Disjunctivity of the Erdős–Borwein constant

Abstract

We prove that every finite binary word occurs infinitely often in the Erdős–Borwein constant, answering a question recorded by Bailey and Crandall in 2002. Every fixed word has at least $N\exp\{-C(\log\log N)^3\}$ occurrences beginning among the first N digits, for all sufficiently large N. The proof extends Campbell's construction for the word 11: one prescribed divisor count encodes the desired word, while congruences and an averaged estimate control the surrounding terms.

1 Introduction

Let $d(n)$ be the number of positive divisors of n. The Erdős–Borwein constant is

$$E = \sum_{m\ge1} \frac{1}{2^m-1} = \sum_{m,j\ge1} 2^{-mj} = \sum_{n\ge1} \frac{d(n)}{2^n}, \tag{1}$$

where absolute convergence justifies regrouping. A real number is *binary disjunctive* if its binary expansion contains every finite binary word, or equivalently if its doubling orbit is dense in $[0,1]$. For a nonempty word w, let $N_w(N)$ count its occurrences beginning at positions $1 \le n \le N$ after the binary point of E.

Theorem 1. *For every nonempty finite binary word w, there are constants $C_w > 0$ and $N_0(w)$ such that*

$$N_w(N) \ge N\exp\bigl\{-C_w(\log\log N)^3\bigr\} \qquad (N \ge N_0(w)). \tag{2}$$

In particular, E is binary disjunctive, and every finite binary word occurs infinitely often.

Erdős [4] proved irrationality by forcing divisibility of consecutive divisor counts and controlling the remaining tail. Bailey and Crandall [2, §5, p. 540] explicitly identify binary disjunctivity of E, which they call 2-density, as an open problem. Campbell [3, Theorem 1 and §4] proves infinitely many occurrences of 11 and asks about arbitrary words. We extend his construction, using the prime-distribution theorem of Alford, Granville, and Pomerance [1] in Vandehey's formulation [5, Proposition 2.1]. The elementary divisor estimate below is Campbell's Lemma 2; its proof is included.

The idea is to make all early terms of a shifted series integers except one. An index of the form $q^{a-1}p$, with distinct primes q, p, has exactly $2a$ divisors. Choosing a encodes the desired word; averaging the tail leaves sufficiently many prime indices where carries cannot alter it.

Throughout, log is natural, and $\log_2$ has base two. Implied constants are absolute unless a subscript indicates dependence. Sums and counts over $m < M$ mean $0 \le m < M$.

2 Preliminaries

Lemma 2 (Alford–Granville–Pomerance)**.** *There are constants $X_0, D_0 > 0$ such that, for every $X > X_0$, there is a set $\mathcal{D}(X)$ of at most D_0 integers, each greater than $\log X$, for which*

$$\pi(X;B,s) \ge \frac{X}{2\varphi(B)\log X}$$

whenever $1 \le B \le X^{1/4}$, $\gcd(s,B) = 1$, and no member of $\mathcal{D}(X)$ divides B. Here $\pi(X;B,s)$ counts primes $p \le X$ with $p \equiv s \pmod B$.

Lemma 3 (Divisor average)**.** *If u, A, M are positive integers,* $\gcd(u, A) = 1$*, and $u + (M-1)A \le Y$ with $Y \ge 1$, then*

$$\sum_{m<M} d(u + mA) \le 2M(1 + \tfrac{1}{2}\log Y) + 2\sqrt{Y}.$$

Proof. Pairing divisors about the square root gives

$$\sum_{m<M} d(u+mA) \le 2 \sum_{h \le \sqrt{Y}} \#\{m < M : h \mid u + mA\} \le 2 \sum_{h \le \sqrt{Y}} (M/h + 1).$$

For $\gcd(h, A) > 1$ there are no solutions; otherwise they form one residue class modulo h. The conclusion follows from $\sum_{h \le z} h^{-1} \le 1 + \log z$ for $z \ge 1$. □

3 Proof of Theorem 1

Fix a word w of length ℓ and integer value $0 \le v < 2^\ell$, including leading zeros. Put

$$I_w = [v2^{-\ell}, (v+1)2^{-\ell}), \qquad r = \ell + 2, \qquad a = 4v + 2.$$

Then $a/2^r = (v + \frac{1}{2})2^{-\ell}$ is the midpoint of I_w. We will arrange that $\{2^{n-1}E\}$ differs from this midpoint by less than $2^{-\ell-1}$, so that w begins at digit n.

Let X tend to infinity, and set

$$k = \lceil 4 \log_2 \log X \rceil, \qquad L = \left\lfloor (\log_2 X)^2 \right\rfloor. \tag{3}$$

In what follows, X is sufficiently large depending on w; in particular, $k > r$ and $L \ge 2k$.

1. Construct the progression

For each $D \in \mathcal{D}(X)$ having a prime divisor in $(L, 2L)$, discard one such prime. This discards at most D_0 primes. By the prime number theorem, $\gg L/\log L$ primes remain, more than k^2. Choose distinct remaining primes q and $p_{j,t}$ for

$$\mathcal{J} = \{0, \dots, k-1\} \setminus \{r\}, \qquad j \in \mathcal{J}, \quad 1 \le t \le j+1.$$

Define

$$P_j = \prod_{t=1}^{j+1} p_{j,t}, \qquad Q = q^{a-1}, \qquad A = q^a \prod_{j \in \mathcal{J}} P_j^2, \qquad B = A/Q = q \prod_{j \in \mathcal{J}} P_j^2.$$

There are at most k^2 selected primes, and

$$\log B = O(k^2 \log L) = O((\log\log X)^3), \qquad Q \le (2L)^{a-1} = (\log X)^{O_w(1)}. \tag{4}$$

Thus $B \le X^{1/4}$. No $D \in \mathcal{D}(X)$ divides B: if it did, all its prime factors would lie in $(L, 2L)$, including the one discarded.

The Chinese remainder theorem gives a unique $0 \le R < A$ such that

$$R + r \equiv Q \pmod{q^a}, \qquad R + j \equiv P_j \pmod{P_j^2} \quad (j \in \mathcal{J}). \tag{5}$$

The congruence for $j = 0$ gives $R > 0$. Since $Q > r$, the integer $s = (R + r)/Q$ satisfies $1 \le s \le B$; also $s \equiv 1 \pmod q$, so $s < B$. For every $p \mid P_j$,

$$Qs = R + r \equiv r - j \not\equiv 0 \pmod p,$$

because $0 < |r - j| < k < L < p$. Hence $\gcd(s, B) = 1$.

Put $M = \lfloor X/B \rfloor + 1$ and $n_m = R + mA$ for $m < M$. Lemma 2 supplies at least

$$\frac{X}{2B \log X} \geq \frac{M}{4 \log X} \tag{6}$$

indices for which $p = s + mB \leq X$ is prime. At these indices, $n_m + r = Qp = q^{a-1}p$, and $p \equiv 1 \pmod{q}$ ensures $p \neq q$. Consequently

$$d(n_m + r) = 2a, \qquad 2^{j+1} \mid d(n_m + j) \quad (j \in \mathcal{J}). \tag{7}$$

The second assertion holds for every $m < M$: by (5), each of the $j + 1$ primes in P_j divides $n_m + j$ to exponent exactly one.

2. Control the tail

Write $T_m = \sum_{j \geq k} d(n_m + j)2^{-j}$ and $Y = 2QX$. For $k \leq j < L$, we have $\gcd(R + j, A) = 1$. Indeed, each prime $p \mid A$ divides an assigned $R + j_p$ with $0 \leq j_p < k$, whereas $0 < j - j_p < L < p$. Furthermore,

$$n_m + j < QB + QX + L \leq Y.$$

By (4), $\log Y = O_w(\log X)$ and $\sqrt{Y} = o(M)$, since $M \geq X^{3/4}$. Lemma 3 therefore gives

$$\sum_{m<M} d(n_m + j) \ll_w M \log X \qquad (k \leq j < L).$$

After multiplying by 2^{-j} and summing, this part of $\sum_{m<M} T_m$ is $O_w(M(\log X)2^{-k})$.

For $j \geq L$, use $d(n_m + j) \leq 2\sqrt{Y + j}$ and $\sum_{j \geq L} \sqrt{j}\, 2^{-j} \leq 4\sqrt{L}\, 2^{-L}$. Since $Y \leq 2^L$ and $L \geq 2k$ for large X, the remaining part is

$$\ll M(\sqrt{Y} + \sqrt{L})2^{-L} \ll M2^{-L/2} \leq M2^{-k}.$$

Thus

$$\sum_{m<M} T_m \ll_w M(\log X)2^{-k}.$$

As $T_m \geq 0$, the number of indices with $T_m \geq 2^{-\ell}$ is at most $2^\ell \sum_{m<M} T_m$, hence

$$\#\{m < M : T_m \geq 2^{-\ell}\} \ll_w \frac{M}{(\log X)^3} = o\left(\frac{M}{\log X}\right).$$

Here we used $2^{-k} \leq (\log X)^{-4}$ from (3). Comparing with (6), at least $M/(8 \log X)$ prime indices also satisfy $T_m < 2^{-\ell}$.

3. Read and count the words

For any such index, put $n = n_m$. By (1) and (7),

$$\begin{aligned} 2^{n-1}E &= \sum_{1\le t<n} d(t)2^{n-1-t} + \sum_{0\le j<k} \frac{d(n+j)}{2^{j+1}} + \frac{T_m}{2} \\ &= \text{an integer} + \frac{a}{2^r} + \frac{T_m}{2}. \end{aligned}$$

Since $0 \le T_m/2 < 2^{-\ell-1}$, the fractional part lies in I_w. Thus w begins at $n = Qp - r < QX$. Distinct prime indices give distinct occurrences, at least $M/(8\log X)$ in total.

To obtain (2) for every sufficiently large N, choose

$$U = \log_2 N, \qquad X = \frac{N}{2(2U^2)^{a-1}}.$$

Then $X \to \infty$ and $Q \le (2L)^{a-1} \le (2U^2)^{a-1}$, so $QX \le N/2$. The preceding count gives

$$N_w(N) \ge \frac{M}{8\log X} \ge \frac{X}{8B\log X}.$$

Finally, $\log(N/X) = O_w(\log\log N)$ and $\log B = O((\log\log N)^3)$ by (4). Absorbing these costs and $\log(8\log X)$ into $C_w(\log\log N)^3$ proves (2). □

References

[1] W. R. Alford, A. Granville, and C. Pomerance, *There are infinitely many Carmichael numbers*, Ann. of Math. (2) **139** (1994), no. 3, 703–722. doi:10.2307/2118576.

[2] D. H. Bailey and R. E. Crandall, *Random generators and normal numbers*, Experimental Mathematics **11** (2002), no. 4, 527–546. doi:10.1080/10586458.2002.10504704.

[3] J. M. Campbell, *On the binary digits of the Erdős–Borwein constant*, arXiv:2605.24160, 2026. arxiv.org/abs/2605.24160.

[4] P. Erdős, *On arithmetical properties of Lambert series*, J. Indian Math. Soc. (N.S.) **12** (1948), 63–66. Author archive.

[5] J. Vandehey, *On an incomplete argument of Erdős on the irrationality of Lambert series*, Integers **13** (2013), Paper A58. arXiv:1206.0340.