000	.000	.000
3	.010	.057	.201	.267	.215	.117	.042	.009	.001	.000	.000
4	.001	.011	.088	.200	.251	.205	.111	.037	.006	.000	.000
5	.000	.001	.026	.103	.201	.246	.201	.103	.026	.001	.000
6	.000	.000	.006	.037	.111	.205	.251	.200	.088	.011	.001
7	.000	.000	.001	.009	.042	.117	.215	.267	.201	.057	.010
8	.000	.000	.000	.001	.011	.044	.121	.233	.302	.194	.075
9	.000	.000	.000	.000	.002	.010	.040	.121	.268	.387	.315
10	.000	.000	.000	.000	.000	.001	.006	.028	.107	.349	.599

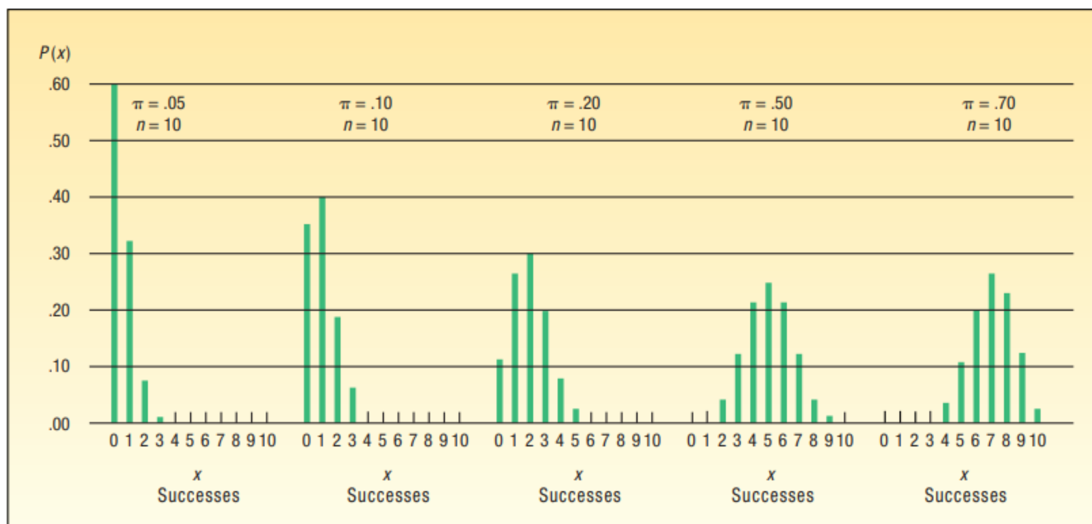


CHART 6–2 Graphing the Binomial Probability Distribution for a π of .05, .10, .20, .50, and .70 and an n of 10

Binomial Distribution table

- If π, the probability of success, remains the same but n becomes larger, the shape of the binomial distribution becomes more symmetrical.

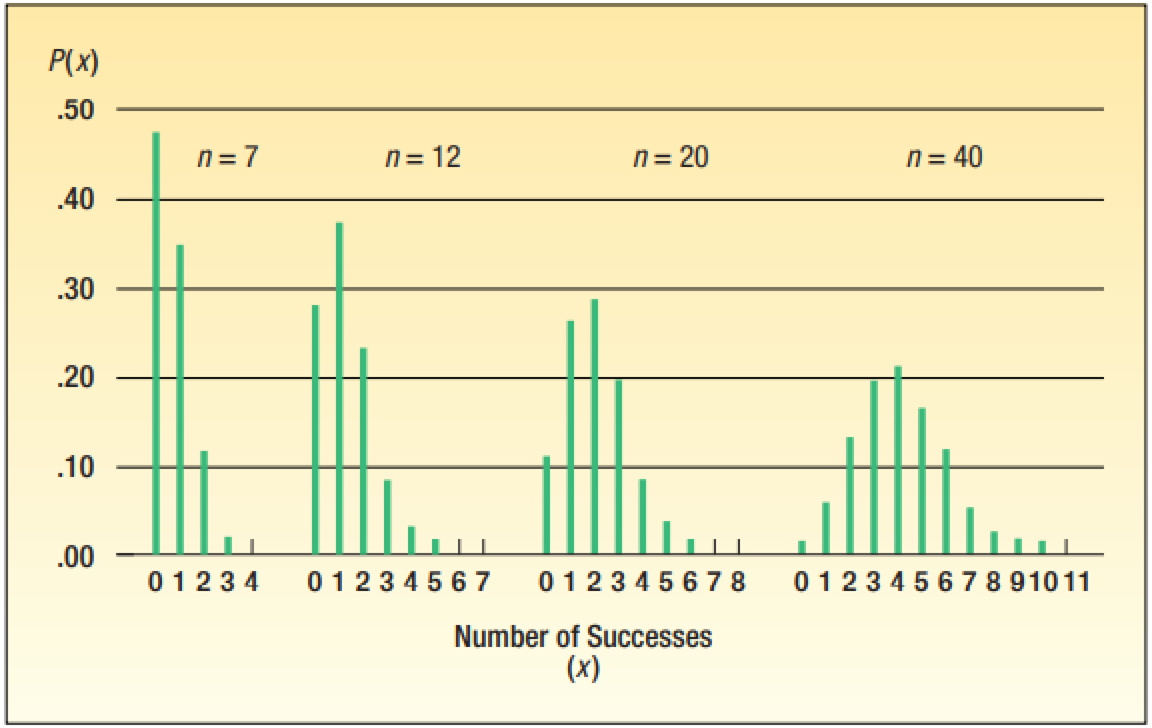


CHART 6–3 Chart Representing the Binomial Probability Distribution for a π of .10 and an n of 7, 12, 20, and 40

Cumulative Binomial Probability Distributions - Example

LO5

EXAMPLE

A study by the Illinois Department of Transportation concluded that 76.2 percent of front seat occupants used seat belts. A sample of 12 vehicles is selected.

What is the probability the front seat occupants in exactly 7 of the 12 vehicles are wearing seat belts?

$$
\begin{aligned}
P(x = 7 \mid n = 12 \text{ and } \pi = .762) &= {}_{12}C_7(.762)^7(1 - .762)^{12-7} \\
&= 792(.149171)(.000764) = .0902
\end{aligned}
$$

What is the probability the front seat occupants in at least 7 of the 12 vehicles are wearing seat belts?

$$
\begin{aligned}
P(x \geq 7 \mid n = 12 \text{ and } \pi = .762 \\
&= P(x = 7) + P(x = 8) + P(x = 9) + P(x = 10) + P(x = 11) + P(x = 12) \\
&= .0902 + .1805 + .2569 + .2467 + .1436 + .0383 \\
&= .9562
\end{aligned}
$$

Hypergeometric Probability Distribution

HYPERGEOMETRIC DISTRIBUTION $$P(x) = \frac{({}_SC_x)({}_{N-S}C_{n-x})}{{}_NC_n} \quad \textbf{[6–6]}$$

where:

- N is the size of the population.
- S is the number of successes in the population.
- x is the number of successes in the sample. It may be 0, 1, 2, 3, ...
- n is the size of the sample or the number of trials.
- C is the symbol for a combination.

HYPERGEOMETRIC PROBABILITY EXPERIMENT

1. An outcome on each trial of an experiment is classified into one of two mutually exclusive categories—a success or a failure.
2. The random variable is the number of successes in a fixed number of trials.
3. The trials are *not independent.*
4. We assume that we sample from a finite population without replacement and $n/N > 0.05$. So, the probability of a success *changes* for each trial.

Example

PlayTime Toys Inc. employs 50 people in the Assembly Department. Forty of the employees belong to a union and ten do not. Five employees are selected at random to form a committee to meet with management regarding shift starting times. What is the probability that four of the five selected for the committee belong to a union?

- N is 50, the number of employees.
- S is 40, the number of union employees.
- x is 4, the number of union employees selected.
- n is 5, the number of employees selected

$$P(4) = \frac{({}_{40}C_4)({}_{50-40}C_{5-4})}{{}_{50}C_5} = \frac{(91{,}390)(10)}{2{,}118{,}760} = .431$$

TABLE 6–4 Hypergeometric Probabilities ($n = 5$, $N = 50$, and $S = 40$) for the Number of Union Members on the Committee

Union Members	Probability
0	.000
1	.004
2	.044
3	.210
4	.431
5	.311
	1.000

Hypergeometric vs Binomial Probabilities

TABLE 6–5 Hypergeometric and Binomial Probabilities for PlayTime Toys Inc. Assembly Department

Number of Union Members on Committee	Hypergeometric Probability, $P(x)$	Binomial Probability ($n = 5$ and $\pi = .80$)
0	.000	.000
1	.004	.006
2	.044	.051
3	.210	.205
4	.431	.410
5	.311	.328
	1.000	1.000

- If selected items are not returned to the population, the binomial distribution can be used to closely approximate the hypergeometric distribution when $n < .05N$. In words, the binomial will suffice if the sample is less than 5 percent of the population.

Poisson Probability Distribution

POISSON PROBABILITY EXPERIMENT

1. The random variable is the number of times some event occurs during a defined interval.
2. The probability of the event is proportional to the size of the interval.
3. The intervals do not overlap and are independent.

POISSON DISTRIBUTION $$P(x) = \frac{\mu^x e^{-\mu}}{x!}$$ **[6–7]**

where:

μ (mu) is the mean number of occurrences (successes) in a particular interval.
e is the constant 2.71828 (base of the Napierian logarithmic system).
x is the number of occurrences (successes).
$P(x)$ is the probability for a specified value of x.

The mean number of successes, μ, can be determined by $n\pi$, where n is the total number of trials and π the probability of success.

MEAN OF A POISSON DISTRIBUTION $$\mu = n\pi$$ **[6–8]**

The variance of the Poisson is also equal to its mean.

Example

To illustrate the Poisson probability computation, assume baggage is rarely lost by Delta Airlines. Most flights do not experience any mishandled bags; some have one bag lost; a few have two bags lost; rarely a flight will have three lost bags; and so on. Suppose a random sample of 1,000 flights shows a total of 300 bags were lost. Thus, the arithmetic mean number of lost bags per flight is 0.3, found by 300/1,000. If the number of lost bags per flight follows a Poisson distribution with $\mu = 0.3$, we can compute the various probabilities using formula (6–7):

For example, the probability of not losing any bags is:

$$P(0) = \frac{(0.3)^0(e^{-0.3})}{0!} = 0.7408$$

In other words, 74 percent of the flights will have no lost baggage. The probability of exactly one lost bag is:

$$P(1) = \frac{(0.3)^1(e^{-0.3})}{1!} = 0.2222$$

Thus, we would expect to find exactly one lost bag on 22 percent of the flights.
Poisson probabilities can also be found in the table in Appendix B.5.

Recall from the previous illustration that the number of lost bags follows a Poisson distribution with a mean of 0.3. Use Appendix B.5 to find the probability that no bags will be lost on a particular flight. What is the probability exactly one bag will be lost on a particular flight? When should the supervisor become suspicious that a flight is having too many lost bags?

TABLE 6–6 Poisson Table for Various Values of μ (from Appendix B.5)

	μ								
x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.9048	0.8187	$\boxed{0.7408}$	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066
1	0.0905	0.1637	0.2222	0.2681	0.3033	0.3293	0.3476	0.3595	0.3659
2	0.0045	0.0164	0.0333	0.0536	0.0758	0.0988	0.1217	0.1438	0.1647
3	0.0002	0.0011	0.0033	0.0072	0.0126	0.0198	0.0284	0.0383	0.0494
4	0.0000	0.0001	0.0003	0.0007	0.0016	0.0030	0.0050	0.0077	0.0111
5	0.0000	0.0000	0.0000	0.0001	0.0002	0.0004	0.0007	0.0012	0.0020
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003
7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Poisson Distribution Example

Example

Coastal Insurance Company underwrites insurance for beachfront properties along the Virginia, North and South Carolina, and Georgia coasts. It uses the estimate that the probability of a named Category III hurricane (sustained winds of more than 110 miles per hour) or higher striking a particular region of the coast (for example, St. Simons Island, Georgia) in any one year is .05. If a homeowner takes a 30-year mortgage on a recently purchased property in St. Simons, what is the likelihood that the owner will experience at least one hurricane during the mortgage period?