Continuous Probability Distributions

Chapter 7

McGraw-Hill/Irwin

Learning Objectives

LO1 List the characteristics of the uniform distribution.

LO2 Compute probabilities by using the uniform distribution.

LO3 List the characteristics of the normal probability distribution.

LO4 Convert a normal distribution to the standard normal distribution.

LO5 Find the probability that an observation on a normally distributed random variable is between two values.

LO6 Find probabilities using the Empirical Rule.

LO7 Approximate the binomial distribution using the normal distribution.

LO8 Describe the characteristics and compute probabilities using the exponential distribution.

LO1 List the characteristics of the uniform distribution.

The Uniform Distribution

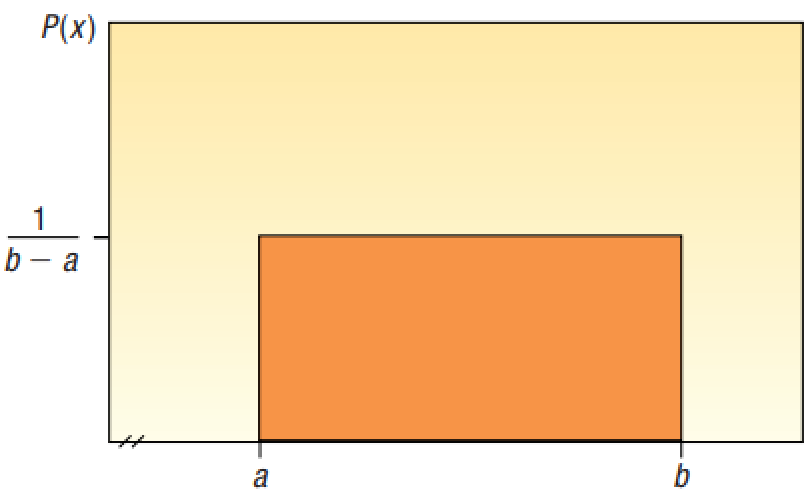
The uniform probability distribution is perhaps the **simplest distribution** for a **continuous random variable**.

This distribution is **rectangular in shape** and is defined by minimum and maximum values.

MEAN OF THE UNIFORM DISTRIBUTION $$\mu = \frac{a + b}{2} \quad [7\text{–}1]$$

STANDARD DEVIATION OF THE UNIFORM DISTRIBUTION $$\sigma = \sqrt{\frac{(b - a)^2}{12}} \quad [7\text{–}2]$$

UNIFORM DISTRIBUTION $$P(x) = \frac{1}{b - a} \quad \text{if } a \leq x \leq b \text{ and } 0 \text{ elsewhere} \quad [7\text{–}3]$$

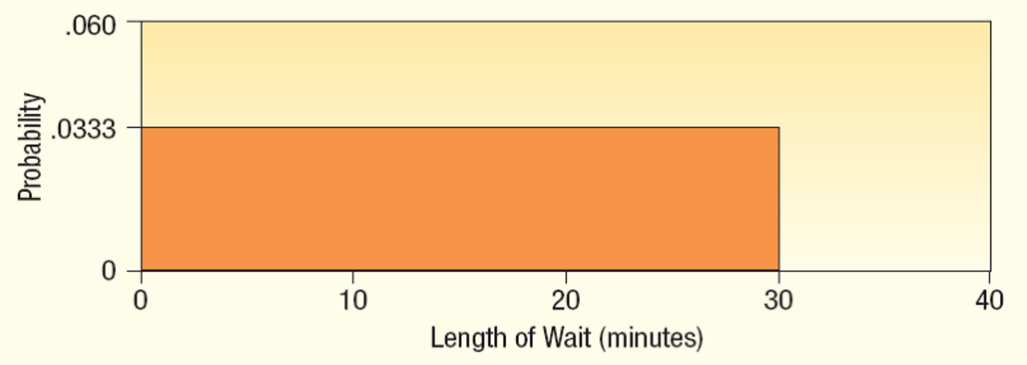


The Uniform Distribution - Example

EXAMPLE

Southwest Arizona State University provides bus service to students while they are on campus. A bus arrives at the North Main Street and College Drive stop every 30 minutes between 6 A.M. and 11 P.M. during weekdays. Students arrive at the bus stop at random times. The time that a student waits is uniformly distributed from 0 to 30 minutes.

1. Draw a graph of this distribution.
2. Show that the area of this uniform distribution is 1.00.
3. How long will a student "typically" have to wait for a bus? In other words what is the mean waiting time? What is the standard deviation of the waiting times?
4. What is the probability a student will wait between 25 and 30 minutes, 10 and 20 minutes?



The times students must wait for the bus is uniform over the interval from 0 minutes to 30 minutes, so in this case *a* is 0 and *b* is 30.

$$\text{Area} = (\text{height})(\text{base}) = \frac{1}{(30-0)}(30-0) = 1.00$$

$$\mu = \frac{a+b}{2} = \frac{0+30}{2} = 15$$

$$\sigma = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{(30-0)^2}{12}} = 8.66$$

$$P(25 < \text{wait time} < 30) = (\text{height})(\text{base}) = \frac{1}{(30-0)}(5) = .1667$$

The Family of Normal Probability Distributions

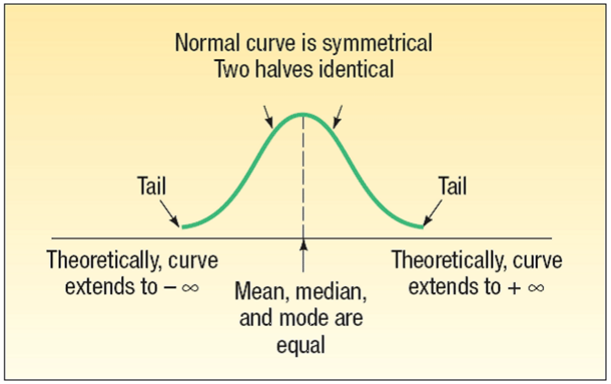
NORMAL PROBABILITY DISTRIBUTION $$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left[\frac{(X-\mu)^2}{2\sigma^2}\right]} \quad \textbf{[7–4]}$$

- μ and σ are the mean and the standard deviation, as usual
- π is a natural mathematical constant.
- You will not need to make any calculations from formula (7–4). Instead you will be using a table, which is given as Appendix B.1, to look up the various probabilities

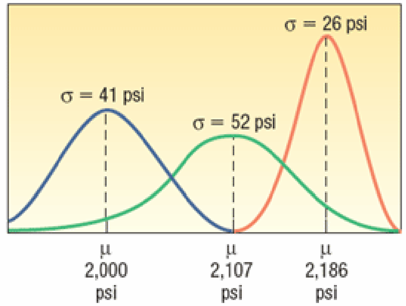
LO3 List the characteristics of the normal probability distribution.

Normal Probability Distribution

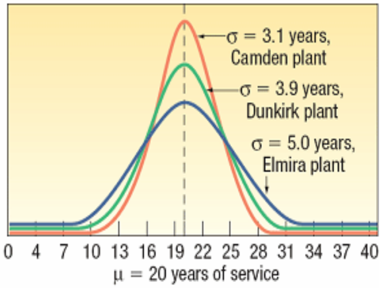
1. It is **bell-shaped** and has a single peak at the center of the distribution.
2. It is **symmetrical** about the mean
3. It is **asymptotic:** The curve gets closer and closer to the *X*-axis but never actually touches it.
4. The location of a normal distribution is determined by the mean, μ, the dispersion or spread of the distribution is determined by the standard deviation, σ .
5. The arithmetic **mean, median, and mode are equal**
6. The total **area under the curve is 1.00**; half the area under the normal curve is to the right of this center point and the other half to the left of it



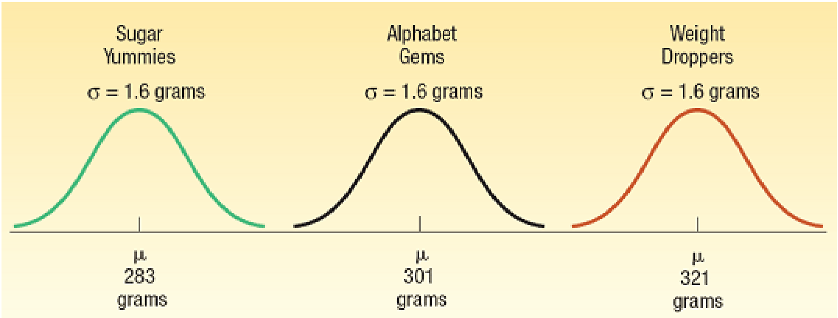
Family of Distributions



Different Means and Standard Deviations



Equal Means and Different Standard Deviations



Different Means and Equal Standard Deviations

LO4 Convert a normal distribution to the standard normal distribution

The Standard Normal Probability Distribution

- **The standard normal distribution** is a normal distribution with a mean of 0 and a standard deviation of 1.
- It is also called the *z* distribution (z values or z scores.
- A *z*-value is the signed distance between a selected value, designated *X*, and the population mean μ, divided by the population standard deviation, σ.
- The formula is:

STANDARD NORMAL VALUE

$$z = \frac{X - \mu}{\sigma} \quad \text{[7–5]}$$

Where:

- X is the value of any particular observation or measurement.
- μ is the mean of the distribution
- σ is the standard deviation of the distribution.

LO4 Convert a normal distribution to the standard normal distribution

The Standard Normal Probability Distribution - Example

TABLE 7–1 Areas under the Normal Curve

z	0.00	0.01	0.02	0.03	0.04	0.05	. . .
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	
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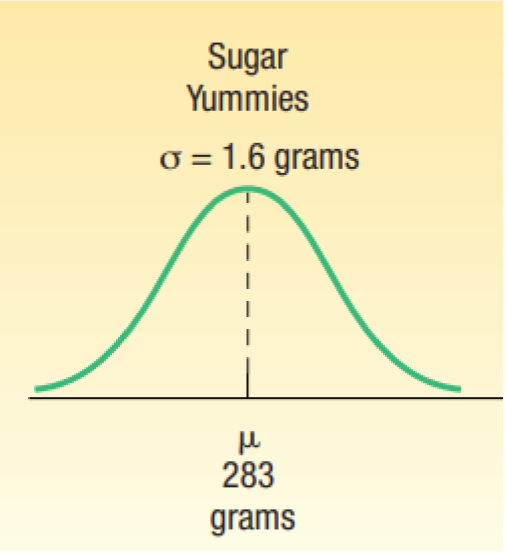
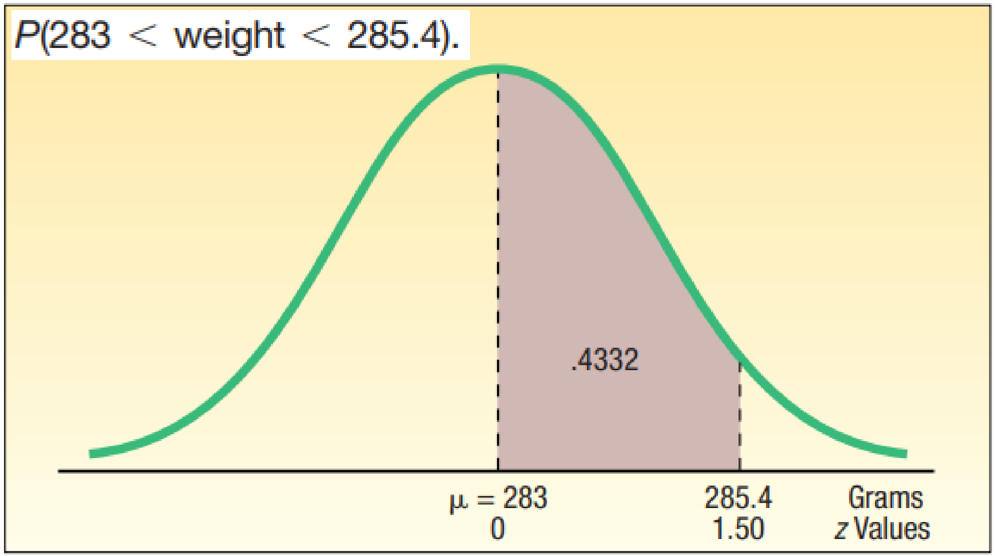




LO5 Find the probability that an observation on a normally distributed random variable is between two values.

The Normal Distribution – Example

The weekly incomes of shift foremen in the glass industry follow the normal probability distribution with a mean of \$1,000 and a standard deviation of \$100.

What is the *z* value for the income, let's call it *X*, of a foreman who earns \$1,100 per week? For a foreman who earns \$900 per week?

For $X = \$1,100$:

$$z = \frac{X - \mu}{\sigma} = \frac{\$1,100 - \$1,000}{\$100} = 1.00$$

For $X = \$900$:

$$z = \frac{X - \mu}{\sigma} = \frac{\$900 - \$1,000}{\$100} = -1.00$$

Normal Distribution – Finding Probabilities

EXAMPLE

The mean weekly income of a shift foreman in the glass industry is normally distributed with a mean of $1,000 and a standard deviation of $100.

What is the likelihood of selecting a foreman whose weekly income is between $1,000 and $1,100? $P(\$1{,}000 < \textit{weekly income} < \$1{,}100)$

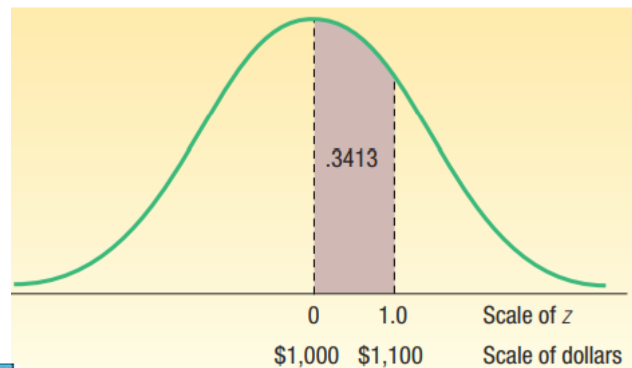
z	0.00	0.01	0.02
.	.	.	.
.	.	.	.
.	.	.	.
0.7	.2580	.2611	.2642
0.8	.2881	.2910	.2939
0.9	.3159	.3186	.3212
1.0	.3413	.3438	.3461
1.1	.3643	.3665	.3686

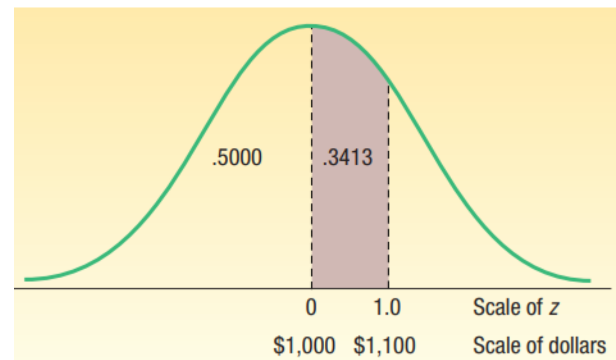
For $X = \$1{,}000$:

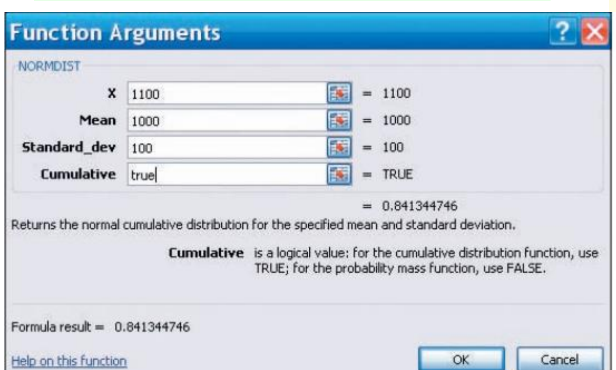
$$z = \frac{X - \mu}{\sigma} = \frac{\$1{,}000 - \$1{,}000}{\$100} = 0.00$$

For $X = \$1{,}100$:

$$z = \frac{X - \mu}{\sigma} = \frac{\$1{,}100 - \$1{,}000}{\$100} = 1.00$$





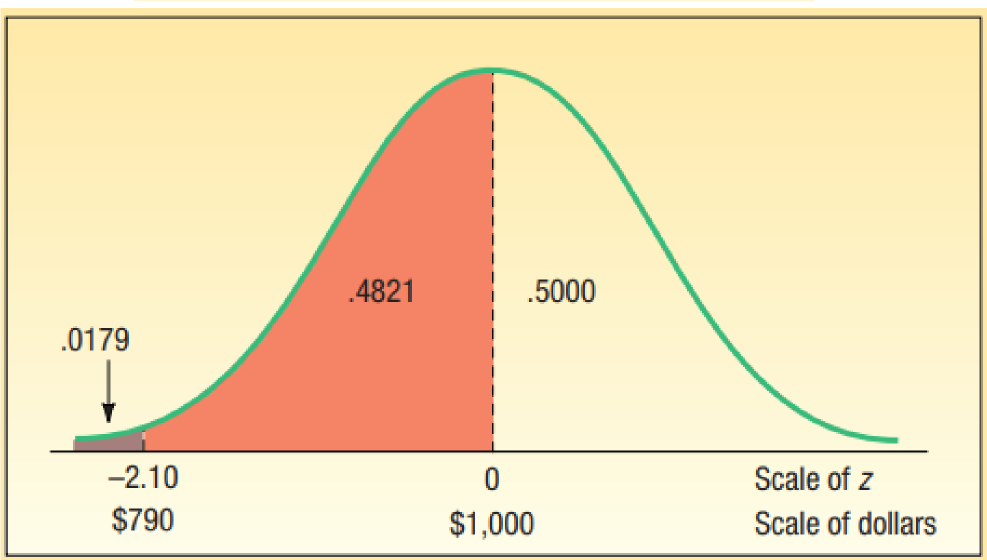


Normal Distribution – Finding Probabilities (Example 2)

Refer to the information regarding the weekly income of shift foremen in the glass industry. The distribution of weekly incomes follows the normal probability distribution with a mean of $1,000 and a standard deviation of $100. What is the probability of selecting a shift foreman in the glass industry whose income is:

1. Between $790 and $1,000?
2. Less than $790?

$$z = \frac{X - \mu}{s} = \frac{\$790 - \$1{,}000}{\$100} = -2.10$$



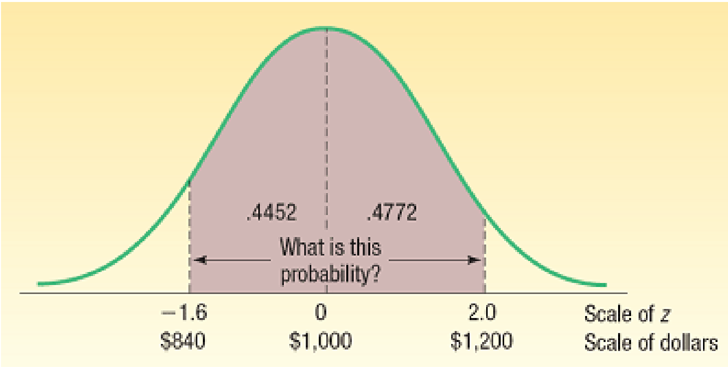
What is the probability of selecting a shift foreman in the glass industry whose income is:
Between $840 and $1,200

For $X = \$840$:

$$z = \frac{X - \mu}{\sigma} = \frac{\$840 - \$1{,}000}{\$100} = -1.60$$

For $X = \$1{,}200$:

$$z = \frac{X - \mu}{\sigma} = \frac{\$1{,}200 - \$1{,}000}{\$100} = 2.00$$



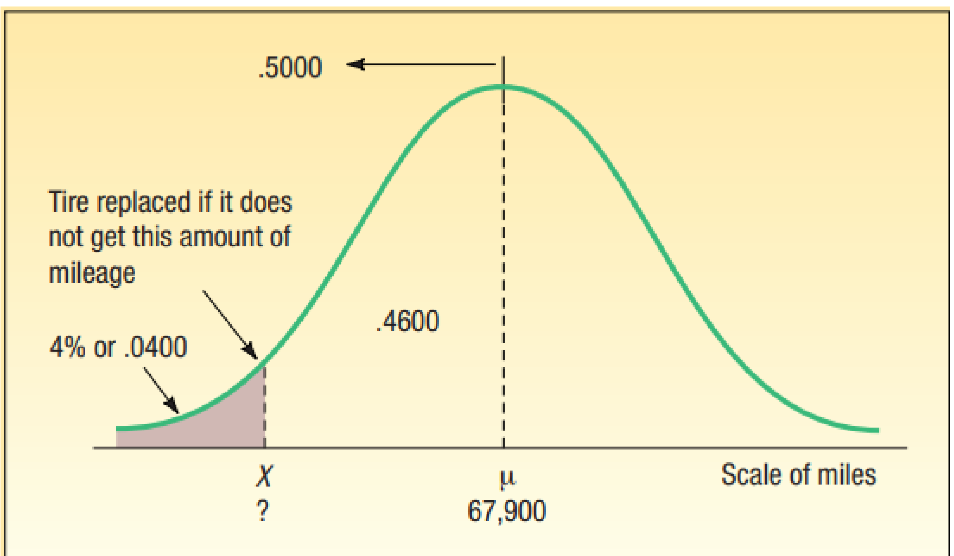
Using Z in Finding X Given Area - Example

LO5

Layton Tire and Rubber Company wishes to set a minimum mileage guarantee on its new MX100 tire. Tests reveal the mean mileage is 67,900 with a standard deviation of 2,050 miles and that the distribution of miles follows the normal probability distribution. Layton wants to set the minimum guaranteed mileage so that no more than 4 percent of the tires will have to be replaced.

What minimum guaranteed mileage should Layton announce?

z ...	.03	.04	.05	.06
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
1.5	.4370	.4382	.4394	.4406
1.6	.4484	.4495	.4505	.4515
1.7	.4582	.4591	$\boxed{.4599}$	.4608
1.8	.4664	.4671	.4678	.4686



Solve X using the formula :

$$z = \frac{x - \mu}{\sigma} = \frac{x - 67{,}900}{2{,}050}$$

The value of z is found using the 4% information

The area between 67,900 and x is 0.4600, found by 0.5000 - 0.0400

Using Appendix B.1, the area closest to 0.4600 is 0.4599, which gives a z alue of -1.75. Then substituting into the equation :

$$-1.75 = \frac{x - 67{,}900}{2{,}050}, \text{ then solving for } x$$

$$-1.75(2{,}050) = x - 67{,}900$$

$$x = 67{,}900 - 1.75(2{,}050)$$

$$x = 64{,}312$$

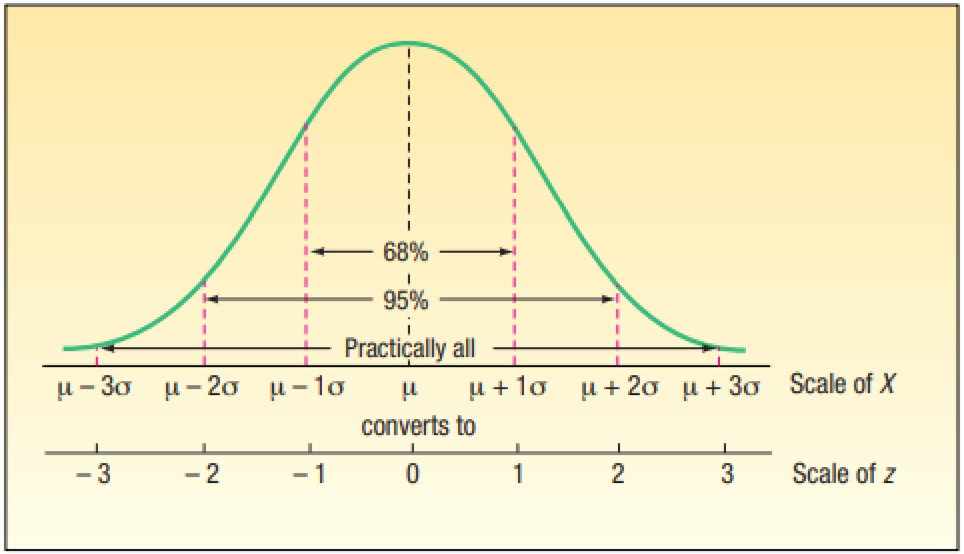
The standard normal probability distribution

1. To find the area between 0 and z or $(-z)$, look up the probability directly in the table.
2. To find the area beyond z or $(-z)$, locate the probability of z in the table and subtract that probability from .5000.
3. To find the area between two points on different sides of the mean, determine the z values and add the corresponding probabilities.
4. To find the area between two points on the same side of the mean, determine the z values and subtract the smaller probability from the larger.

The Empirical Rule

The Empirical Rule (Chapter 3):

1. About 68 percent of the area under the normal curve is within one standard deviation of the mean. This can be written as $\mu \pm 1\sigma$.
2. About 95 percent of the area under the normal curve is within two standard deviations of the mean, written $\mu \pm 2\sigma$.
3. Practically all of the area under the normal curve is within three standard deviations of the mean, written $\mu \pm 3\sigma$



The Empirical Rule - Example

As part of its quality assurance program, the Autolite Battery Company conducts tests on battery life. For a particular D-cell alkaline battery, the mean life is 19 hours. The useful life of the battery follows a normal distribution with a standard deviation of 1.2 hours.

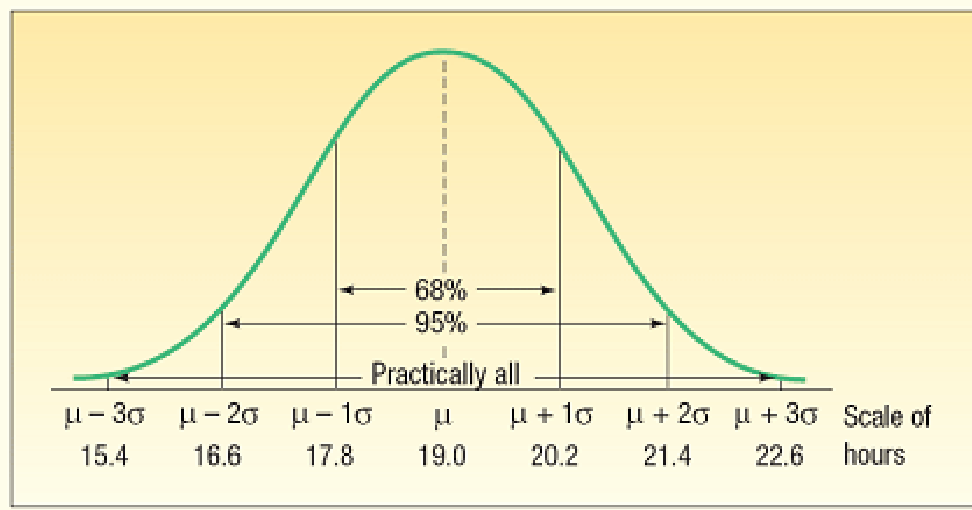
Answer the following questions.

1. About 68 percent of the batteries failed between what two values?
2. About 95 percent of the batteries failed between what two values?
3. Virtually all of the batteries failed between what two values?

We can use the results of the Empirical Rule to answer these questions.

1. About 68 percent of the batteries will fail between 17.8 and 20.2 hours by 19.0 ± 1(1.2) hours.
2. About 95 percent of the batteries will fail between 16.6 and 21.4 hours by 19.0 ± 2(1.2) hours.
3. Virtually all failed between 15.4 and 22.6 hours, found by 19.0 ± 3(1.2)

This information is summarized on the following chart.



LO7 Approximate the binomial distribution using the normal distribution.

Normal Approximation to the Binomial

- The normal distribution (a continuous distribution) yields a good approximation of the binomial distribution (a discrete distribution) for large values of n.
- The normal probability distribution is generally a good approximation to the binomial probability distribution when $n\pi$ and $n(1-\pi)$ are both greater than 5. This is because as n increases, a binomial distribution gets closer and closer to a normal distribution.

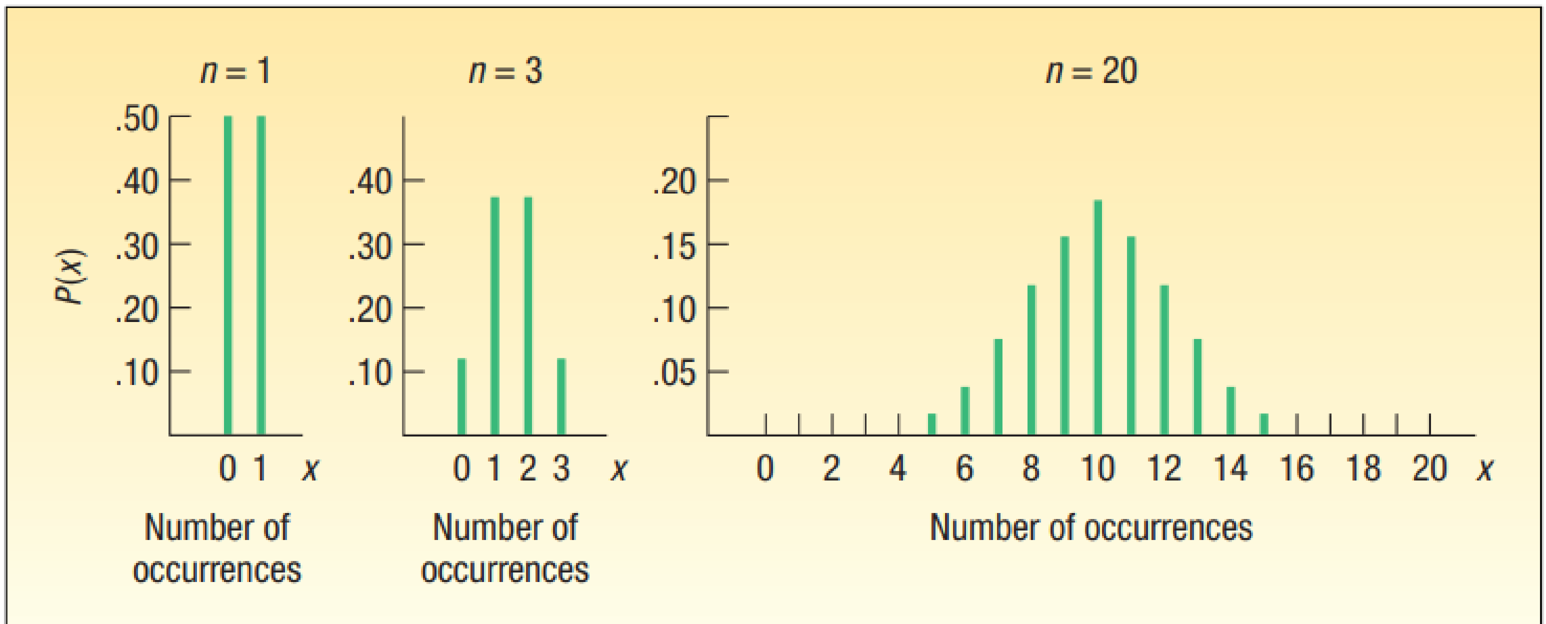


CHART 7–7 Binomial Distributions for an n of 1, 3, and 20, Where $\pi = .50$

Continuity Correction Factor

CONTINUITY CORRECTION FACTOR The value .5 subtracted or added, depending on the question, to a selected value when a discrete probability distribution is approximated by a continuous probability distribution.

Only four cases may arise. These cases are:

1. For the probability at least X occurs, use the area above $(X - .5)$.
2. For the probability that more than X occurs, use the area above $(X + .5)$
3. For the probability that X or fewer occurs, use the area below $(X + .5)$.
4. For the probability that fewer than X occurs, use the area below $(X - .5)$

Normal Approximation to the Binomial - Example

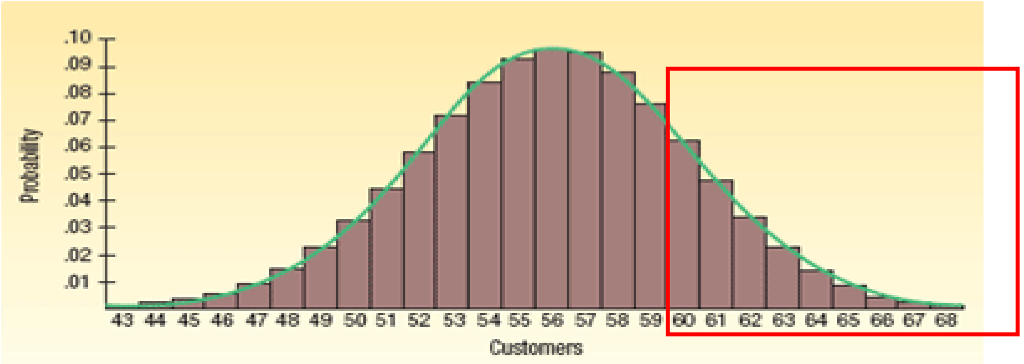
Suppose the management of the Santoni Pizza Restaurant found that 70 percent of its new customers return for another meal. For a week in which 80 new (first-time) customers dined at Santoni's, what is the probability that 60 or more will return for another meal?

$$P(x) = {}_{n}C_{x}\,(\pi)^{x}\,(1 - \pi)^{n-x}$$

$$P(x = 60) = {}_{80}C_{60}\,(.70)^{60}\,(1 - .70)^{20} = .063$$

$$P(x = 61) = {}_{80}C_{61}\,(.70)^{61}\,(1 - .70)^{19} = .048$$

Number Returning	Probability	Number Returning	Probability
43	.001	56	.097
44	.002	57	.095
45	.003	58	.088
46	.006	59	.077
47	.009	60	.063
48	.015	61	.048
49	.023	62	.034
50	.033	63	.023
51	.045	64	.014
52	.059	65	.008
53	.072	66	.004
54	.084	67	.002
55	.093	68	.001



$P(X \geq 60) = 0.063+0.048+ \ldots + 0.001 = 0.197$

Normal Approximation to the Binomial - Example

Suppose the management of the Santoni Pizza Restaurant found that 70 percent of its new customers return for another meal. For a week in which 80 new (first-time) customers dined at Santoni's, what is the probability that 60 or more will return for another meal?

$$\mu = n\pi = 80(.70) = 56$$

$$\sigma^2 = n\pi(1-\pi) = 80(.70)(1-.70) = 16.8$$

$$\sigma = \sqrt{16.8} = 4.10$$

$$z = \frac{X-\mu}{\sigma} = \frac{59.5-56}{4.10} = 0.85$$

Step 1. Find the mean and the variance of a binomial distribution and find the *z* corresponding to an X of 59.5 $(x - .5$, the correction factor)

Step 2: Determine the area from 59.5 and beyond.

Step 3: Calculate the area beyond 59.5 by subtracting $.3023$ from $.5000$ $(.5000 - .3023 = .1977)$.